Non-toric examples of elliptic canonical bases

§ Introduction

v: grading shift

Typical situation: $\mathcal{A}$: graded h.w. cat $\circlearrowleft$ $\mathbb{D}$ duality

$\left\{ \begin{array}{l} \text{a } \Lambda \text{ : poset} \\ \text{a } \{\nabla_\lambda\}_{\lambda \in \Lambda} \text{ : costandard} \\ \quad \{\Delta_\lambda\} \text{ : standard} \end{array} \right.$

s.t. "....." $\mathbb{D}\nabla_\lambda = \Delta_\lambda$

$\mathbb{D}v = v^{-1}\mathbb{D}$

canonical basis : in $K(\mathcal{A}) \circlearrowleft \mathbb{Z}[v, v^{-1}]$

= classes of simple modules

$\{L_\lambda\}_{\lambda \in \Lambda}$

Basic problem compute $[L_\lambda]$ in terms of $\{[\nabla_\lambda]\}_{\lambda \in \Lambda}$

Kazhdan-Lusztig type algorithm

$[L_\lambda]$ is uniquely determined by

1. $\mathbb{D}[L_\lambda] = [L_\lambda]$
2. $[L_\lambda] \in [\nabla_\lambda] + \sum_{\mu < \lambda} v^{-1}\mathbb{Z}[v^{-1}][\nabla_\mu]$

· K-theoretic canonical basis

Lusztig : defined "canonical bases"

in $K_{eq}(T^*(G/B))$

by · defing bar involution directly

· use inner product to characterize (up to sign)

instead triangular property

Lusztig's conj (proved by Bezrukavnikov-Mirković)

canonical bases "controls" rep th of $\mathfrak{g}_{\overline{\mathbb{F}}_p}$ char >> 0

<u>Observation</u> costandard/standard bases are given by <u>K-theoretic stable/unstable bases</u>

$\rightsquigarrow$. one can generalize this picture to conical symplectic resolution (equipped with duality $(X, X^!)$ symplectic A^{G} X^A: finite)

. one can introduce slope/quantization param $s \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$

<u>Main conj</u> $\forall s$: <u>generic</u>

1. canonical basis $\mathcal{B}_{X,s} = \{\mathcal{E}^s_\lambda\}_{\lambda \in \Lambda}$ $\quad (\Lambda = X^A \times X^*(A))$

is a $\mathbb{Z}[v, v^{-1}]$-basis $K_{A \times \mathbb{C}^*_v}(X)$

Moreover, $\exists \mathcal{T}_s$: tilting bdl on X

s.t. $\mathcal{E}^s_\lambda =$ [direct summand of $\mathcal{T}_s$]

(up to equiv shift)

2. $\mathcal{A}_S := \mathrm{End}(\mathcal{T}_S)^{op}$ $\mathcal{A}_S^!$: Koszul dual

$$D^b\mathrm{Coh}^{A\times\mathbb{C}_v^\times}(X) \xrightarrow{\sim} D^b\mathcal{A}_S\text{-gmod}^A$$
$$\downarrow \wr \text{ Koszul duality}$$
$$D^b\mathcal{A}_S^!\text{-gmod}^A$$

$\mathcal{A}_S^!\text{-gmod}^A$: graded h.w. by
cat Stable basis $\longleftrightarrow$ costandard
Unstab (slope) $\longleftrightarrow$ standard
$\mathcal{E}_\lambda^s$ $\longleftrightarrow$ simple module

Goal Define and calculate elliptic analogue

easy: elliptic bar involution

problem triangular property does not generalize

add some natural "expected properties"
until it is characterized uniquely

1st step : toric case (done)

$\rightsquigarrow$ extract some properties

2nd step : very small nontoric case

$X = T^*Gr(2,4)$, $X^!$

§ Symplectic duality

Data / Notation / Assumption

- X : conical symp resol $\mathbb{C}^\times_v$

  i.e.

- X : smooth alg symp var
  wt (symp form) $= 2$
- $\mathbb{C}[X]$: nonnegatively graded, $\mathbb{C}[X]_0 = \mathbb{C}$
- $X \longrightarrow \operatorname{Spec} \mathbb{C}[X]$: resol of sing

- $A \curvearrowright X$: Hamiltonian torus action
  commute with $\mathbb{C}^\times$-action

  s.t. X^A : finite

- for $p \in X^A$, $\Phi(p) := \{$ A-wts appearing in $T_pX\}$ with multiplicity "equivariant root"
- define $n_p : \Phi(p) \to \mathbb{Z}$ by $T_pX = \sum_{\alpha \in \Phi(p)} v^{n_p(\alpha)-1} a^\alpha$
- choose $\mathcal{C} \subset \pi_0(X_*(A) \otimes \mathbb{R} \setminus \{$root hyperplanes$\})$ chamber

$\rightsquigarrow \Phi(p) = \Phi_+(p) \sqcup \Phi_-(p)$

$T_pX = N_{p,-} + N_{p,+}$

· choose P : free $\mathbb{Z}$-module "tautological line bundles"

& $\mathcal{L} : P \longrightarrow \mathrm{Pic}^{A \times \mathbb{C}^\times}(X)$

s.t. $P \longrightarrow \mathrm{Pic}^{A \times \mathbb{C}^\times}(X) \longrightarrow \mathrm{Pic}(X)$: surjective

· $\mathrm{Amp} :=$ pullback of ample cone of $X \subset P \otimes_{\mathbb{Z}} \mathbb{R}$

· $T_X^{1/2} \in K_{A \times \mathbb{C}^\times}(X)$ polarization

s.t. $T_X = T_X^{1/2} + \hbar^{-1} (T_X^{1/2})^\vee$

$\det T_X \cong \mathcal{L}(\ ^{-1} \hbar_A)$ as A-equiv line bdl

· similar data for another con. symp resol $X^!$

Def $X = (X, A, \dots)$ & $X^! = (X^!, A^!, \dots)$

forms a dual pair if

∃ identification $X^A \cong (X^!)^{A^!}$ (order reversing)

$X_*(A) \cong P^!$,

$P \cong X_*(A^!)$ s.t.

· $\mathcal{C} = \mathrm{Amp}^!$, $\mathrm{Amp} = \mathcal{C}^!$,

$\forall p \in X^A, \lambda \in P, \mu \in P'$

$$\langle wt_A \mathcal{L}(\lambda)|_p, \mu \rangle + \langle wt_{A'} \mathcal{L}'(\mu)|_p, \lambda \rangle = 0$$

$$wt_{\mathbb{C}^\times} \mathcal{L}(\lambda)|_p = \sum_{\beta \in \Phi'_+(p)} \langle \beta, \lambda \rangle$$

$$wt_{\mathbb{C}^\times} \mathcal{L}'(\mu)|_p = \sum_{\alpha \in \Phi_+(p)} \langle \alpha, \mu \rangle$$

$$\sum_{\alpha \in \Phi_+(p)} w_p(\alpha) + \sum_{\beta \in \Phi'_+(p)} w'_p(\beta) = 0$$

$$w_p(\alpha) + \langle \kappa_{X'}, \alpha \rangle \equiv 1 \mod 2 \quad \forall \alpha \in \Phi_+(p)$$

$$w'_p(\beta) + \text{- - - - -}$$

<u>Def</u> $-X := (X, A, -\mathcal{C}, \cdots\cdots)$

$X_{flop} := (\text{dual of } -X^!)$

§ Elliptic bar involution

- $\tau \in \mathcal{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im} \tau > 0\}$

$q = e^{2\pi i \tau} \rightsquigarrow \mathbb{C}^\times / q^{\mathbb{Z}}$: elliptic curve

- theta fcn $\vartheta(x) := (x^{1/2} - x^{-1/2}) \prod_{m=1}^{\infty} (1 - q^m x)(1 - q^m x^{-1})$

$$\left(\vartheta\left(\sum x_i - \sum y_j\right) = \frac{\prod \vartheta(x_i)}{\prod \vartheta(y_j)} \right)$$

• $\widetilde{K}(X)_{loc} := \bigoplus_{p \in X^A} \Big($ multivalued merom fcn on $\mathfrak{h} \times A \times A^! \times \mathbb{C}^\times_v \Big)$

param: $\tau \quad a \quad z \quad v$

for $p \in X^A$ $\exists$ $\mathrm{Stab}^{ell}_X(p) = \left(\mathrm{Stab}^{ell}_X(p)|_{p'}\right)_{p' \in X^A} \in \widetilde{K}(X)_{loc}$

(renormalized) elliptic stable basis ([Aganagic-Okounkov])

s.t. • $\mathrm{Stab}^{ell}_X(p)|_p = \vartheta(N_{p,-})\,\vartheta(N^!_{p,-})$

$\mu \in P^!$ • $\left(\dfrac{\mathrm{Stab}^{ell}_X(p)|_{p'}}{\vartheta(N_{p',-})}\right)_{a \mapsto q^\mu a} = \dfrac{\mathcal{L}^!(\mu)|_{p'}}{\mathcal{L}^!(\mu)|_p} \cdot$

$$\left(\frac{\mathrm{Stab}^{ell}_{X}(p)|_{p'}}{\vartheta(N'_{p,-})}\right)_{z\mapsto q^{\lambda}z} = \frac{\mathcal{L}(\lambda)|_{p}}{\mathcal{L}(\lambda)|_{p'}} \cdot (\,) \qquad \lambda \in P$$

$$\left(\qquad\right)_{v\mapsto qv} = (\,)\left(\qquad\right)$$

support condition

Def (elliptic bar involution)

$$\beta^{ell}_X : \widetilde{K}(X)_{loc} \to K(X)_{loc} \quad : v\text{-semilinear} \quad \beta v = v^{-1}\beta$$

by $$\beta^{ell}_X\left(\mathrm{Stab}^{ell}_X(p)\right) = (-1)^{\frac{\dim X}{2}}\, \mathrm{Stab}^{ell}_{-X}(p)$$

Conj β^{ell}_X : involution

§ K-theory limits

$$\hat{a}_s(w) := \begin{cases} 1 & \langle s, wtw\rangle \notin \mathbb{Z} \\ w^{1/2} - w^{-1/2} & \langle s, wtw\rangle \in \mathbb{Z} \end{cases}$$

- $s \in P \otimes_{\mathbb{Z}} \mathbb{R}$ (slope param) is called <u>generic</u> if $\langle s, \beta \rangle \notin \mathbb{Z}$ $\forall \beta \in \Phi^1(p)$

<u>Def/Prop</u> ((renormalized) K-theoretic stable basis) [Aganagic-Okounkov]

For $s \in P \otimes_{\mathbb{Z}} \mathbb{R}$ ~~gen~~ X

$$\lfloor x \rfloor' := \begin{cases} \lfloor x \rfloor + 1 & x \notin \mathbb{Z} \\ x & x \in \mathbb{Z} \end{cases}$$

$$\mathrm{Stab}^K_{X,s}(p) := (-1)^{2}\, v^{\sum_{\beta \in \Phi^1_X(p)} \lfloor \langle s, \beta \rangle \rfloor'}\, \hat{a}_s(N^1_{p,-})$$

$$\times \lim_{\xi \to 0} \xi^{\cdot} \left(\sqrt{\mathcal{L}(-\hbar_X)}\, \frac{\mathrm{Stab}^{\mathrm{ell}}_{\alpha}(p)}{\Theta(N^1_{p,-})} \right) \Bigg|_{z = \xi^s}$$

When s: generic, this coincides with (ren.) K-theoretic stable basis

<u>Def</u> (K-theoretic bar inv)

$$\beta^K_{X,s}\left(\mathrm{Stab}^K_{X,s}(p)\right) = (-1)^{\frac{\dim X}{2}}\,\mathrm{Stab}^K_{-X,s}(p)$$

If s: generic, $\beta^K_{X,s}$: inv.

$\exists$ canonical basis $\mathbb{B}^K_{X,s} = \{ q^{\mu} \mathcal{E}^K_{X,s}(p) \}$

$$\widetilde{\mathbb{B}}_X := \bigcup_{s:\text{generic}} \mathbb{B}^K_{X,s}$$

<u>Conj</u> For any $s \in P \otimes_{\mathbb{Z}} \mathbb{R}$

$$s_{\pm} := s \pm \epsilon \cdot x \quad (x \in \mathrm{Amp},\ 0 < \epsilon \ll 1) \text{ \& generic}$$

$$\exists!\ wc_{s_+, s_-} : \mathbb{B}^K_{X, s_+} \xrightarrow{\sim} \mathbb{B}^K_{X, s_-}$$

$$\exists!\ \Sigma^K_{X,s}(\varphi) = \Sigma^K_{X,s_+}(\varphi) + \sum_{\substack{\beta \in P^\vee \\ \langle \beta, s \rangle \in \mathbb{Z} \\ 0 < \langle \beta, x \rangle < \langle \beta_{max}, x \rangle}} z^{-\beta} \sigma_\beta + z^{-\beta_{max}} \cdot wc_{s_+, s_-}\left(\Sigma^K_{X,s_+}(\varphi)\right)$$

s.t. $\left(\begin{array}{l} \cdot\ \beta^K_{X,s}\left(\Sigma^K_{X,s}(\varphi)\right) = \Sigma^K_{X,s_+}(\varphi) \\ \cdot\ \sigma_\beta \in v^{-1}\mathbb{Z}[v^{-1}] \cdot \langle \Sigma \rangle_{\Sigma \in \mathbb{B}^K_{s_+}} \\ \qquad\quad v\mathbb{Z}[v] \langle \Sigma \rangle_{\Sigma \in \mathbb{B}^K_{s_-}} \end{array} \right.$

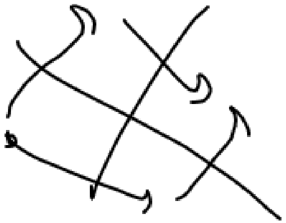
$$\left(\begin{array}{lll} \dfrac{wC_{s_+,s_-}(\Sigma_{s_+})}{\text{triangular w.r.t. some filt.}} = V^{?}\,\Sigma_{s_+} + (\text{lower}) \\ \qquad\qquad\qquad\qquad \mathbb{B}_{x,s_+} \qquad \mathbb{B}_{x,s_-} \end{array} \right)$$

Define equiv rel on $\tilde{\mathbb{B}}_X$ by $\mathcal{E} \sim wc_{s,s'}(\mathcal{E})$

$(\mathcal{E} \in \mathbb{B}^{t}_{X,s})$

Set $\Xi_X := \mathbb{X}^{t}(\mathbb{A}) \backslash \tilde{\mathbb{B}}_X / \sim$

(expect : $|\Xi_X| = |X^A|$)



§ Expected property

<u>Goal</u> $\forall \lambda \in \Xi_X$ associate $\mathcal{E}_X^{ell}(\lambda) \in \widetilde{K}(X)_{loc}$

<u>Property</u> A (K-theory limit)

$\exists\, \alpha_\lambda \in X^*(A)\otimes\mathbb{Q},\ \exists\, \beta_\lambda \in P^\vee\otimes\mathbb{Q}$ s.t.

- $\left(a^{-\alpha_\lambda}\, \hbar^{-\beta_\lambda} \sqrt{\mathcal{L}(-\kappa_\theta)} \otimes \mathcal{E}_X^{ell}(\lambda)\right)$: single valued holomorphic
- $\forall s \in P\otimes\mathbb{R}$, $\lim_{q \to 0} \varphi^\theta\left(\quad \right)\Big|_{z = q^s} = a_\lambda^\theta\, \mathcal{E}^K_{X,s}(\lambda)$

Property A' : the same as Prop A but only for generic s)

Property B (duality) $\exists \overset{\tau}{\gamma}(v)$: hol fcn on $\mathfrak{h} \times \mathbb{C}_v^\times$

and identification $E_X \cong E_{X^!} = B$ s.t.

$\overset{\tau}{\gamma}(v^{-1}) = \overset{\tau}{\gamma}(v)$

$$\overset{\tau}{\gamma}(v)\, \mathrm{Stab}_X^{ell}(p)|_{p'} = \sum_{\lambda \in B} \pm\, \mathcal{E}_{X^!}^{ell}(\lambda)|_{p'} \cdot \mathcal{E}_X^{ell}(\lambda)|_p$$

<u>Property C</u> (chamber indep)

$$\left(\begin{array}{l} \cdot\ \mathcal{E}^{ell}_{-X}(\lambda) = \underline{\mathcal{E}^{ell}_{X}(\lambda)} \\ \cdot\ \mathcal{E}^{ell}_{X_{flop}}(\lambda)\big|_p = \mathcal{E}^{ell}_{X}(-1)\big|_p \end{array} \right.$$

<u>Rem</u> Prop B+C $\Rightarrow \beta^{ell}_X(\mathcal{E}^{ell}_X(\lambda)) = \mathcal{E}^{ell}_X(\lambda)$

- $\mu \in P$, $\mathcal{L}(\mu) \otimes - : \widetilde{B}_X \to \widetilde{B}_X$
  descends to an action $P \curvearrowright E_X$
  $(\mu, \lambda) \mapsto \lambda + \mu$

<u>Property D</u> (Kähler q-diff eq)
$\exists$ pairing $(-,-) : P^\vee \times P^\vee \to \mathbb{Z}$ s.t.
$\forall \lambda \in E_X$, $\forall \mu \in P \subset P^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$

$$\mathcal{E}^{ell}_X(\lambda)\Big|_{z \mapsto \tau^{\mu} z} = \pm\, \tau^{-\frac{(\mu,\, G_\lambda \mu)}{2}} z^{-G_\lambda \mu} \mathcal{L}(-\mu) \otimes \mathcal{E}^{ell}_X(\lambda+\mu)$$

where $G_\lambda : P^\vee \longrightarrow P^\vee$: self adj

$$G_{\lambda+\mu} = G_\lambda \quad (P \ni \mu)$$

Property E' (speciality in v)

$\forall p \in X^A$, $\exists m_p \in \mathbb{Z}$ s.t. $\forall \lambda \in \Xi_X$

$$\mathcal{E}^{ell}_X(\lambda)\Big|_{p,\, v \mapsto \hbar v} = \pm \underbrace{\hbar^{-\frac{m_p}{2}} v^{-m_p} z^{(\cdot)} a^{(\cdot)}}_{\text{indep on } \lambda} \mathcal{E}^{ell}_X(\lambda)\Big|_p$$

<u>Property F'</u> (modularity) $\qquad (x \in P^\vee \otimes \mathbb{C})$

$$\chi_\lambda(\tau, x) := \mathcal{E}^{ell}_X(\lambda)\Big|_{v=a=1,\ z=\exp(2\pi i x)}$$

$\{\chi_\lambda(\tau, x)\}$ forms a vector-valued weak Jacobi forms

<u>Main result</u>: $X = T^*Gr(2,\mathbb{C})$, $X^!$

$\exists \{\mathcal{E}^{ell}_X(\lambda)\}$ satisfying Prop $\underline{A}'$, B, C, D, E', F'