Elliptic Quantum Toroidal Algebra $U_{q,t,p}(\mathfrak{gl}_{1,tor})$ and Affine Quiver Gauge Theories

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MSJ-SI 2023

International Workshop and Mini School on

Elliptic Integrable Systems, Representation Theory and Hypergeometric Functions

in

Tokyo

Past Workshops:

2004 Kyoto, 2008 Bonn, 2013 Leiden, 2017 Vienna, 2019 Stockholm

Review of $U_{q,p}(\widehat{\mathfrak{g}})$ ($\widehat{\mathfrak{g}}$: affine Lie alg) and $W_{p,p^*}(\mathfrak{g})$

- elliptic (p: nome) and dynamical analogue of $U_q(\widehat{\mathfrak{g}})$
  in the Drinfeld realization

- defined by gen. and rel.
  - elliptic Drinfeld currents $E_j(z), F_j(z), \psi_j^{\pm}(z)$
  - Π_j : dynamical parameters (⇔ Kähler param.)

- Hopf algebroid str. as a co-alg. str.,

  $\Delta : U_{q,p} \to U_{q,p} \widetilde{\otimes} U_{q,p}$ alg.hom.

- $U_{q,p}(\widehat{\mathfrak{gl}}_N) \cong E_{q,p}(\widehat{\mathfrak{gl}}_N)$ (H.K '18)

  $E_{q,p}(\widehat{\mathfrak{gl}}_N)$: central extension of Felder's EQG $RLL = LLR^*$

Level-k $U_{q,p}(\widehat{\mathfrak{g}})$ realizes $W_{p,p^*}(\mathfrak{g})$ $(p^* = pq^{-2k})$

In general,

$W_{p,p^*}(\mathfrak{g})$: the deformation of the coset type W-alg.

$$(\widehat{\mathfrak{g}})_{r-h^\vee-k} \oplus (\widehat{\mathfrak{g}})_k \supset (\widehat{\mathfrak{g}})_{r-h^\vee}$$

- For simply laced $\widehat{\mathfrak{g}}$:

  the coset type W-alg. $\cong$ the Hamiltonian reduction type W-alg.

  Hence, $W_{p,p^*}(\mathfrak{g}) \Leftrightarrow$ Frenkel-Reshetikhin's $W_{q,t}(\mathfrak{g})$

  by $(p, p^*) \leftrightarrow (q, t)$ (at least for $k = 1$)

- For non-simply laced $\widehat{\mathfrak{g}}$: e.g. $\widehat{\mathfrak{g}} = B_N^{(1)}$

  $W_{p,p^*}(B_N)$ gives the deformation of Fateev-Lukyanov's WB_N-alg.

  Note also $WB_N \cong$ Hamiltonian reduction of $B^{(1)}(0|N)$ (Ito '90)

Example

level-1 $U_{q,p}(\widehat{\mathfrak{sl}}_N)$		$W_{q,t}(\mathfrak{sl}_N)$ $\quad \beta = \frac{r-1}{r}$	
$p = q^{2r}$	$\leftrightarrow$	q	
$p^* = pq^{-2} = q^{2(r-1)}$	$\leftrightarrow$	t	
$F_j(z)$	$\leftrightarrow$	$S_j^+(z)$	
$E_j(z)$	$\leftrightarrow$	$S_j^-(z)$	
$\sum_\mu \Phi_\mu^D(p^{-1}z)^{-1}\Phi_\mu^D(z)$	$\leftrightarrow$	$T(z) = \sum_\mu \Lambda_\mu(z)$	$(\mu = 1, \cdots, N)$
$\prod_{m=1}^{a} \Phi_m^D(zq^{A_m})$	$\leftrightarrow$	$V_+^a(z)$	(Awata-Yamada '10)
$\prod_{m=1}^{a} \Psi_m^{*D}(zq^{A'_m})$	$\leftrightarrow$	$V_-^a(z)$	$a = 1, \cdots, N-1$
$\prod_{m=1}^{a}\prod_{k=1}^{l-1} \Phi_m^D(zq^{A_m}p^{*B_k})$	$\leftrightarrow$	$V_u^a(z)$	$u = p^l$

e.g. Intertwining Operators (Vertex Operators) of the $U_{q,p}(\widehat{\mathfrak{sl}}_N)$-modules

$V_z = V[[z, z^{-1}]]$: evaluation rep. ass.w. the vector rep. $V = \bigoplus_{\mu=1}^{N} \mathbb{C}v_\mu$

$\mathcal{F}_\lambda$: level-1 rep. with the h.w. λ

$$\Phi(z) \ : \ \mathcal{F}_\lambda \to V_z \,\widetilde{\otimes}\, \mathcal{F}_{\lambda'}$$

s.t. $\Phi(z)x = \Delta(x)\Phi(z) \qquad \forall x \in U_{q,p}$

Components are defined by $\Phi(z) = \sum_\mu v_\mu \widetilde{\otimes} \Phi_\mu(z)$.

Different coproducts Δ (standard) and Δ^D (the Drinfeld copro.) yield diffrent intertwiners $\Phi(z)$ and $\Phi^D(z)$, respectively.

- $\Phi^D(z) \ \rightsquigarrow \ W_{p,p^*}(\mathfrak{g})$
- $\Phi(z) \ \rightsquigarrow$ (elliptic) q-KZ eq. , $\mathrm{Stab}_{\mathfrak{C}}(F)$

Integral Solution to the Elliptic Q-KZ eq. via $U_{q,p}(\widehat{\mathfrak{sl}}_N)$

Theorem 1.1 (H.K '17, '18) *Under the zero-weight condition,*

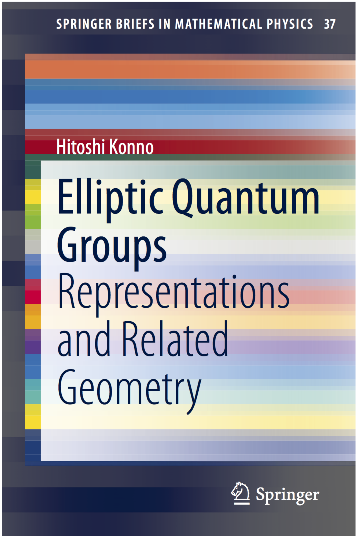
$$\mathrm{tr}_{\mathcal{F}_\lambda}\left(Q^{-d}\Phi_{\mu_1}(z_1)\cdots\Phi_{\mu_n}(z_n)\right)=\oint_{\mathbb{T}^M}\underline{d\boldsymbol{t}}\ \Phi(\boldsymbol{t},\boldsymbol{z})\ \mathrm{Stab}_{\mathfrak{C}}(F_I;p),$$

$$\Phi(\boldsymbol{t},\boldsymbol{z})=\prod_{l=1}^{N-1}\exp\left\{\frac{1}{\log p}\log(\Pi_l/\Pi_{l+1})\log\left(\prod_{a=1}^{\lambda^{(l)}}t_a^{(l)}\right)\right\}\times\prod_{l=1}^{N-1}\left(\prod_{a=1}^{\lambda^{(l)}}t_a^{(l)}\right)^{\lambda_l-h_{\alpha_l}}$$

$$\times\prod_{l=1}^{N-1}\left[\prod_{a=1}^{\lambda^{(l)}}\prod_{b=1}^{\lambda^{(l+1)}}\frac{\Gamma(t_a^{(l)}/t_b^{(l+1)};p,Q)}{\Gamma(p^*t_a^{(l)}/t_b^{(l+1)};p,Q)}\prod_{1\le a<b\le\lambda^{(l)}}\frac{\Gamma(p^*t_a^{(l)}/t_b^{(l)},p^*t_b^{(l)}/t_a^{(l)};p,Q)}{\Gamma(t_a^{(l)}/t_b^{(l)},t_b^{(l)}/t_a^{(l)};p,Q)}\right],$$

$$t_a^{(N)}=z_a\qquad(a=1,\cdots,n)$$

- This is an elliptic and dynamical analogue of Mimachi '96.
- Geometrically, this is an elliptic analogue of the vertex function with descendent in $\mathrm{K}_T(T^*Fl_n)$. See also Aganagic-Frenkel-Okounkov "Quantum q-Langlands Corresp." '17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Taking Residues at "$\mathbb{t} = \mathbb{z}_J p^{-d_J}$"

$$\mathrm{tr}_{\mathcal{F}_\lambda}\left(Q^{-d}\Phi_{\mu_1}(z_1)\cdots\Phi_{\mu_n}(z_n)\right)$$
$$= \sum_{d_{l,a}\in\mathfrak{D}}\prod_{l=1}^{N-1}\left[\left(p^{*\lambda_{l+1}}\Pi_{l+1}/\Pi_l\right)^{\sum_{a=1}^{\lambda^{(l)}} d_{l,a}}\right.$$
$$\left.\times\prod_{a=1}^{\lambda^{(l)}}\prod_{b=1}^{\lambda^{(l+1)}}\frac{\theta_Q(pz_{j_b^{(l+1)}}/p^* z_{j_a^{(l)}};p)_{d_{l,a}-d_{l+1,b}}}{\theta_Q(pz_{j_b^{(l+1)}}/z_{j_a^{(l)}};p)_{d_{l,a}-d_{l+1,b}}}\prod_{1\le a\neq b\le\lambda^{(l)}}\frac{\theta_Q(p^* z_{j_a^{(l)}}/z_{j_b^{(l)}};p)_{d_{l,b}-d_{l,a}}}{\theta_Q(z_{j_a^{(l)}}/z_{j_b^{(l)}};p)_{d_{l,b}-d_{l,a}}}\right]$$
$$\times \mathrm{Stab}_{\mathfrak{C}}(F_I;p)|_{F_J}$$

$$J \ : \ J^{(1)}\subset J^{(2)}\subset\cdots\subset J^{(N-1)}\subset[1,n]$$
$$J^{(l)}=\{j_1^{(l)}<\cdots<j_{\lambda^{(l)}}^{(l)}\}$$

For the full flag case, this gives an elliptic analogue of the 3d vortex partition function for the $T[U(N)]$ theory.

Trigonometric Limit $Q \to 0$

$$\mathrm{tr}_{\mathcal{F}_\lambda}\left(Q^{-d}\Phi_{\mu_1}(z_1)\cdots\Phi_{\mu_n}(z_n)\right)$$
$$\to \langle 0|\Phi_{\mu_1}(z_1)\cdots\Phi_{\mu_n}(z_n)|0\rangle = \oint_{\mathbb{T}^M} \underline{d\mathbb{t}}\ \Phi^{trig.}(\mathbb{t},\mathbb{z})\ \mathrm{Stab}_{\mathfrak{C}}(F_I;p),$$

$$\Phi^{trig.}(\mathbb{t},\mathbb{z})$$
$$= \prod_{l=1}^{N-1}\exp\left\{\frac{1}{\log p}\log(\Pi_l/\Pi_{l+1})\log\left(\prod_{a=1}^{\lambda^{(l)}} t_a^{(l)}\right)\right\}\times\prod_{l=1}^{N-1}\left(\prod_{a=1}^{\lambda^{(l)}} t_a^{(l)}\right)^{\lambda_l - h_{\alpha_l}}$$
$$\times\prod_{l=1}^{N-1}\left[\prod_{a=1}^{\lambda^{(l)}}\prod_{b=1}^{\lambda^{(l+1)}}\frac{(p^* t_a^{(l)}/t_b^{(l+1)};p)_\infty}{(t_a^{(l)}/t_b^{(l+1)};p)_\infty}\prod_{1\le a<b\le\lambda^{(l)}}\frac{(t_a^{(l)}/t_b^{(l)},t_b^{(l)}/t_a^{(l)};p)_\infty}{(p^* t_a^{(l)}/t_b^{(l)},p^* t_b^{(l)}/t_a^{(l)};p)_\infty}\right]$$

- This is the vertex function in $K_T(T^*Fl_n)$
- The full flag case gives an integral formula for the Macdonald functions (Mimachi '96, Mimachi-Noumi '96, Noumi-Shiraishi '12)
- The sum of residues at "$\mathbb{t} = \mathbb{z}_J p^{-d_J}$" gives the 3d vortex partition function for the $T[U(N)]$ theory.

Rev. of $U_{q,t}(\mathfrak{gl}_{1,tor})$ and the Linear Quiver Gauge Theories

$U_{q,t}(\mathfrak{gl}_{1,tor})$ has been applied to the instanton calculus and the study of the AGT corresp. for the 5d & 6d lifts of the 4d $\mathcal{N}=2$ SUSY gauge theories. The essence is summarized as

- $W_{q,t}(\mathfrak{g})$ is realized on $U_{q,t}(\mathfrak{gl}_{1,tor})\otimes\cdots\otimes U_{q,t}(\mathfrak{g}_{1,tor})$

  (Miki '07, Feigin-Hashizume-Hoshimo-Shiraishi-Yanagida '09, Berstein-Feigin-Merzon '15, Awata et.al. '16)

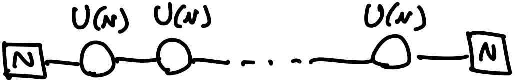
- Intertwiners of $U_{q,t}(\mathfrak{gl}_{1,tor})$ w.r.t. the Drinfeld copro. realize the refiened topological vertex (Awata-Feigin-Shiraishi '12)

- A certain block of composition of such intertwiners realizes the intertwiner of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ (Zenkevich '18, Fukuda-Okubo-Shiraishi '19, '20)

e.g. Triality in 5d Lift of $\mathcal{N}=2$ SUSY Gauge Theories

Aganagic-Haouzi-Shakirov '14, Nedelin-Pasquetti-Zenkevich'19, $\cdots$

e.g. $U(N)^{N-1}$ theory with N fundamental and anti-fundamental hypermultiplets
(A_{N-1} type linear quiver gauge theory)



5d Nekrasov Instanton PF

Higgsing $\swarrow\!\!\swarrow$. $\searrow\!\!\searrow$ Higgsing (AGT corresp.?)

3d Vortex PF $\Longleftrightarrow$ Corr. fnc. of the vertex ops.

$\longleftarrow$ taking residues

of $W_{p,p^*}(\mathfrak{sl}_N)$ ("q-conf. block")

$\Updownarrow$

$\langle 0|\Phi_{\mu_1}(z_1)\cdots\Phi_{\mu_N}(z_N)|0\rangle$

in $U_{q,p}(\widehat{\mathfrak{sl}}_N)$

Triality in 6d Lift of $\mathcal{N}=2$ SUSY Gauge Theories

Nieri '15, Iqbal-Kozcaz-Yau '15, $\cdots$

e.g. $U(N)$ th. with N fundamental and anti-fundamental hypermultiplets
(A_1 type linear quiver gauge theory)

6d Nekrasov Instanton PF

Higgsing $\swarrow$ $\searrow$ Higgsing (AGT corresp.?)

4d Vortex PF $\Longleftrightarrow$ Trace of the vertex ops. of $W_{p,p^*}(\mathfrak{sl}_2)$

$\overleftarrow{\text{taking residues}}$

("elliptic conf. block")

$\Updownarrow$

$$\mathrm{tr}\left(Q^{-d}\Phi_{\mu_1}(z_1)\cdots\Phi_{\mu_N}(z_N)\right)$$

in $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

This can be understood by the "trace" of $U_{q,t}(\mathfrak{gl}_{1,tor})$!
It should be distinguished from the elliptic quantum toroidal algebra !

This talk : Elliptic Quantum Toroidal Algebra $U_{q,t,p}(\mathfrak{gl}_{1,tor})$

Main points:

- intertwiners $\Phi(u)$, $\Psi^*(u)$ w.r.t. the Drinfeld copro. and their "sifted inverse" $\Phi^*(u)$, $\Psi(u)$
- $U_{q,t,p}(\mathfrak{gl}_{1,tor})$ realizes the Jordan quiver W-alg. $W_{p,p^*}(\Gamma(\widehat{A}_0))$
  $SU(4)$ Ω-deformation parameters : $q,\ t,\ p,\ p^*$ s.t. $q/t = p^*/p$
- Nekrasov instanton PFs of the 5d & 6d lifts of the 4d $\mathcal{N} = 2^*$ gauge theories

Elliptic Quantum Toroidal Algebra $U_{q,t,p}(\mathfrak{gl}_{1,tor})$

Elliptic Quantum Toroidal Algebra $U_{q,t,p}(\mathfrak{gl}_{1,tor})$

Let $q, t, p \in \mathbb{C}^*$, $|q|, |t|, |p| < 1$.
We define $U_{q,t,p}(\mathfrak{gl}_{1,tor})$ to be a $\mathbb{C}$-algebra generated by

$$\alpha_m, \quad x_n^\pm, \quad K^{\pm 1}, \quad \gamma^{\pm 1/2} \quad (m \in \mathbb{Z} \setminus \{0\},\ n \in \mathbb{Z}).$$

Let us set

$$\psi^+(z) = K \exp\left(-\sum_{m>0} \frac{p^m}{1-p^m} \alpha_{-m} (\gamma^{-1/2} z)^m\right) \exp\left(\sum_{m>0} \frac{1}{1-p^m} \alpha_{-m} (\gamma^{-1/2} z)^{-m}\right),$$

$$\psi^-(z) = K^{-1} \exp\left(-\sum_{m>0} \frac{1}{1-p^m} \alpha_{-m} (\gamma^{1/2} z)^m\right) \exp\left(\sum_{m>0} \frac{p^m}{1-p^m} \alpha_{-m} (\gamma^{1/2} z)^{-m}\right),$$

$$x^\pm(z) = \sum_{n \in \mathbb{Z}} x_n^\pm z^{-n}$$

: elliptic Drinfeld currents

$\gamma^{1/2}$, K : central,

$$[\alpha_m,\alpha_n]=-\frac{\kappa_m}{m}(\gamma^m-\gamma^{-m})\gamma^{-m}\frac{1-p^m}{1-p^{*m}}\delta_{m+n,0},\qquad p^*=p\gamma^{-2}$$

$$[\alpha_n,x^+(z)]=-\frac{\kappa_n}{n}\frac{1-p^n}{1-p^{*n}}(\gamma^{-3n/2}z)^n x^+(z,p),$$

$$[\alpha_n,x^-(z)]=\frac{\kappa_n}{n}(\gamma^{-1/2}z)^n x^-(z,p),$$

$$[x^+(z),x^-(w)]=\frac{(1-q)(1-1/t)}{(1-q/t)}\left(\delta(\gamma^{-1}z/w)\psi^+(w)-\delta(\gamma z/w)\psi^-(\gamma^{-1}w)\right),$$

$$x^+(z)x^+(w)=g(z/w;p^*)^{-1}x^+(w)x^+(z),$$

$$x^-(z)x^-(w)=g(z/w;p)x^-(w)x^-(z),$$

$$g(p^*\frac{w}{z},p^*)^{-1}g(p^*\frac{u}{w},p^*)^{-1}g(p^*\frac{u}{z},p^*)^{-1}\left(\frac{w}{u}+\frac{w}{z}-\frac{z}{w}-\frac{u}{w}\right)x^+(z)x^+(w)x^+(u)$$
$$+\text{permutations in } z,w,u=0,$$

$$g(p\frac{w}{z},p)\,g(p\frac{u}{w},p)\,g(p\frac{u}{z},p)\left(\frac{w}{u}+\frac{w}{z}-\frac{z}{w}-\frac{u}{w}\right)x^-(z)x^-(w)x^-(u)$$
$$+\text{permutations in } z,w,u=0,$$

where

$$\kappa_m=(1-q^m)(1-t^{-m})(1-(t/q)^m),$$

$$g(z;p)=\frac{\theta_p(q^{-1}z)\theta_p((q/t)z)\theta_p(tz)}{\theta_p(qz)\theta_p((q/t)^{-1}z)\theta_p(t^{-1}z)},\qquad \theta_p(z)=(z;p)_\infty(p/z;p)_\infty$$

Hopf Algebroid Structure via Δ^D

Let $\widetilde{\otimes}$ denote a tensor product with extra condition

$$F(z;p^*)a\widetilde{\otimes}b = a\widetilde{\otimes}F(z;p)b, \qquad p^* = p\gamma^{-2}$$

The following gives the Drinfeld coproduct for $\mathcal{U}_{q,t,p} = U_{q,t,p}(\mathfrak{gl}_{1,tor})$.

$$\begin{aligned}
\Delta^D(\gamma^{\pm 1/2}) &= \gamma^{\pm 1/2}\widetilde{\otimes}\gamma^{\pm 1/2}, \\
\Delta^D(\psi^{\pm}(z)) &= \psi^{\pm}(\gamma_{(2)}^{\mp 1/2}z)\widetilde{\otimes}\psi^{\pm}(\gamma_{(1)}^{\pm 1/2}z) \\
\Delta^D(x^{+}(z)) &= 1\widetilde{\otimes}x^{+}(\gamma_{(1)}^{-1/2}z) + x^{+}(\gamma_{(2)}^{1/2}z)\widetilde{\otimes}\psi^{-}(\gamma_{(1)}^{-1/2}z), \\
\Delta^D(x^{-}(z)) &= x^{-}(\gamma_{(2)}^{-1/2}z)\widetilde{\otimes}1 + \psi^{+}(\gamma_{(2)}^{-1/2}z)\widetilde{\otimes}x^{-}(\gamma_{(1)}^{1/2}z).
\end{aligned}$$

Here $\gamma_{(1)} = \gamma\widetilde{\otimes}1, \gamma_{(2)} = 1\widetilde{\otimes}\gamma$.

$U_{q,t,p}(\mathfrak{gl}_{1,tor})$-modules

Vector Representation $V(u)$ $u \in \mathbb{C}^*$

Definition 3.1

We say that an $\mathcal{U}_{q,t,p}$-module has level $(k,l) \in \mathbb{C}^2$ if $\gamma^{1/2}$ acts by $q^{k/2}$ and K acts by $(q/t)^{l/2}$.

For $u \in \mathbb{C}^*$, $V(u) :=$ Span$_{\mathbb{C}}$ $\{\ [u]_j\ (j \in \mathbb{Z})\ \}$ has a level-$(0,0)$ $\mathcal{U}_{q,t,p}$-module structure by

$$x^+(z)[u]_j = a^+(p)\delta(q^j u/z)[u]_{j+1},$$
$$x^-(z)[u]_j = a^-(p)\delta(q^{j-1} u/z)[u]_{j-1},$$
$$\psi^\pm(z)[u]_j = \left.\frac{\theta_p(q^j t^{-1} u/z)\theta_p(q^{j-1} t u/z)}{\theta_p(q^j u/z)\theta_p(q^{j-1} u/z)}\right|_\pm [u]_j$$

where

$$a^\pm(p) = (1 - t^{\pm 1})\frac{(p(t/q)^{\pm 1};p)_\infty (pt^{\mp 1};p)_\infty}{(p;p)_\infty (pq^{\mp 1};p)_\infty}.$$

Semi-Infinite Tensor Product Rep. $\mathcal{F}_u^{(0,1)}$ (q-Fock Rep.)

In the trig. case: Feigin^2-Jimbo-Miwa-Mukhin '11

Applying Δ^D repeatedly, one can extend the level (0,0) $\mathcal{U}_{q,t,p}$- module structure inductively to the semi-infinite tensor product rep. of level (0,1)

$$\mathcal{F}_u^{(0,1)} \subset V(u)\widetilde{\otimes}V(u(t/q)^{-1})\widetilde{\otimes}V(u(t/q)^{-2})\widetilde{\otimes}\cdots$$

spanned by vectors

$$|\lambda\rangle_u = [u]_{\lambda_1-1}\widetilde{\otimes}[u(t/q)^{-1}]_{\lambda_2-2}\widetilde{\otimes}[u(t/q)^{-2}]_{\lambda_3-3}\widetilde{\otimes}\cdots ,$$
$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots) \qquad \lambda_l = 0 \text{ for } l \gg 1.$$

Theorem 3.2

The following gives a level-(0,1) action of $\mathcal{U}_{q,t,p}$ on $\mathcal{F}_u^{(0,1)}$.

$$x^+(z)|\lambda\rangle_u = a^+(p) \sum_{i=1}^{\ell(\lambda)+1} \delta(u_i/z) \prod_{j=1}^{i-1} \frac{\theta_p(tu_i/u_j)\theta_p(qu_i/tu_j)}{\theta_p(qu_i/u_j)\theta_p(u_i/u_j)} |\lambda + \mathbf{1}_i\rangle_u,$$

$$x^-(z)|\lambda\rangle_u = a^-(p)(q/t)^{1/2} \sum_{i=1}^{\ell(\lambda)} \delta(q^{-1}u_i/z) \prod_{j=i+1}^{\ell(\lambda)} \frac{\theta_p(qu_j/tu_i)}{\theta_p(u_j/tu_i)} \prod_{j=i+1}^{\ell(\lambda)+1} \frac{\theta_p(tu_j/u_i)}{\theta_p(qu_j/u_i)} |\lambda - \mathbf{1}_i\rangle_u,$$

$$\psi^+(z)|\lambda\rangle_u = (q/t)^{1/2} \prod_{j=1}^{\ell(\lambda)} \frac{\theta_p(t^{-1}u_j/z)}{\theta_p(q^{-1}u_j/z)} \prod_{j=1}^{\ell(\lambda)+1} \frac{\theta_p(tu_j/qz)}{\theta_p(u_j/z)} |\lambda\rangle_u,$$

$$\psi^-(z)|\lambda\rangle_u = (q/t)^{-1/2} \prod_{j=1}^{\ell(\lambda)} \frac{\theta_p(tz/u_j)}{\theta_p(qz/u_j)} \prod_{j=1}^{\ell(\lambda)+1} \frac{\theta_p(qz/tu_j)}{\theta_p(z/u_j)} |\lambda\rangle_u,$$

where we set $u_i = q^{\lambda_i} t^{-i+1} u$.

Geometric Interpretation

Conjecture 3.1

The action in Theorem 3.2 gives a level-(0,1) action of $\mathcal{U}_{q,t,p}$ *on* $\bigoplus_{|\lambda|} \mathrm{E}_T(\mathrm{Hilb}_{|\lambda|}(\mathbb{C}^2))$ *with* $T = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \ni (q,t,u)$ *via*

$$|\lambda\rangle_u\text{'s} \quad \Leftrightarrow \quad \text{fixed point classes } [\lambda]\text{'s}$$

Moreover,

$$[\lambda] = \sum_{\mu} \mathrm{Stab}_{\mathfrak{C}}^{-1}(\mu)\Big|_{\lambda} \mathrm{Stab}_{\mathfrak{C}}(\mu)$$

gives an elliptic analogue of Macdonald symmetric function.

Cf. $U_{q,t}(\mathfrak{gl}_{1,tor}) \curvearrowright \bigoplus_n \mathrm{K}_T(\mathrm{Hilb}_n(\mathbb{C}^2))$ Feigin-Tsymbaliuk '11

$U_{q,p}(\widehat{\mathfrak{gl}}_N) \curvearrowright \bigoplus_\lambda \mathrm{E}_T(T^* Fl_\lambda)$ H.K '18

Theorem 3.3 *(Level $(0,0)$ rep. in terms of the elliptic Ruijsenaars op.)*
(Cf. Miki '07 trig.case)

On $\mathbb{C}[x_1^{\pm 1}, \cdots, x_N^{\pm 1}]$,

$$x^+(z) = a^+(p) \sum_{i=1}^{N} \prod_{j \neq i} \frac{\theta_p(t x_i/x_j)}{\theta_p(x_i/x_j)} \delta(x_i/z) T_{q,x_i},$$

$$x^-(z) = -a^-(p) \sum_{i=1}^{N} \prod_{j \neq i} \frac{\theta_p(t^{-1} x_i/x_j)}{\theta_p(x_i/x_j)} \delta(q^{-1} x_i/z) T_{q,x_i}^{-1},$$

$$\psi^\pm(z) = \prod_{j=1}^{N} \frac{\theta_p(t^{-1}x_j/z)\theta_p(q^{-1}t x_j/z)}{\theta_p(x_j/z)\theta_p(q^{-1}x_j/z)}\Bigg|_{\pm},$$

or

$$\alpha_m = \frac{(1-t^{-m})(1-(q/t)^{-m})}{m} \sum_{j=1}^{N} x_j^m \qquad (m \in \mathbb{Z} \backslash \{0\}).$$

In particular, the zero-mode $x_0^+ = \oint_{|z|=0} \frac{dz}{2\pi i z} x^+(z)$ *acts as the elliptic Ruijsenaars difference operator.*

Level $(1,N)$ Representation $\mathcal{F}_u^{(1,N)}$ of $U_{q,p}(\mathfrak{gl}_{1,tor})$

Proposition 3.4 (Feigin-Hashizume-Hoshino-Shiraishi-Yanagida'09)

The following gives a level $(1,N)$ *representation on the Fock module* $\mathcal{F}_u^{(1,N)}$ *of* α_m *carrying a vacuum weight* $u \in \mathbb{C}^*$ *.*

$$x^+(z) = uz^{-N}\left(\frac{t}{q}\right)^{\frac{3N}{4}} \exp\left\{-\sum_{n>0}\frac{(t/q)^{n/4}}{1-(t/q)^n}\alpha_{-n}z^n\right\}\exp\left\{\sum_{n>0}\frac{(t/q)^{3n/4}}{1-(t/q)^n}\alpha_n z^{-n}\right\},$$

$$x^-(z) = u^{-1}z^{N}\left(\frac{t}{q}\right)^{-\frac{3N}{4}} \exp\left\{\sum_{n>0}\frac{(t/q)^{n/4}}{1-(t/q)^n}\alpha'_{-n}z^n\right\}\exp\left\{-\sum_{n>0}\frac{(t/q)^{3n/4}}{1-(t/q)^n}\alpha'_n z^{-n}\right\},$$

$$\psi^+(z) = \left(\frac{t}{q}\right)^{-\frac{N}{2}} \exp\left\{-\sum_{n>0}\frac{p^n(t/q)^{-n/4}}{1-p^n}\alpha_{-n}z^n\right\}\exp\left\{\sum_{n>0}\frac{(t/q)^{n/4}}{1-p^n}\alpha_n z^{-n}\right\},$$

$$\psi^-(z) = \left(\frac{t}{q}\right)^{\frac{N}{2}} \exp\left\{-\sum_{n>0}\frac{(t/q)^{n/4}}{1-p^n}\alpha_{-n}z^n\right\}\exp\left\{\sum_{n>0}\frac{p^n(t/q)^{-n/4}}{1-p^n}\alpha_n z^{-n}\right\},$$

where

$$\alpha'_m = \frac{1-p^{*m}}{1-p^m}\gamma^m\alpha_m \quad (m \in \mathbb{Z}_{\neq 0}), \quad \gamma^{1/2} = (t/q)^{1/4}, \text{ hence } \quad p^* = pq/t.$$

Intertwining Operators of $U_{q,t,p}(\mathfrak{gl}_{1,tor})$-modules

Type I $\Phi(u) : \mathcal{F}^{(1,N+1)}_{-uv} \to \mathcal{F}^{(0,1)}_{u} \widetilde{\otimes} \mathcal{F}^{(1,N)}_{v}$

$$\Delta^D(x)\Phi(u) = \Phi(u)x \qquad (\forall x \in \mathcal{U}_{q,t,p})$$

Theorem 4.1

$$\Phi(u) = \sum_{\lambda} |\lambda\rangle_u \widetilde{\otimes} \Phi_\lambda(u),$$

$$\Phi_\lambda(u) = \frac{q^{n(\lambda')} t^*(\lambda, u, v, N) N_\lambda(p)}{c_\lambda} \Phi_\emptyset(u) \prod_{(i,j)\in\lambda} \widetilde{x}^-((t/q)^{1/4} t^{-i+1} q^{j-1} u),$$

$$\Phi_\emptyset(u) = \exp\left\{ -\sum_{m>0} \frac{\alpha'_{-m}}{\kappa_m} ((t/q)^{1/2} u)^m \right\} \exp\left\{ \sum_{m>0} \frac{\alpha'_m}{\kappa_m} ((t/q)^{1/2} u)^{-m} \right\}.$$

where $\widetilde{x}^-(z) = u z^{-N} (t/q)^{3N/4} x^-(z), \quad N_\lambda(0) = N'_\lambda(0) = 1,$

$$< P_\lambda, P_\lambda >_{q,t} = \frac{c'_\lambda}{c_\lambda} \quad \rightsquigarrow \quad \frac{c'_\lambda N_\lambda(p)}{c_\lambda N'_\lambda(p)} = \frac{\prod_{\square\in\lambda} \theta_p(q^{a(\square)+1} t^{\ell(\square)})}{\prod_{\square\in\lambda} \theta_p(q^{a(\square)} t^{\ell(\square)+1})} =: \frac{c'_\lambda(p)}{c_\lambda(p)},$$

$$t^*(\lambda, u, v, N) = (q^{-1} v)^{-|\lambda|} (-u)^{N|\lambda|} f_\lambda(q,t)^N.$$

Cf. Trig. case : Awata-Feigin-Shiraishi '12

Type II dual $\Psi^*(v) : \mathcal{F}_u^{(1,N)} \widetilde{\otimes} \mathcal{F}_v^{(0,1)} \to \mathcal{F}_{-uv}^{(1,N+1)}$

$$x\Psi^*(v) = \Psi^*(v)\Delta^D(x) \qquad (\forall x \in \mathcal{U}_{q,t,p})$$

Theorem 4.2

$$\Psi^*_\lambda(v)\xi = \Psi^*(v)(\xi\widetilde{\otimes}|\lambda\rangle'_v) \qquad \xi \in \mathcal{F}_u^{(1,N)}, \qquad |\lambda\rangle'_v = \frac{c'_\lambda(p)}{c_\lambda(p)}|\lambda\rangle_v,$$

$$\Psi^*_\lambda(v) = \frac{q^{n(\lambda')}t(\lambda,u,v,N)}{c_\lambda N'_\lambda(p)}\Psi^*_\emptyset(v)\prod_{(i,j)\in\lambda}\widetilde{x}^+((t/q)^{1/4}t^{-i+1}q^{j-1}v),$$

$$\Psi^*_\emptyset(u) = \exp\left\{\sum_{m>0}\frac{\alpha_{-m}}{\kappa_m}((t/q)^{1/2}u)^m\right\}\exp\left\{-\sum_{m>0}\frac{\alpha_m}{\kappa_m}((t/q)^{1/2}u)^{-m}\right\}.$$

where $\widetilde{x}^+(z) = u^{-1}z^N(t/q)^{-3N/4}x^+(z)$,

$$t(\lambda,u,v,N) = (-uv)^{|\lambda|}(-v)^{-(N+1)|\lambda|}f_\lambda(q,t)^{-N-1}.$$

Type I "shifted inverse" $\Phi^*(v) : \mathcal{F}_u^{(0,1)} \widetilde{\otimes} \mathcal{F}_v^{(1,N)} \to \mathcal{F}_{-uv}^{(1,N+1)}$

We define $\Phi^*(u)$ by

$$\Phi^*_\lambda(v)\xi = \Phi^*(v)(|\lambda\rangle_u \widetilde{\otimes} \xi) \qquad \xi \in \mathcal{F}_v^{(1,N)}$$

$$\Phi^*_\lambda(u) = \frac{q^{n(\lambda')} t(\lambda, v, p^{-1}u, N) N'_\lambda(p)}{c'_\lambda} : \widetilde{\Phi}_\lambda(p^{-1}u)^{-1} :$$

$$\widetilde{\Phi}_\lambda(u) =: \Phi_\emptyset(u) \prod_{(i,j)\in\lambda} \widetilde{x}^-((t/q)^{1/4} t^{-i+1} q^{j-1} u) :$$

Then $\Phi^*(u)$ satisfies similar relations to the type I dual intertwining relations "$x\Phi^*(v) = \Phi^*(v)\Delta^D(x)$", but not exactly the same !

Hopefully, there is a suitable dual representation to $\mathcal{F}_u^{(0,1)}$, and $\Phi^*(u)$ can be interpreted as the dual intertwining operator to $\Phi(u)$ w.r.t. such dual representation.

The same happens for the shifted inverse of the type II dual VO.

$U_{q,t,p}(\mathfrak{gl}_{1,tor})$ and Affine Quiver W-alg. $W_{p,p^*}(\Gamma(\widehat{A}_0))$

Remember : $U_{q,p}(\widehat{\mathfrak{g}})$ realizes $W_{p,p^*}(\mathfrak{g})$

e.g. the level-1 elliptic currents $E_i(z), F_i(z)$ of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$

$\Longleftrightarrow$ screening currents $S_i^+(z), S_i^-(z)$ of $W_{p,p^*}(\mathfrak{gl}_N)$

Theorem 5.1 (H.K '14)

Let $\Phi^D(z) : \mathcal{V} \to V_z \widetilde{\otimes} \mathcal{V}'$ *s.t.* $\Phi^D(z)x = \Delta^D(x)\Phi^D(z) \quad \forall x \in U_{q,p}$
Then the generating functions $T(z)$ *of* $W_{p,p^*}(\mathfrak{gl}_N)$ *is realized as*

$$T(z) = \sum_{\mu=1}^{N} \Phi_\mu^D(p^{-1}z)^{-1}\Phi_\mu^D(z)$$

We have the same statement for $D_N^{(1)}$ and $B_N^{(1)}$.

$U_{q,t,p}(\mathfrak{gl}_{1,tor})$ and Affine Quiver W-algebra $W_{p,p^*}(\Gamma(\widehat{A}_0))$

Screening currents :

The level $(1,N)$ rep. of $\mathcal{U}_{q,t,p}$: $\gamma^{1/2}=(t/q)^{1/4}, \quad p^*=p\,q/t$

Setting $s^+_m=\frac{(t/q)^{m/2}}{1-(t/q)^m}\alpha_m,\ s^-_m=\frac{(t/q)^{m/2}}{1-(t/q)^m}\alpha'_m$, we have

$$x^{\pm}((t/q)^{1/4}z)=\left(\frac{u}{z^N}\left(\frac{t}{q}\right)^{N/2}\right)^{\pm1}:\exp\left\{\pm\sum_{m\neq0}s^{\pm}_m\ z^{-m}\right\}:,$$

and

$$[s^+_m,s^+_n]=-\frac{1}{m}\frac{1-p^m}{1-p^{*m}}(1-q^{-m})(1-t^m)\delta_{m+n,0},$$

$$[s^-_m,s^-_n]=-\frac{1}{m}\frac{1-p^{*-m}}{1-p^{-m}}(1-q^{-m})(1-t^m)\delta_{m+n,0}.$$

One of $x^{\pm}((t/q)^{1/4}z)$ coincides with the screening current of the affine quiver W-algebra $W_{p,p^*}(\Gamma(\widehat{A}_0))$ in Kimura-Pestun'15 with the $SU(4)$ Ω-deformation parameters p,p^*,q,t s.t. $p/p^*=t/q$ (Nekrasov '16).

Generating Function

$$T(u) \;=\; \Phi^*(u)\Phi(u) = \sum_\lambda \Phi^*_\lambda(u)\Phi_\lambda(u) = \sum_\lambda \mathcal{C}_\lambda(q,t,p) :\widetilde{\Phi^*}_\lambda(u)\widetilde{\Phi}_\lambda(u):$$

One finds

$$:\widetilde{\Phi^*}_\lambda(u)\widetilde{\Phi}_\lambda(u) :=: \prod_{\square\in A(\lambda)} Y(u/q^{\square}) \prod_{\blacksquare\in R(\lambda)} Y((q/t)u/q^{\blacksquare})^{-1} :$$

where $q^{\square} = t^{i-1}q^{-j+1}$ for $\square = (i,j) \in \lambda$ etc. ,

$$Y(u) =: \exp\left\{ \sum_{m\neq 0} y_m u^{-m} \right\} :$$

with $y_m = \dfrac{1-p^m}{\kappa_m}(q/t)^{m/2}\alpha'_m$ satisfying

$$[y_m, y_n] = -\frac{1}{m}\frac{(1-p^{*m})(1-p^{-m})}{(1-q^m)(1-t^{-m})}\delta_{m+n,0}.$$

Hence this operator part coincides with the one of $W_{p,p^*}(\Gamma(\widehat{A}_0))$ in Kimura-Pestun'15.

OPE coefficient × normalization factors of $\Phi^*_\lambda(u), \Phi_\lambda(u)$ yields

$$\mathcal{C}_\lambda(q,t,p) = \mathcal{C}\ \mathfrak{q}^{|\lambda|} Z^{\widehat{A}_0}_\lambda(t,q^{-1},p)$$

where

$$\mathfrak{q} = p^{*-1}p^{N-1}(t/q)^{1/2}, \qquad \mathcal{C} = \frac{(p^{-1}t;q,t,p)_\infty}{(q;q,t,p)_\infty},$$

$$Z^{\widehat{A}_0}_\lambda(t,q^{-1},p^*) = \prod_{\square\in\lambda} \frac{(1-pq^{a(\square)+1}t^{\ell(\square)})(1-pq^{-a(\square)}t^{-\ell(\square)-1})}{(1-q^{a(\square)+1}t^{\ell(\square)})(1-q^{-a(\square)}t^{-\ell(\square)-1})}.$$

Note that $\sum_{\lambda,|\lambda|=k} Z^{\widehat{A}_0}_\lambda(t,q^{-1},p^*)$ is the equivariant Hirzebruch χ_y-genus with $y=p$ of $\mathrm{Hilb}_k(\mathbb{C}^2)$ (Li-Liu-Zhou'04). Hence

$$\begin{aligned} T(u) &= \sum_\lambda \Phi^*_\lambda(u)\Phi_\lambda(u) \\ &= \mathcal{C}\sum_\lambda \mathfrak{q}^{|\lambda|} Z^{\widehat{A}_0}_\lambda(t,q^{-1},p) :\prod_{\square\in A(\lambda)} Y(u/q^{\square}) \prod_{\blacksquare\in R(\lambda)} Y((q/t)u/q^{\blacksquare})^{-1} : \end{aligned}$$

coincides with the gen. fnc. of $W_{p,p^*}(\Gamma(\widehat{A}_0))$ in Kimura-Pestun'15.

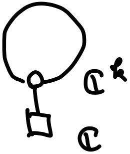
Instanton Calculus in the 5d & 6d Lifts of the 4d $\mathcal{N} = 2^*$ Theories

The 5d & 6d lifts of the $\mathcal{N}=2^*\ U(1)$ Theory

$$T(u)=\mathcal{C}\sum_{\lambda}\mathfrak{q}^{|\lambda|}Z_{\lambda}^{\widehat{A}_0}(t,q^{-1},p):\prod_{\square\in A(\lambda)}Y(u/q^{\square})\prod_{\blacksquare\in R(\lambda)}Y((q/t)u/q^{\blacksquare})^{-1}:$$

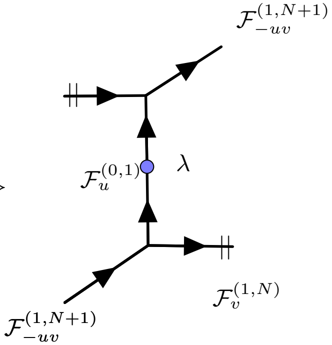
From this we immediately obtain the instanton PF of the 5d lift of the $\mathcal{N}=2^*\ U(1)$ Theory

$$\langle 0|T(u)|0\rangle=\mathcal{C}\sum_{\lambda}\mathfrak{q}^{|\lambda|}Z_{\lambda}^{\widehat{A}_0}(t,q^{-1},p)=\mathcal{C}\sum_{k\geq 0}\mathfrak{q}^{k}\underbrace{\sum_{\substack{\lambda\\|\lambda|=k}}Z_{\lambda}^{\widehat{A}_0}(t,q^{-1},p)}_{\chi_p\text{-genus of }\mathrm{Hilb}_k(\mathbb{C}^2)}.$$



This result and $T(u) = \sum_\lambda \Phi^*_\lambda(u)\Phi_\lambda(u)$ indicate

$$T(u) \quad \Longleftrightarrow$$



(Hollowood-Iqbal-Vafa'08)

The 5d & 6d lifts of the $\mathcal{N}=2^*$ $U(1)$ Theory

The trace gives the instanton PF of the 6d lift of the $\mathcal{N}=2^*$ $U(1)$ theory

$$\mathrm{tr}_{\mathcal{F}^{(1,N+1)}_{-uv}} Q^{-d} T(u) = \mathcal{C}_Q \sum_{\lambda} \mathfrak{q}^{|\lambda|} Z^{\widehat{A}_0}_{\lambda}(t, q^{-1}, p; Q), \quad v \in \mathbb{C}^*$$

where

$$\mathcal{Z}^{\widehat{A}_0}_{\lambda}(t, q^{-1}, p; Q) = \prod_{\Box \in \lambda} \frac{\theta_Q(p\, q^{a(\Box)+1} t^{\ell(\Box)})\theta_Q(p q^{-a(\Box)} t^{-\ell(\Box)-1})}{\theta_Q(q^{a(\Box)+1} t^{\ell(\Box)})\theta_Q(q^{-a(\Box)} t^{-\ell(\Box)-1})}.$$

Note :

$\sum_{\lambda, |\lambda|=k} \mathcal{Z}^{\widehat{A}_0}_{\lambda}(t, q^{-1}, p; Q)$ is the equivariant elliptic genus of $\mathrm{Hilb}_k(\mathbb{C}^2)$.

(Haghighat-Iqbal-Kozcaz-Lockhart-Vafa'15)

The 5d & 6d Lifts of the $\mathcal{N}=2^*$ $U(M)$ Theory

$$T(u_1)\cdots T(u_M)=\mathcal{C}_M\sum_{k=0}^{\infty}\mathfrak{q}_M^k\sum_{\substack{\lambda^{(1)},\cdots,\lambda^{(M)}\\ \sum|\lambda^{(j)}|=k}}\prod_{i,j}^{M}Z_{\lambda^{(i)},\lambda^{(j)}}(u_{j,i};t,q^{-1},p)$$

$$\times :\prod_{l=1}^{M}\prod_{\square\in A(\lambda^{(l)})}Y(u_l/q^{\square})\prod_{\blacksquare\in R(\lambda^{(l)})}Y((q/t)u_l/q^{\blacksquare})^{-1}:,$$

where $u_{j,i}=u_j/u_i$,

$$\mathfrak{q}_M=\mathfrak{q}p^{-(M-1)}=p^{*-1}p^{M+N}(t/q)^{1/2},$$

$$\mathcal{Z}_{\lambda,\mu}(u;t,q^{-1},p)=\prod_{\square\in\lambda}\frac{(1-p\,uq^{a_\mu(\square)+1}t^{\ell_\lambda(\square)})}{(1-uq^{a_\mu(\square)+1}t^{\ell_\lambda(\square)})}\prod_{\blacksquare\in\mu}\frac{(1-puq^{-a_\lambda(\blacksquare)}t^{-\ell_\mu(\blacksquare)-1})}{(1-uq^{a_\lambda(\blacksquare)}t^{-\ell_\mu(\blacksquare)-1})}$$

$T(u_1)\cdots T(u_M)$ gives the higher dim. extension of $W_{p,p^*}(\Gamma(\widehat{A}_0))$.

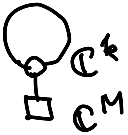
We obtain the instanton PF of the 5d lift of the $\mathcal{N}=2^*$ $U(M)$ theory

$$\langle 0|T(u_1)\cdots T(u_M)|0\rangle = \mathcal{C}_M \sum_{k=0}^{\infty} \mathfrak{q}_M^k \chi_p(\mathfrak{M}_{k,M}),$$

where

$$\chi_p(\mathfrak{M}_{k,M}) = \sum_{\substack{\lambda^{(1)},\cdots,\lambda^{(M)} \\ \sum|\lambda^{(j)}|=k}} \prod_{i,j}^{M} Z_{\lambda^{(i)},\lambda^{(j)}}(u_{j,i};t,q^{-1},p)$$

is the χ_y $(y=p)$ genus of the moduli space of rank M instantons with charge k.



(Haghighat-Iqbal-Kozcaz-Lockhart-Vafa '15)

The trace gives the instanton PF of the 6d lift of the $\mathcal{N}=2^* \ U(M)$ theory

$$\mathrm{tr}_{\mathcal{F}^{(1,N+1)}_{-u_1 v_1}} Q^{-d} T(u_1)\cdots T(u_M) = \mathcal{C}_{Q,M} \sum_{k=0}^{\infty} \mathfrak{q}_M^k \, \mathcal{E}_{p,Q}(\mathfrak{M}_{k,M}),$$

where

$$\mathcal{E}_{p,Q}(\mathfrak{M}_{k,M}) = \sum_{\substack{\lambda^{(1)},\cdots,\lambda^{(M)} \\ \sum_j |\lambda^{(j)}|=k}} \prod_{i,j=1}^{M} \mathcal{Z}_{\lambda^{(i)},\lambda^{(j)}}(u_{j,i}; t, q^{-1}, p; Q),$$

$$\mathcal{Z}_{\lambda^{(i)},\lambda^{(j)}}(u; t, q^{-1}, p; Q)$$
$$= \prod_{\square\in\lambda^{(i)}} \frac{\theta_Q(p\, u q^{a_{\lambda^{(j)}}(\square)+1} t^{\ell_{\lambda^{(i)}}(\square)})}{\theta_Q(u q^{a_{\lambda^{(j)}}(\square)+1} t^{\ell_{\lambda^{(i)}}(\square)})} \prod_{\blacksquare\in\lambda^{(j)}} \frac{\theta_Q(p u q^{-a_{\lambda^{(i)}}(\blacksquare)} t^{-\ell_{\lambda^{(j)}}(\blacksquare)-1})}{\theta_Q(u q^{a_{\lambda^{(i)}}(\blacksquare)} t^{-\ell_{\lambda^{(j)}}(\blacksquare)-1})}.$$

Note :

$\mathcal{E}_{p,Q}(\mathfrak{M}_{k,M})$ gives the equivariant elliptic genus of the moduli space of rank M instantons with charge k.

Hence we have shown a new AGT correspondence :

Instanton PF of the 5d $\&$ 6d lifts of the 4d $\mathcal{N}=2^*$ th.

$\Longleftrightarrow$ corr. fnc. of $W_{p,p^*}(\Gamma(\widehat{A}_0))$

via a realization of $W_{p,p^*}(\Gamma(\widehat{A}_0))$ by $U_{q,t,p}(\mathfrak{gl}_{1,tor})$