A Weak Categorical Quantum Toroidal Algebra Action on Moduli Space of Stable Sheaves

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Overview

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- We consider a projective smooth algebraic surface S over $\mathbb{C}$.
- Schiffmann-Vasserot (when $S = \mathbb{P}^2$) and Neguț (for all algebraic surfaces S) constructed the quantum toroidal algebra $U_{q_q,q_2}(\ddot{gl}_1)$ action on the Grothendieck group of moduli space of stable sheaves over S.
- We construct a weak categorification of above the quantum toroidal algebra action.

Description of Our Results

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- We consider an ample divisor $H \subset S$, and fix $r > 0$, $E \in H^2(S, \mathbb{Z})$. (we also assume some technical conditions)
- Let $\mathcal{M}$ be the moduli space of (Giesker) H-stable sheaves $\mathcal{F}$ such that $rank(\mathcal{F}) = r, c_1(\mathcal{F}) = E$.
- We consider the nested moduli space

  $$\mathfrak{Z}_{k,k+1} := \{(\mathcal{F}_{-1} \subset_x \mathcal{F}_0) \text{ stable coherent sheaves for some } x \in S\}$$

  such that $c_2(\mathcal{F}_0) = k$. Here $\mathcal{F}_{-1} \subset_x \mathcal{F}_0$ means that $\mathcal{F}_{-1} \subset \mathcal{F}_0$ and $\mathcal{F}_{-1}/\mathcal{F}_0 \cong \mathbb{C}_x$. There is a tautological line bundle $\mathcal{L}$ on $\mathfrak{Z}_{k,k+1}$, such that for each closed point $(\mathcal{F}_0 \subset_x \mathcal{F}_1)$, the fiber of $\mathcal{L}$ at this closed point is $\mathcal{F}_1/\mathcal{F}_0$.
- We denote $\mathcal{U}_k$ the universal sheaf of $\mathcal{M}_k \times S$ and $q = K_S$ the canonical line bundle of S.

The Main Theorem

Theorem

Consider the correspondences $e_i, f_i : D^b(\mathcal{M}) \to D^b(\mathcal{M} \times S)$ *induced from:*

$$e_i := \mathcal{L}^i \mathcal{O}_{\mathfrak{Z}_{k,k+1}} \in D^b(\mathcal{M}_k \times \mathcal{M}_{k+1} \times S)$$
$$f_i := \mathcal{L}^{i-r} \mathcal{O}_{\mathfrak{Z}_{k,k+1}} \in D^b(\mathcal{M}_{k+1} \times \mathcal{M}_k \times S)$$

Then we could construct the triangles in $D^b(\mathcal{M} \times \mathcal{M} \times S \times S)$*:*

$$\begin{cases} \cdots \to R\Delta_*(q^{r-1} det(\mathcal{U}_k)^{-1} \mathcal{U}_k) \to e_i f_{-i+1} \to R\iota_*(f_{-i+1} e_i) \to \cdots \\ \cdots \to R\iota_*(f_{-i-1} e_i) \to e_i f_{-i-1} \to R\Delta_*(det(\mathcal{U}_k)^{-1} \mathcal{U}_k^\vee) \to \cdots \end{cases}$$

and

$$e_i f_{-i} \cong R\iota_* f_i e_j \bigoplus_{a=-r+1}^{0} R\Delta_*(q^{-a} det(\mathcal{U}_k)^{-1} \mathcal{O}_{\mathcal{M}_k \times S})[1-2a-r].$$

The case when $|i+j|>1$

- The morphism $\iota : \mathcal{M} \times \mathcal{M} \times S \times S \to \mathcal{M} \times \mathcal{M} \times S \times S$ maps (x, z, s_1, s_2) to (x, z, s_2, s_1) and $\Delta : \mathcal{M} \times S \to \mathcal{M} \times \mathcal{M} \times S \times S$ is the diagonal embedding.
- More generally, we construct morphisms $e_i f_j \to R\iota_*(f_j e_i)$ when $i+j<0$ and morphisms $R\iota_*(f_j e_i) \to e_i f_j$ when $i+j>0$ such that the cones are filtered by combinations of symmetric and wedge product of the universal sheaf $\mathcal{U}_k$ and its derived dual.

Techinical Assumptions

- We assume the following assumptions:

$$\textbf{Assumption A:} \quad \gcd(r, E \cdot H) = 1,$$
$$\textbf{Assumption S:} \quad \mathcal{M} \text{ is smooth.}$$

- Assuming Assumption A, for any short exact sequence which does not split:

$$0 \to \mathcal{F} \to \mathcal{F}' \to \mathbb{C}_x \to 0,$$

  the sheaf $\mathcal{F}$ is stable if and only if $\mathcal{F}'$ is stable.

- Assuming Assumption A, we have

$$\text{slope stable} = \text{semistable} = \text{stable} = \text{slope semistable}$$

Quantum toroidal algebra and its representations

Quantum Toroidal Algebra $U_{q_1,q_2}(\ddot{gl}_1)$

1. Quantum toroidal algebra has many different presentations, including:
   1. The Ding-Iohara-Miki algebra
   2. The Hall algebra of coherent sheaves on an elliptic curve
   3. The stable limit of spherical DAHA.
2. It is an affinization of the q-Heisenberg algebra.
3. Let $\mathbb{K} = \mathbb{Q}(q_1, q_2)$, and $q = q_1 q_2$. The (level r presentation of) quantum toroidal algebra $U_{q_1,q_2}(\ddot{gl}_1)$ is the $\mathbb{K}$-algebra with generators:

$$\{E_k, F_k, H_l^{\pm}\}_{k \in \mathbb{Z}, l \in \mathbb{N}}$$

such that $H_0^- = q^r, H_0^+ = 1$, and

The Presentation of the Quantum Toroidal Algebra

$$(z-wq_1)(z-wq_2)(z-\frac{w}{q})E(z)E(w) = (z-\frac{w}{q_1})(z-\frac{w}{q_2})(z-wq)E(w)E(z)$$

$$(z-wq_1)(z-wq_2)(z-\frac{w}{q})E(z)H^{\pm}(w) = (z-\frac{w}{q_1})(z-\frac{w}{q_2})(z-wq)H^{\pm}(w)E(z)$$

$$[[E_{k+1}, E_{k-1}], E_k] = 0 \quad \forall k \in \mathbb{Z}$$

together with the opposite relations for $F(z)$ instead of $E(z)$, as well as:

$$[E(z), F(w)] = \delta(\frac{z}{w})(1-q_1)(1-q_2)(\frac{H^+(z)-H^-(w)}{1-q}) \tag{1}$$

where

$$E(z) = \sum_{k\in\mathbb{Z}} \frac{E_k}{z^k}, \quad F(z) = \sum_{k\in\mathbb{Z}} \frac{F_k}{z^k}, \quad H^{\pm}(z) = \sum_{l\in\mathbb{N}\cup\{0\}} \frac{H_l^{\pm}}{z^{\pm l}}$$

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n \in \mathbb{Q}\{\{z\}\}.$$

The Action on the Grothendieck group of $\mathcal{M}$

- Schiffmann-Vasserot (when $S = \mathbb{P}^2$) and Neguț (for all surfaces S) constructed a quantum toroidal algebra (with level r presentation) action on the Grothendieck group of moduli spaces of stable sheaves with rank r, such that E_i and F_j are represented by Fourier-Mukai correspondences e_i and f_j.
- The operators h_1 and h_1^- are induced by the universal sheaf and its derived dual.
- The action factors through the deformed $\mathcal{W}$-algebra, ${}_q\mathcal{W}(gl_r)$.
- When $S = \mathbb{P}^2$ and let $\mathcal{M}$ be the moduli space of torsion free framed sheaves, $K_0(\mathcal{M})$ is a Verma module, and the fundamental class is the Whittaker vector.

The incarnation with AGT correspondence

- It induced a proof of a version of the AGT conjecture concerning pure $N = 2$ gauge theory for the group $G = G^L = GL(r)$.
- Another proof was obtained by Maulik-Okounkov (for $G = GL(r)$) and Braverman-Finkelberg-Nakajima (for G is simply laced) through the stable envelope.

Weak Categorifications

- The weak categorification of the positive part of $U_{q_1,q_2}(\ddot{gl}_1)$ is also obtained by Neguț.
- Another categorification of the positive part was obtained by Porta-Sala, through the categorified Hall algebra.
- Our main theorem categorified the commutator relation of the positive and negative part, and thus induces a weak categorification for the whole algebra.

The Proof of the Main Theorem

A Quick Review of the Rank 1 Case

- We consider the following moduli spaces:

$$\mathfrak{Z}_- = \{(\mathcal{F}_0 \subset_y \mathcal{F}_1, \mathcal{F}_0' \subset_x \mathcal{F}_1) | \mathcal{F}_0, \mathcal{F}_0', \mathcal{F}_1 \text{ are stable, } \}$$
$$\mathfrak{Z}_+ = \{(\mathcal{F}_0 \supset_x \mathcal{F}_{-1}, \mathcal{F}_0' \supset_y \mathcal{F}_{-1}) | \mathcal{F}_0, \mathcal{F}_0', \mathcal{F}_{-1} \text{ are stable, } \}$$
$$\mathfrak{Y} = \{(\mathcal{F}_{-1} \subset_x \mathcal{F}_0 \subset_y \mathcal{F}_1, \mathcal{F}_{-1} \subset_y \mathcal{F}_0' \subset_x \mathcal{F}_1) | \mathcal{F}_0, \mathcal{F}_0', \mathcal{F}_1 \text{ are stable, } \}$$

- The scheme $\mathfrak{Y}$ is smooth and $\mathfrak{Z}_-$ is a (canonical) rational singularity.
- When $r = 1$, the scheme $\mathfrak{Z}_+$ is equi-dimensional (and also Cohen-Macaulay), and the irreducible component other than $\mathfrak{Z}_{k,k+1}$ is also a (canonical) rational singularity.

New Challenges for the Higher Rank Cases

- When $r > 1$, the scheme $\mathfrak{Z}_+$ is no longer equi-dimensional. We have to consider the derived enhancement $\mathbb{R}\mathfrak{Z}_+$ to compute the composition of the Fourier-Mukai transforms.
- We don't have a Kodaira vanishing theorem (also Kawamata vanishing theorem, "canonical=rational" theorem) for derived schemes.
- The derived structure on $\mathfrak{Y}$ does not follow from the "naive" intersection, as the dimension does not match.

Derived Blow-ups

The derived blow-up of a closed embedding of two derived schemes was constructed by Hekking very recently (the case that the embedding is quasi-smooth had been constructed by Rydh-Khan).

Conjecture

We have $\mathfrak{Y} \cong Bl_{\mathfrak{Z}_{k,k+1}} \mathbb{R}\mathfrak{Z}_+$ and $\mathfrak{Y} \cong Bl_{\mathfrak{Z}_{k-1,k}} \mathfrak{Z}_-$. Moreover, the derived structure on $\mathfrak{Y}$ induced from the derived blow-up is trivial.

Theorem (Z.)

There exists a smooth locally free sheaf V on an ambient variety $X \supset \mathfrak{Z}_{k,k+1}$, and a global section $s \in \Gamma(V, X)$, such that $s|_{\mathfrak{Z}_{k,k+1}} = 0$. It induces a global section $s' \in \Gamma(V \otimes \mathcal{O}(-D), Bl_{\mathfrak{Z}_{k,k+1}} X)$. ($D$ is the exceptional divisor) Moreover,

- *$\mathbb{R}\mathfrak{Z}_+$ is the derived loci of s*
- *$\mathfrak{Y}$ is the derived loci of s'.*

The Discrepancy of $\mathfrak{Z}_-$

We have the following (weaker) theorem:

Theorem (Z.)

Let W be the regular locus of $\mathfrak{Z}_-$, then $W \cap \mathfrak{Z}_{k-1,k}$ is non empty and $\alpha_-^{-1}(W) = Bl_{W \cap \mathfrak{Z}_{k-1,k}} W$, where α_- is the forgetful morphism from $\mathfrak{Y}$ to $\mathfrak{Z}_-$.

As a corollary, $\mathfrak{Z}_-$ is a terminal singularity (and thus a rational singularity), with discrepancy $r+1$.
Finally, the split of commutator relations of e_i and f_{-i} follows from the fact that $D^b(\mathcal{M} \times \mathcal{M} \times S \times S)$ is Karoubian.

Future Directions

Semistablity or Stability?

- The Assumption A is too strict.
- In general, what we need is that
  1. A twisted universal sheaf on $\mathcal{M} \times S$
  2. The stability (or semi-stability) condition does not change by modify a finite length sheaf.
- We expect to remove the smooth condition of $\mathcal{M}$ (to quasi-smooth) by different arguments in the future.
- What if $r = 0$? It does not make sense when we consider framed sheaves but makes the sense when considering the stable sheaves.

Biadjoint relations

- The Fourier-Mukai transforms e_k and f_{-k} are biadjoint up to the shift of degree and $q = K_S$.
- The short exact sequence on $\mathfrak{Z}_{k,k+1} \times S$:

$$0 \to \mathcal{U}_{k+1} \to \mathcal{U}_k \to \mathcal{O}_{\mathfrak{Z}_{k,k+1}} \to 0$$

  where Γ is the closed embedding.
- It induces a triangle $he_i \to e_i h \to e_{i+1}$. We expect the morphism $e_i f_{-i+1} \to h$ in our main theorem follows from the triangle through biadjoint relations (and also for other morphisms).
- Based on that, we could study the 2-morphism relations in the categorification.

Char $p > 2$ Case

- The recent progress of the MMP in positive characteristic makes it possible to generalize the action to positive characteristic case.
- Kovacs proved that, in char $p > 0$ case, a Cohen-Macaulay klt singularity is also a rational singularity. Cohen-Macaulay is necessary, as Torato proved that even a terminal singularity might not be a rational singularity in char $p > 0$ case.
- We should consider SAG but not DAG, as the dg algebra works poorly for higher homotopy theory.

An Intrinsic Proof

- Our proof still relies on an embedding of $\mathfrak{Y}$ into an ambient variety.
- We don't have a functorial description of the derived blow-up when the embedding is not quasi-smooth now.