The Hodge standard conjecture for self-products of K3 surfaces

(Joint with Tetsushi Ito and Teruhisa Koshikawa)

§ Grothendieck's standard conjectures

k : field

X : proj smooth variety $/k$ of dim d

$$H^i(X) := H^i_{et}(X_{\bar{k}}, \mathbb{Q}_\ell)$$

$\bar{k}$: alg closure

$\ell \neq ch(k)$

$\exists$ cycle class map

$$cl^i : Z^i(X)_{\mathbb{Q}} \longrightarrow H^{2i}(X) \quad (= H^{2i}(X)(i)) \quad \text{omit Tate twists}$$

$Z^i(X)_{\mathbb{Q}}$: $\mathbb{Q}$-vector sp of algebraic cycles on X of codim i

$$A^i(X) := \mathrm{Im}(cl^i) \subset H^{2i}(X)$$

$A^i(X)$: $\mathbb{Q}$-subspace

(NOT $\mathbb{Q}_\ell$-sub)

Conj $I(X)$ (Hodge standard conjecture)

L : ample div on X

$i \leq \frac{d}{2}$

$$P^i(X) := \{ \alpha \in A^i(X) \mid \underbrace{\alpha \cdot L^{d-2i+1}}_{\in A^{d-i+1}(X)} = 0 \}$$

$P^i(X)$: primitive part

Then

$$P^i(X) \times P^i(X) \to \mathbb{Q}$$
$$(\alpha, \beta) \mapsto (-1)^i \alpha \cdot \beta \cdot L^{d-2i}$$

is positive definite $\square$

Conj B(X) (Lefschetz standard)

L : ample , $i \le d$

Hard Lefschetz thm

$\downarrow$

The inverse of the isom $H^i(X) \xrightarrow{\sim} H^{2d-i}(X)$ is algebraic

$$x \mapsto x \cup L^{d-i}$$

(i.e. induced by an element in $A^i(X \times X)$)

$\square$

Conj C(X) (Künneth standard)

$$H^*(X) := \bigoplus_i H^i(X)$$

Then $H^*(X) \twoheadrightarrow H^i(X) \hookrightarrow H^*(X)$ is algebraic $\square$

Conj D(X)

l-adic homological equiv = numerical equiv

i.e. the kernel of $cl^i : Z^i(X)_{\mathbb{Q}} \to H^{2i}(X)$

$$\| $$

$$\{ \alpha \in Z^i(X)_{\mathbb{Q}} \mid \alpha \cdot \beta = 0 \text{ for } {}^\forall \beta \in Z^{d-i}(X)_{\mathbb{Q}} \}$$

$\square$

Rem

"$\cap$" is known

Standard conjectures are widely open!

Known results

surfaces, curves. OK

- $I(X) + B(X) \Rightarrow C(X), D(X)$

- $I(X)$ is known if $\dim X \leq 2$ (Hodge index thm)
  or
  $ch(k) = 0$ (Hodge theory)

- $B(X)$ is known if $\dim X \leq 2$ or X: abelian variety

  Moreover $B(X) \Rightarrow B(X \times \cdots \times X)$

- If $k = \mathbb{F}_q$: finite fld, then $C(X)$ is known
  (by Weil conjecture)

Rem

Historically, standard conj was introduced by Grothendieck to prove Weil conj.

In fact $I(X \times X) + B(X) \Rightarrow$ Weil conj for X

(cf. positivity of Rosati involutions $\Rightarrow$ Weil conj for abelian vars)

However, Weil conj was proved by Deligne

(and it is used to study standard conj's)

$\square$

§ Main results

X : K3 surface / k

$\overset{\Leftrightarrow}{\text{def}}$ X : proj smooth surface s.t. $H^1(X, \mathcal{O}_X) = 0$
$\Omega^2_X \cong \mathcal{O}_X$

$X^2 := X \times X$

Main thm

$I(X^2)$ holds true.

Cor

All standard conjectures hold for X^2

also for $X^{[2]}$: Hilbert scheme of 2-points on X $\square$

§ Sketch of the proof of Thm

We may assume $k = \mathbb{F}_q$ by specialization arguments.

Starting point :

Thm (I - Ito - Koshikawa , 2018)

X : K3 / $\mathbb{F}_q$

(1) The Tate conjecture for X^2 is true,

i.e.

$$cl^i : Z^i(X^2) \otimes \mathbb{Q}_\ell \twoheadrightarrow H^{2i}(X)(i)^{\mathrm{Frob}_q = \mathrm{id}} \quad \forall i$$

(2) $D(X^2)$ is true (special case of Cor) $\square$

$\rightsquigarrow$ Study the (pure) motive of X

Important part : the transcendental motive $t(X) \subset h^2(X)$ of X

in the category of numerical motives

Properties of $t(X)$

- ℓ-adic realization (cohomology) of $t(X)$ is $=: T(X)_\ell$

  the orthogonal complement of $\mathrm{Pic}(X_{\overline{\mathbb{F}}_q})_{\mathbb{Q}_\ell} \hookrightarrow H^2(X)$

- $\mathrm{End}(t(X))$ is a finite dimensional semisimple $\mathbb{Q}$-alg (due to Jannsen)

- $\mathrm{End}(t(X)) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} \mathrm{End}_{\mathrm{Frob}_q}(T(X)_\ell)$ (by Tate conj for X^2)

- $\mathrm{End}(t(X))$ admits a natural involution i

  induced by switching two factors of $X \times X$.

$T(X)_\ell$ admits cup product $\langle\ ,\ \rangle$

For $f \in \mathrm{End}(t(X))$, $i(f)$ is the adjoint of f w.r.t $\langle , \rangle$

$(\langle f(x), y\rangle = \langle x, i(f)(y)\rangle)$

Rem $\mathrm{End}(T(X))$ is a K3 analogue of $\mathrm{End}(A)\otimes\mathbb{Q}$ for (simple) abelian var $A/\mathbb{F}_q$

In fact, we can prove :

$$\mathrm{End}(T(X)) \text{ is a division alg } \quad (\text{if } t(X) \neq 0)$$

Main thm can be deduced from the following :

Prop i is a positive involution i.e. in alg $\mathrm{End}(t(X))$

$$0 \neq f \in \mathrm{End}(t(X)) \Rightarrow \mathrm{Tr}(f \cdot i(f)) > 0$$

$\square$

Fact (positivity of Rosati involution)

A : abelian variety $\quad \lambda : A \to A^\vee$: polarization

dual ab var

$$i : \mathrm{End}(A)\otimes\mathbb{Q} \to \mathrm{End}(A)\otimes\mathbb{Q} \quad \text{is a positive involution}$$
$$f \mapsto \lambda^{-1}\cdot f^\vee \cdot \lambda$$

$\square$

Proof of Prop

May assume $t(X) \neq 0$ ($\Leftrightarrow$ X is not supersingular)

$$H_X = \{ f \in \mathrm{End}(t(X))^\times \mid f\cdot i(f) = 1 \} \quad : \text{algebraic gp}/\mathbb{Q}$$

"motivic isometries"

i : positive $\Leftrightarrow$ $H_X(\mathbb{R})$: compact

Use the Kuga-Satake period map to show this ①

$\mathcal{M}$: (smooth loci of) moduli space of (polarized) K3 surfaces $/\mathbb{Z}_{(p)}$

$KS : \mathcal{M} \to \underset{\sim}{Sh} \;/\mathbb{Z}_{(p)}$ smooth scheme

↑ étale map

integral model of Shimura variety attached to $SO(\Lambda)$

$X \longmapsto (A, \lambda)$: polarized abelian variety (Kuga-Satake) (with additional structures)

We can attach algebraic group $/\mathbb{Q}$

$\mathbb{G}_m \subset H_A \subset (\mathrm{End}(A)\otimes\mathbb{Q})^\times$

cf $\mathbb{G}_m \subset GSpin(\Lambda) \subset GL(Cl(\Lambda))$ ↑ Clifford alg

Fact $\Rightarrow$ $(H_A/\mathbb{G}_m)(\mathbb{R})$: compact

Thus, it suffices to construct :

a surjection $\tau : H_A \twoheadrightarrow H_X$ of alg gps $/\mathbb{Q}$ with $\mathbb{G}_m \subset \ker\tau$

($\Rightarrow$ $(H_A/\mathbb{G}_m)(\mathbb{R}) \twoheadrightarrow (H_X)(\mathbb{R})$)

◎ Construction of τ (sketch)

Take $f \in H_A(\mathbb{Q}) \subset \mathrm{End}(A)\otimes\mathbb{Q}$ ⑧

We want to attach an alg cycle $\tau(f) \in H_X(\mathbb{Q})$

Step 1 (After enlarging $\mathbb{F}_q$)

$\exists$ $\mathcal{O}_K \supset \mathbb{Z}_q$: complete DVR res fld $= \mathbb{F}_q$

$\exists$ $A_{\mathcal{O}_K}$: abelian scheme $/\mathcal{O}_K$, lift of A

$\exists$ $\tilde{f} \in \mathrm{End}(A_{\mathcal{O}_K})\otimes\mathbb{Q}$: lift of f

Step 2 (Fix $K \hookrightarrow \mathbb{C}$)

KS is étale

$\Rightarrow$ $A_{\mathcal{O}_K}$ and $\tilde{f}$ induce a lift $X_{\mathcal{O}_K}$ of X

$\boxed{\text{cf. } GSpin \to SO(H^2(X))}$ and $g : H^2(X_{\mathbb{C}}, \mathbb{Q}) \xrightarrow{\sim} H^2(X_{\mathbb{C}}, \mathbb{Q})$

Hodge isometry

Step 3 g is algebraic (due to Mukai and Buskin)

$\tau(f) :=$ specialization of g $\in H_X(\mathbb{Q})$

$\square$ Prop

Thm

§ Higher powers $X^n := \underbrace{X \times \cdots \times X}_{n}$

We can prove $I(X^n)$ $\forall n$ for certain K3 surf X $/\mathbb{F}_q$

Def $X / \mathbb{F}_q$ is neat

$\underset{\text{def}}{\Longleftrightarrow}$ $\forall x \in H^{2i}(X^n)(i)^{\mathrm{Frob}_q = id}$ is generated by
Tate cycle $\forall n \geq 3$ $\forall i$ pull-backs of Tate cycles on X^2

via projections $X^n \to X^2$

Rem Assume X : neat (9)

Then Tate conj for X^2

$\Rightarrow$ — " — for X^n $\forall n$

Thm X : neat

Then $I(X^n)$ holds true for $\forall n$

can be reduced to $I(X^2)$

Thm If the Picard number $\rho(X_{\overline{\mathbb{F}}_q}) \geqq 18$, then X is neat.

Rem $\exists$ non-neat K3

(constructed via Honda-Tate theory for K3)

(Taelman and K.I.)

Rem We don't know if

$$\underset{\forall n}{\underbrace{I(X^n)}} \Rightarrow \underset{\forall n}{I(X^{[n]})} \quad \text{holds or not.}$$

$I(\ )$ may depend on the choice of ample L in general

$\underbrace{I(X^{[2]})}_{\dim 4}$ does not depend — " —