OPE coefficients in Argyres-Douglas theories

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Based on 2112.11899,

with

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Introductory part

Introduction

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The operator ϕ_r has U(1) charge and conformal dimension r.

More precisely, we want to compute their OPE coefficients

$$\phi_a(x) \times \phi_b(0) \sim \lambda_{a\,b\,c}\, \phi_c(0) \;+\; \cdots$$

Methods

I will show two methods to approach this problem

1. Supersymmetric localization

   The partition function Z can be computed in terms of a finite dimensional integral. This can be used to compute correlators

$$\langle \phi_r \, \bar{\phi}_r \rangle = \frac{1}{Z_{S^4}} \int \mathrm{d}a \left(u(a)^{2r} + \text{unmixing} \right) \left| Z_{\Omega\text{-background}} \right|^2 .$$

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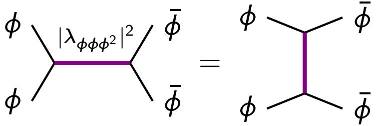
2. Conformal bootstrap

   Use the axioms of unitarity, operator product expansion and crossing to constrain a four-point function



$$\Longrightarrow \quad L \leq |\lambda_{\phi\phi\phi^2}|^2 \leq U \, .$$

Note on normalizations

In a conformal field theory two-point functions are completely fixed (provided we normalize operators appropriately)

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = \frac{\delta_{ij}}{(x_{12}^2)^{\Delta}}\,,$$

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The dynamical information then completely resides in the three-point functions, which are fixed up to a constant

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \frac{\lambda_{ijk}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}\,x_{13}^{\Delta_1+\Delta_3-\Delta_2}\,x_{23}^{\Delta_2+\Delta_3-\Delta_1}}\,.$$

Note on normalizations (cont.d)

With chiral operators people typically set $\lambda_{ijk} = 1$ so the dynamical data is transferred to the two-point functions

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = \frac{G_i\,\delta_{ij}}{(x_{12}^2)^{\Delta}}\,,$$

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This is why we are going to compute two-point functions with localization and compare (the above ratio) with tree-point functions from the bootstrap.

Conformal bootstrap

Three simple principles go into the conformal bootstrap

1. Operator product expansion (OPE)
   The OPE of two operators is convergent away from other insertions

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \cdots \rangle = \sum_k \lambda_{12k} \left[\begin{matrix} 1 \\ 2 \end{matrix} \!\!\succ\!\!\!-\!\!\!-\; k \right] (x_{12}, \partial_1) \, \langle \mathcal{O}_k \mathcal{O}_3 \cdots \rangle \,,$$

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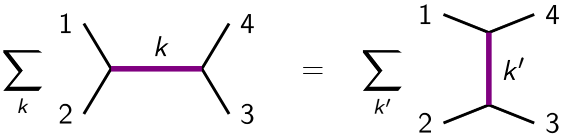
2. Crossing
   Taking the OPE in different channels in a four-point function should not matter



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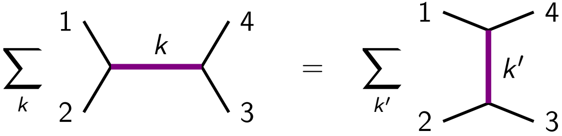

3. Unitarity

   The OPE coefficients are real thus their squares are positive.

Conformal bootstrap (cont.d)

Taking the OPE twice leads to the conformal block expansion

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_{\mathcal{O}}}} \sum_{\Delta,\ell} a_{\Delta,\ell}\, g_{\Delta,\ell}(u,v)\,,$$

where $g_{\Delta,\ell}$, the conformal blocks, are known kinematic functions, $a_{\Delta,\ell} = \lambda^2_{\mathcal{O}\mathcal{O}\mathcal{O}_{\Delta,\ell}}$ are (positive) OPE coefficients squared and

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,, \qquad x_{ij} = x_i - x_j\,,$$

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Crossing leads to a sum rule

$$\sum_{\Delta,\ell} a_{\Delta,\ell}\left(v^2 g_{\Delta,\ell}(u,v) - u^2 g_{\Delta,\ell}(v,u)\right) = 0\,,$$

on which we can act with a functional α of the form

$$\alpha[F] = \sum_{n,m} \alpha_{n,m}\, \partial_u^n \partial_v^m F(u,v)|_{u=v=1/4}\,.$$

Bounds on OPE coefficients

Let us denote as $F_{\Delta,\ell} = v^2 g_{\Delta,\ell}(u,v) - (u \leftrightarrow v)$. The action of α reads

$$\sum_{\Delta,\ell \neq (0,0),(\Delta_\star,\ell_\star)} a_{\Delta,\ell}\, \alpha\big[F_{\Delta,\ell}\big] + \alpha\big[F_{0,0}\big] = -a_{\Delta_\star,\ell_\star} \alpha\big[F_{\Delta_\star,\ell_\star}\big]\,.$$

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We can obtain two-sided bounds as follows [Caracciolo, Castedo Echeverri, Harling, Serone (2014)]

- **Upper bound**: maximize $\alpha\big[F_{0,0}\big]$ over the space $\alpha_{m,n}$ subject to

$$\begin{array}{ll} \mathbf{1.}\ \alpha\big[F_{\Delta,\ell}\big] \geq 0 & \\ \mathbf{2.}\ \alpha\big[F_{\Delta_\star,\ell_\star}\big] = 1 & \end{array} \quad \Longrightarrow \quad a_{\Delta_\star,\ell_\star} \leq -\alpha\big[F_{0,0}\big]$$

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Supersymmetric localization

Suppose we want to compute the path integral of an observable $\mathcal{O}(\Phi, \Psi)$

$$Z = \int \mathcal{D}\Psi \mathcal{D}\Phi \, \mathcal{O}(\Phi, \Psi) \,,$$

which is invariant under some supersymmetry transformation

$$\delta_Q \mathcal{O} = 0 \,, \qquad \text{with} \qquad \delta_Q \Phi = \Psi \,, \quad \delta_Q \Psi = V(\Phi) \,, \quad \delta_Q^2 = 0 \,.$$

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We can deform this integral by adding a Q-exact term

$$Z(t) = \int \mathcal{D}\Psi \mathcal{D}\Phi \, \mathcal{O}(\Phi, \Psi) \, e^{-t \, \delta_Q W(\Phi, \Psi)} \,, \qquad Z(0) = Z \,.$$

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However, the above function is actually independent on t

$$Z'(t) = -\int \delta_Q W\, \mathcal{O}\, e^{-t\,\delta_Q W} = -\int \delta_Q \left(W\, \mathcal{O}\, e^{-t\,\delta_Q W} \right) = 0\,,$$

therefore we can compute $\lim_{t\to\infty} Z(t) = Z(0)$ instead.

Supersymmetric localization (cont.d)

If $W = V(\Phi) \cdot \Psi$ then the dominant field configuration in the $t \to \infty$ limit will satisfy $V(\Phi) = 0$.

With an appropriate choice of Q the theory can be put on S^4 and the relevant configurations are Coulomb branch vacua

$$\text{vector multiplet}: \; \phi = \mathrm{diag}\,(a_1, \ldots, a_r)\,,$$

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All in all we have [Pestun (2012); Gerchkovitz, Gomis, Ishtiaque, Karasik, Komargodski, Pufu (2017)]

$$Z_{S^4} = \int_{\mathfrak{h}} d^r a \left| e^{-S_{\Omega,\text{cl}}(a)} Z_{\Omega,\text{1-loop}}(a,m,q,R)\, Z_{\Omega,\text{inst}}(a,m,q,R) \right|^2 .$$

$Z_{\Omega,\text{1-loop}}$ is known exactly and $Z_{\Omega,\text{inst}}$ is the Nekrasov partition function [Nekrasov (2003)] known as a power series in $q = e^{2\pi \mathrm{i} \tau}$.

Seiberg-Witten basics

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Example: SU(2) SQCD with one flavor.

$$\Sigma \ : \qquad y^2 + (x^2 - u)\, y + \frac{\Lambda^3}{4}(x - m) = 0\,,$$

with Λ the dynamically generated scale and m the mass of the hyper. This curve comes with a one-form

$$\lambda = x \frac{\mathrm{d}y}{y}\,.$$

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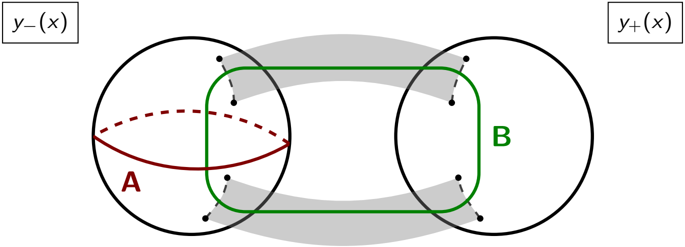

Seiberg-Witten basics (cont.d)

The curve can be used to compute a and a_D (the masses of monopole and dyon in the IR) as a function of u

$$a = \frac{1}{2\pi \mathrm{i}} \oint_{\mathbf{A}} \lambda \,, \qquad a_D = \frac{1}{2\pi \mathrm{i}} \oint_{\mathbf{B}} \lambda \,.$$

When both of these go to zero simultaneously, we reach a CFT in the IR known as the Argyres-Douglas (AD) fixed point. In this case it is the (A_1, A_2) theory [Argyres, Douglas (1995)].

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Define the Seiberg-Witten (SW) prepotential $\mathcal{F}_0$ as

$$a_D = -\frac{1}{2\pi \mathrm{i}} \frac{\partial \mathcal{F}_0}{\partial a} \,.$$

What does SW have to do with localization?

The partition function can be reorganized as an expansion in $1/R$ (the radius of the S^4)

$$|Z_\Omega(a, m, q, R)| = \exp R^2 \left(\mathcal{F}_0(a, m, q) + \sum_{g \geq 1} \mathcal{F}_g(a, m, q) R^{-2g} \right).$$

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Each term in the expansion contains the contribution of all instantons.

The term $\mathcal{F}_0$ is precisely the SW prepotential [Russo (2014)]. Furthermore, the AD point is a saddle

$$a_D \Big|_{\text{AD fixed point}} = 0 = -\frac{1}{2\pi\mathrm{i}} \frac{\partial \mathcal{F}_0}{\partial a} .$$

Therefore we can expand around the AD point.

The term $\mathcal{F}_1$

Also $\mathcal{F}_1$ can be obtained purely from SW data [Shapere, Tachikawa (2008)] (set $\Lambda = 1$ for brevity now)

$$\mathcal{F}_1 = -\frac{1}{2}\log\left[\det\left(\frac{\partial a}{\partial u}\right)\right] + \frac{1}{12}\log\Delta\,,$$

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Elliptic curves 101

Given a curve $y^2 + p(x)y + q(x) = 0$, the solution $y(x)$ has branch points given by the zeros of

$$w(x)^2 = p(x)^2 - 4\,q(x) \equiv \prod_i (x - e_i)\,.$$

The discriminant is defined to have zeros where the branch points collide

$$\Delta(x) = \prod_{i<j}(e_i - e_j)^2\,.$$

Main part

Plan

- ✓ Pedagogical introduction to the conformal bootstrap
- ✓ Very limited introduction to localization and SW geometry

1. Localization approach
2. Bootstrap approach
3. Relation with the large charge expansion

Summary for those who just joined in

We want to compute OPE coefficients of Coulomb branch operators for strongly interacting $\mathcal{N} = 2$ SCFTs in $4d$

$$\phi_a(x) \times \phi_b(0) \sim \lambda_{a\,b\,c}\, \phi_c(0) + \cdots$$

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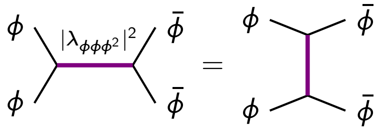
We are going to use two approaches

1. Supersymmetric localization $\longrightarrow$ approximation

$$\langle \phi_r\,\bar{\phi}_r \rangle = \frac{1}{Z_{S^4}} \int \mathrm{d}a\,\left(u(a)^{2r} + \text{unmixing}\right) \left|Z_{\Omega\text{-background}}\right|^2 .$$

2. Conformal bootstrap $\longrightarrow$ rigorous upper and lower bounds



$$\Longrightarrow \quad L \leq |\lambda_{\phi\phi\phi^2}|^2 \leq U\,.$$

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The partition function can be expanded in $1/R = \epsilon_1 = \epsilon_2$

$$|Z_\Omega(a, m, q, R)| = \exp R^2 \left(\mathcal{F}_0(a, m, q) + \sum_{g \geq 1} \mathcal{F}_g(a, m, q) R^{-2g} \right),$$

with $\mathcal{F}_0$ being the Seiberg-Witten prepotential, and $\mathcal{F}_1$ being known

$$\mathcal{F}_0 = -\pi \mathrm{i} \sum_{s=1}^{\text{rank}} a_s(u)\, a_D^s(u)\,,$$

$$\mathcal{F}_1 = -\frac{1}{2} \log \left[\det \left(\frac{\partial a_s}{\partial u_n} \right) \right] + \frac{1}{12} \log \Delta(u)\,.$$

Here $\Delta(u)$ is the discriminant of the Seiberg-Witten curve.

Localization approach

Rank one

Rank one is the easiest and most under control case. There are six examples of AD-type theories [Argyres, Plesser, Seiberg, Witten (1996); Minahan, Nemeschansky (1997)]

	H_0	H_1	H_2	E_6	E_7	E_8
r	$6/5$	$4/3$	$3/2$	3	4	6

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For now, let us ignore the $\mathcal{F}_{g>1}$ terms (we will return to them later). We thus need to compute the integral

$$\int \mathrm{d}a \left| \exp\left(R^2 \mathcal{F}_0(u(a)) - \frac{1}{2}\log\frac{\partial a}{\partial u} + \frac{1}{12}\log\Delta(u(a)) \right) \right|^2 .$$

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The function $u(a)$, while calculable, is a complicated object. However, the coefficient in $\mathcal{F}_1$ is just right so that

$$\int \mathrm{d}u \left| \Delta(u)^{\frac{1}{6}} \, e^{2R^2 \mathcal{F}_0(u)} \right| .$$

Rank one (cont.d)

Expanding around the AD point u_* we find

$$\Delta(u) = (u - u_*)^{\frac{12}{r}(r-1)}\,, \qquad \mathcal{F}_0(u) = f_* - \frac{c_0^2}{2}(u - u_*)^{\frac{2}{r}}\,.$$

For $H_{0,1,2}$, $\frac{12}{r}(r-1)$ is also $N_f + 1$ where N_f is the number of flavor of the UV theory.

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Expanding around the AD point u_* we find

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After changing integration variable to $\tilde{u} = u - u_*$ (recall $u = \langle \phi_r \rangle$) the localization integral becomes

$$C_{ij} \equiv \langle \phi_r^i(\mathbf{N})\, \bar{\phi}_r^j(\mathbf{S}) \rangle = \frac{e^{2R^2 f_*}}{Z_{S_R^4}} \int_0^\infty \mathrm{d}u\, e^{-R^2 c_0^2 \tilde{u}^{\frac{2}{r}}}\, \tilde{u}^{\,i+j+\frac{2}{r}(r-1)} .$$

In particular, it only depends on r! (c_0 gives an overall factor.)

Unmixing

We were expecting a diagonal matrix of two-point functions $\langle \phi_r^i \bar{\phi}_r^j \rangle = \delta_{ij} G_j / R^{2rj}$ and instead we got C_{ij}.

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where the sum is over operators of dimensions less than ϕ_r^i and ϕ_r^j.

Only for rank one it is possible to write a general expression [Gerchkovitz, Gomis, Ishtiaque, Karasik, Komargodski, Pufu (2017)]

$$G_n = \frac{\det_{i,j\leq n} C_{ij}}{\det_{i,j\leq n-1} C_{ij}} \, .$$

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Other authors deal with it by only considering a subset of operators such as $\left[\mathrm{tr}(\phi^2)\right]^n$ [Beccaria, Galvagno, Hasan (2020)].

Pure gauge theories

The easiest examples are given by pure gauge SU(N) theories. They give rise to the (A_1, A_{N-1}) AD theory.

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$$\Sigma\ :\quad y^2 + y\,P_N(x) + \frac{1}{4}\Lambda^{2N} = 0\,,$$

$$\text{with}\quad P_N(x) = x^N - u_2 x^{N-2} - u_3 x^{N-3} - \ldots - u_N\,.$$

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In particular we consider only rank two, namely (A_1, A_4) and (A_1, A_5)

	(A_1, A_4)	(A_1, A_5)
r_1	$8/7$	$5/4$
r_2	$10/7$	$3/2$

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We readily obtain for $N = 5$ and 6

$$\tilde{w}(\tilde{x})^2 \sim \tilde{x}^N - u\tilde{x} - v + O(\nu^N)\,,$$
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Denote the roots of $\tilde{w}(\tilde{x})$ as e_i

$$\tilde{w}(\tilde{x})^2 = \prod_{i=1}^{N}(\tilde{x} - e_i)\,.$$

Choice of cycles

Given a choice of cycles we can compute $a_s(u,v)$, $a_D^s(u,v)$ and thus the potentials $\mathcal{F}_0(u,v)$ and $\mathcal{F}_1(u,v)$

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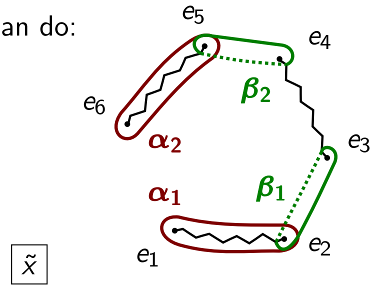

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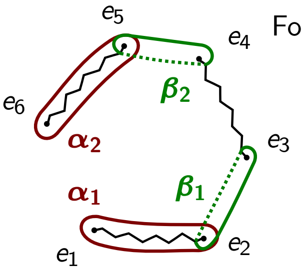

For $N = 5$ it's a bit trickier.

$\tilde{x}$

Final result

The trick is to expand the one form λ in powers of u

$$\lambda = \mathrm{d}\tilde{x} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{3}{2}\right)}{n!\,\Gamma\left(\frac{3}{2}-n\right)} (-\tilde{x}\,u)^n (\tilde{x}^N - v)^{\frac{1}{2}-n}\,.$$

Then the periods (γ_i is either α_s or β_s) are easy to compute

$$\oint_{\gamma_i} \lambda = 2\int_{e_i}^{e_{i+1}} \lambda\,, \qquad e_i = v^{\frac{1}{N}}\, e^{\frac{2\pi \mathrm{i}}{N}}\,.$$

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The result is an integral that can be performed numerically

$$C_{u^m v^n}(A_1, A_4) = \int \mathrm{d}\kappa \mathrm{d}V \left|5^5 + 4^4 \kappa^5\right|^{\frac{1}{6}} e^{-V^{\frac{7}{5}} f(\kappa)} \kappa^m\, V^{n+\frac{4}{5}m+\frac{22}{15}} \,,$$

$$C_{u^m v^n}(A_1, A_5) = \int \mathrm{d}\kappa \mathrm{d}V \left|6^6 - 5^5 \kappa^6\right|^{\frac{1}{6}} e^{-V^{\frac{4}{3}} f(\kappa)} \kappa^m\, V^{n+\frac{5}{6}m+\frac{5}{3}} \,,$$

for $V \propto v$, $\kappa \propto u v^{-r_1/r_2}$, $f(\kappa) = -2V^{-2/r_2} \mathrm{Re}\,\mathcal{F}_0$.

Results

For rank one we find

SCFT	H_0	H_1	H_2	E_6	E_7	E_8
$\lambda^2_{u\,u\,u^2}$	2.098	2.241	2.421	4.514	6.755	15.12
$\lambda^2_{u\,u^2\,u^3}$	3.300	3.674	4.175	12.05	24.01	95.33
$\lambda^2_{u^2\,u^2\,u^4}$	7.206	8.624	10.72	67.01	222.2	2443.4

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Whereas for rank two

SCFT	(A_1, A_4)	(A_1, A_5)
$\lambda^2_{u\,u\,u^2}$	1.87	1.93
$\lambda^2_{u\,v\,uv}$	1.04	1.04
$\lambda^2_{v\,v\,v^2}$	2.23	2.20

Bootstrap approach

General strategy

The bootstrap approach lets us put upper and lower bounds on the OPE coefficients of isolated operators.

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We consider four-point functions of the form

$$\langle \bar{\phi}_r(x_1)\phi_r(x_2)\phi_r(x_3)\bar{\phi}_r(x_4)\rangle = \frac{f(z,\bar{z})}{(x_{12}^2)^r(x_{34}^2)^r}\,,$$

with $x_{ij}^2 = |x_i - x_j|^2$ and

$$\frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = u = z\bar{z}\,, \qquad \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = v = (1-z)(1-\bar{z})\,.$$

Here r is a chiral primary with $U(1)_r$ charge and conformal dimension r.

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Here r is a chiral primary with $U(1)_r$ charge and conformal dimension r.

This setup was studied in [Beem, Lemos, Liendo, Rastelli, Rees (2016); Lemos, Liendo (2016)].

General strategy (cont.d)

The OPE coefficients $\lambda^2_{\phi_r\phi_r\mathcal{O}}$ appear in the superconformal block expansion of $f(z,\bar{z})$.

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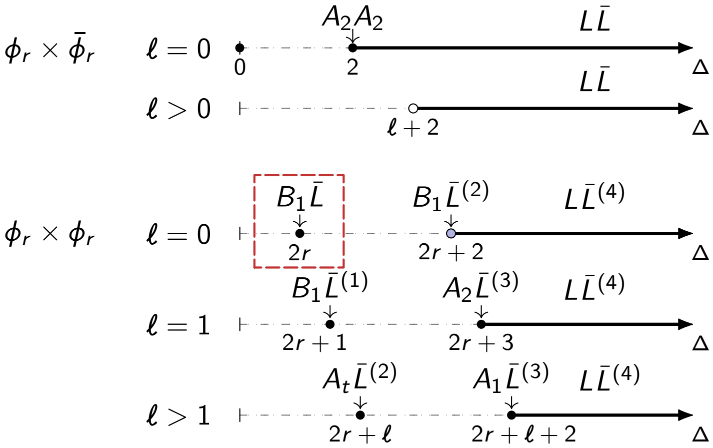
In order to obtain the OPE coefficients $\lambda^2_{\phi_{r_1} \phi_{r_2} \mathcal{O}}$ $(r_1 \neq r_2)$ we need to consider a system of mixed correlators which also includes

$$\langle \bar{\phi}_{r_1}(x_1) \phi_{r_1}(x_2) \phi_{r_2}(x_3) \bar{\phi}_{r_2}(x_4) \rangle \,.$$

The idea is the same, just more complicated, thus I will not discuss it.

Supermultiplets in the OPE

The following multiplets are exchanged in the OPE



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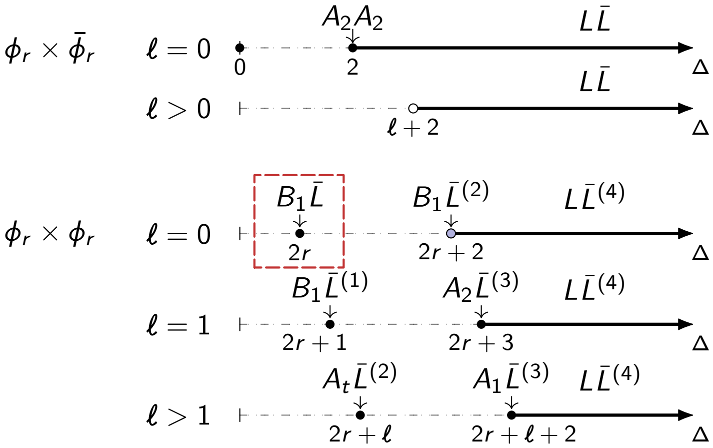

The operator in the rectangle is the one whose OPE we want to bound. The number in parentheses e.g. $L\bar{L}^{(4)}$ is the order of the lowest super-descendant which contributes to the four-point function.

Superconformal blocks

The exchange of each operator in the diagram of before corresponds to a superconformal block contributing to the four-point function.

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$$\begin{aligned}\langle\bar{\phi}_r(x_1)\phi_r(x_2)\phi_r(x_3)\bar{\phi}_r(x_4)\rangle &= \sum_{(\Delta,\ell)\in\bar{\phi}\times\phi}\lambda^2_{\phi\bar{\phi}\mathcal{O}_{\Delta,\ell}}\,\mathcal{G}_{\Delta,\ell}(z,\bar{z})\,,\\ &= \sum_{(\Delta,\ell)\in\phi\times\phi}|\lambda_{\phi\phi\tilde{\mathcal{O}}_{\Delta,\ell}}|^2\,g_{\Delta,\ell}(1-z,1-\bar{z})\,.\end{aligned}$$

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\end{aligned}
$$

As it is always the case for chiral operators, the superblocks are usual scalar blocks with some shifts

$$
\mathcal{G}_{\Delta,\ell}(z,\bar{z}) = (z\bar{z})^{-1} g_{\Delta+2,\ell}(z,\bar{z})\,.
$$

In the crossed channels they are literally just scalar blocks.

Bounds on OPE coefficients

The crossing equations can be recast into a vector form

$$\sum_{\Delta,\ell} |\lambda_{\phi\phi\tilde{\mathcal{O}}}|^2 \, \vec{V}^{\text{charged}}_{\Delta,\ell} + \sum_{\Delta,\ell} (\lambda_{\phi\bar{\phi}\mathcal{O}})^2 \, \vec{V}^{\text{neutral}}_{\Delta,\ell} = -\vec{V}^{\text{neutral}}_{0,0} - \frac{r^2}{6c} \, \vec{V}^{\text{neutral}}_{2,0} \,,$$

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where the entries of $V^{\cdots}_{\Delta,\ell}$ are $\pm$ combinations of a block and its crossed.

We look for functionals $\alpha = \sum \alpha_{n,m} \partial_z^n \partial_{\bar{z}}^m |_{z=\bar{z}=1/2}$ that solve the problem

Fix c, then maximize (minimize) $B_\pm \equiv \alpha\left[\vec{V}^{\text{neut.}}_{0,0} + \frac{r^2}{6c} \, \vec{V}^{\text{neut.}}_{2,0}\right]$ *subject to*

1. $\alpha[\vec{V}^{\text{charged}}_{2r,0}] = \pm 1$
2. $\alpha[\text{others}] \geq 0$

Bounds on neutral sector

One could also assume a specific value of $\lambda_{\phi\phi\phi^2}$ and c and put an upper bound on the gap between the stress tensor and the first neutral unprotected operator.

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This is done by doing a binary search on Δ_{gap} while solving the problem

There exists an α such that

1. $\alpha\left[\vec{V}^{\text{neutral}}_{0,0} + |\lambda_{\phi\phi\phi^2}|^2 \vec{V}^{\text{charged}}_{2r,0} + \frac{r^2}{6c}\vec{V}^{\text{neutral}}_{2,0}\right] = 1$
2. $\alpha[\vec{V}^{\text{neutral}}_{\Delta,0}] \geq 0 \quad \forall\, \Delta \geq \Delta_{\text{gap}} > 2$
3. $\alpha[\text{others}] \geq 0$

Results for OPE coefficients

For rank one we find (some were previously known from [Lemos, Liendo (2016); Gimenez-Grau, Liendo (2020)]). Red numbers disallow the approximate localization result

	H_0	H_1	H_2	H_0 (*localization*)
$\lambda^2_{u\,u\,u^2}$	2.167	2.359	2.698	2.098
	2.142	2.215	2.298	
$\lambda^2_{u\,u^2\,u^3}$	3.637	4.445		
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While for higher rank

	(A_1, A_4)	(A_1, A_5)	(A_1, A_4) (*loc.*)	(A_1, A_5) (*loc.*)
$\lambda^2_{u\,u\,u^2}$	2.102 2.024	2.231 2.055	1.87	1.93
$\lambda^2_{u\,v\,uv}$	1.125 0.981	1.233 0.960		
$\lambda^2_{v\,v\,v^2}$	2.533 2.181	2.709 2.195		

Improving the OPE results

For higher rank we can improve the results and bound the 3d region $\{\lambda_{uuu^2}, \lambda_{vvv^2}, \lambda_{uv\,uv}\}$

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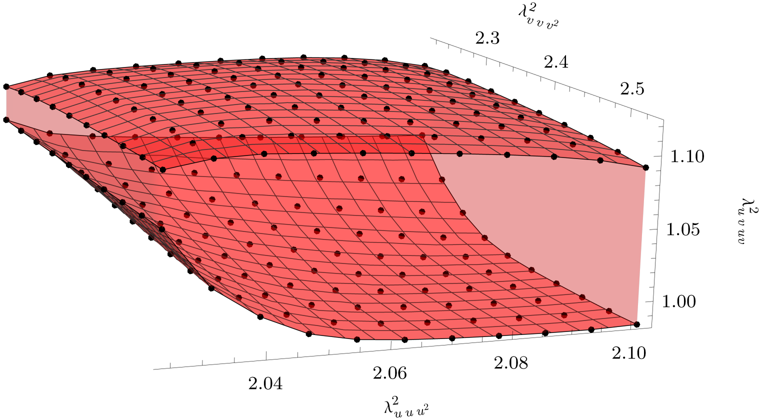
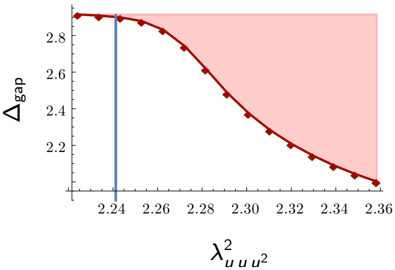
See for instance (A_1, A_4)

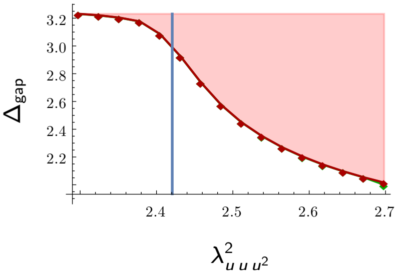


Results for neutral unprotected sector

Take the theories H_1 and H_2. Fixing λ_{uuu^2} we find the upper bounds (red is disallowed)



Theory H_1, $r = 4/3$



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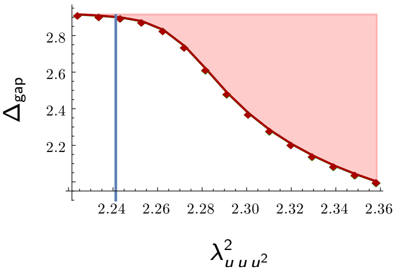

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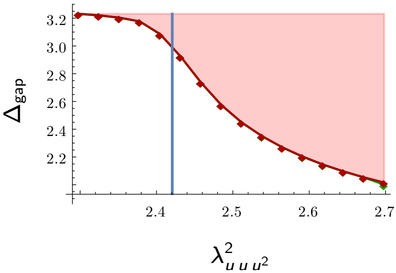

Theory H_2, $r = 3/2$

It seems that the true theory favors a large gap as compared to the free theory value, as expected from their strongly coupled nature.

Large charge expansion

Why does the approximation work at all?

We have dropped infinitely many terms from

$$|Z_\Omega(a, m, q, R)| = \exp R^2 \left(\mathcal{F}_0(a, m, q) + \sum_{g \geq 1} \mathcal{F}_g(a, m, q) R^{-2g} \right).$$

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In the CFT point they should all be contributing at the same order since R can be rescaled away from the integral

$$\mathcal{F}_g(u) \sim c_{\mathcal{F}_g} (a - a_*)^{2-2g} \quad \big(\times \log R(a - a_*) \quad \text{if } g = 1 \big).$$

So just take $a \to a_* + y/R$.

Why does the approximation work at all?

We have dropped infinitely many terms from

$$|Z_\Omega(a, m, q, R)| = \exp R^2 \left(\mathcal{F}_0(a, m, q) + \sum_{g \geq 1} \mathcal{F}_g(a, m, q) R^{-2g} \right).$$

In the CFT point they should all be contributing at the same order since R can be rescaled away from the integral

$$\mathcal{F}_g(u) \sim c_{\mathcal{F}_g} (a - a_*)^{2-2g} \quad \big(\times \log R(a - a_*) \quad \text{if } g = 1 \big).$$

So just take $a \to a_* + y/R$.

Yet somehow throwing most of them away still gives reasonably good results: there has to be a sense in which this is an expansion.

Why does the approximation work at all? (cont.d)

For rank one we have integrals of the form

$$C_{ij} = \int_0^\infty \mathrm{d}y \, y^{r(i+j+3)-3} e^{-y^2} \, \exp\left(-\frac{c_{\mathcal{F}_2}}{y^2} - \frac{c_{\mathcal{F}_3}}{y^4} - \dots\right) .$$

Why does the approximation work at all? (cont.d)

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For large $i, j \sim n$ the integral localizes in the positive solution of

$$\frac{\mathrm{d}}{\mathrm{d}y}\left((r(n+3)-3)\log y - y^2\right) = 0\,.$$

which grows like $\sqrt{n}$. Therefore all terms $1/y^k$ are suppressed by powers of n.

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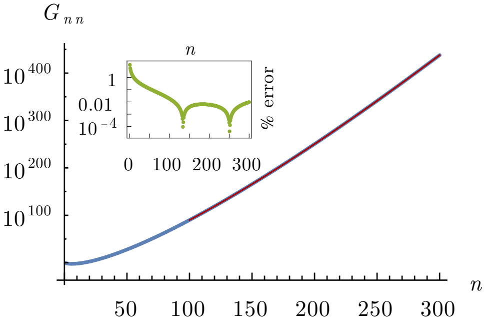
Note: this is hand wavy because to get G_n we need to take a ratio of determinants where all C_{ij}'s enter.

Comparison

Indeed we can compare the OPE coefficients of heavy operators obtained from localization with the asymptotic formula [Hellerman, Maeda (2017); Hellerman, Maeda, Orlando, Reffert, Watanabe (2019)]

$$G_n \underset{n\to\infty}{\sim} \tilde{\mathcal{Y}}\,\Gamma(nr+1)\left(\frac{N_{\mathcal{O}}}{2\pi R}\right)^{2nr}(nr)^{\alpha} + O\left(\frac{1}{n^{\#}}\right),$$



$r = \frac{6}{5}$

Possible future developments

We did this comparison numerically but it would be interesting to see if both the asymptotics and the corrections can be derived by studying the integral

$$G_n = \frac{\det_{i,j\leq n} C_{ij}}{\det_{i,j\leq n-1} C_{ij}}\,, \quad \text{with}$$

$$C_{ij} = \int_0^\infty \mathrm{d}y\, y^{r(i+j+3)-3} e^{-y^2} \exp\left(-\frac{c_{\mathcal{F}_2}}{y^2} - \frac{c_{\mathcal{F}_3}}{y^4} - \ldots\right)\,.$$

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Conversely, could we use this to put bounds on the $c_{\mathcal{F}_g}$'s? A simple Laurent expansion around $y \sim \sqrt{n}$ shows that at any given order in $1/n$ only finitely many $c_{\mathcal{F}_g}$ contribute.

Conclusions

Conclusions

- I discussed the computation of OPE coefficients of Coulomb branch operators in Argyres-Douglas theories
- I showed two complementary approaches: localization and bootstrap
- The localization result could also be used as input in the bootstrap problem
- I argued that the localization method works well because the large charge expansion unexplainably works well even at low dimensions
- I left with some open questions on how to make the link more precise (*w.i.p.*)

Thank you!