Representation theory in triangulated categories

Rings Λ : either

- finite dimensional algebras / field k : A
- commutative (complete) Noetherian (local) ring : R

(• common generalization as module-finite alg's)

Representation theory = study of the category of modules, and its relatives

- $\operatorname{mod}\Lambda = \{$ finitely generated Λ-modules $\}$
- "nice" subcats, derived / singlarity / cluster cats, ...

Thm The Krull-Schmidt thm holds in $\operatorname{mod}\Lambda$

$\forall X \in \operatorname{mod}\Lambda \qquad \exists! \ X = \bigoplus_{i=1}^{n} X_i$ with X_i : indec.

Def A ~~A~~ is called representation-finite if $\operatorname{mod}\Lambda$ has only finitely many indecomposable objects up to isomorphism.

Suppose R is Cohen-Macaulay ring ($\Leftrightarrow$ depth R = dim R)

and consider

$$\operatorname{CM} R = \{ \text{(maximal) Cohen-Macaulay}^{R} \text{ modules} \} \subset \operatorname{mod} R$$

@ This $\operatorname{CM} R$ is an exact category, with enough proj and inj
(Quillen)

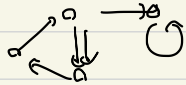
proj $= R$, inj $= \omega_R$

$\operatorname{Ext}^1_R(\operatorname{CM}, \omega_R) = 0$

Def R is representation-finite if CMR has $<\infty$ - many indec's / iso

Example of rep. finite A

- quiver Q (= oriented graph) $\rightsquigarrow$ kQ : path algebra

$\circ \to \circ \to \circ$  (gl-dim $kQ \leq 1$)

$\Updownarrow$

- Q : finite acyclic $\rightsquigarrow$ kQ : finite hereditary algebra
- any fin. dim hereditary / $k = \bar{k}$ is Morita equivalent to kQ (have the equiv. module cat) for $\exists$ Q
- module over kQ = rep. of quiver Q

Thm (Gabriel, 1972)
Q : finite quiver. Then kQ is rep. finite iff Q is a Dynkin quiver i.e. underlying graph of Q is ADE

Examples of rep. finite R

(hierarchy) regular $\Rightarrow$ Gorenstein $\Rightarrow$ Cohen-Macaulay

regular $\Updownarrow$ (gl. dim $R = d$) Gorenstein $\Updownarrow$ (id $R = d$)

- R is regular $\Leftrightarrow$ $\mathrm{CM}\,R = \mathrm{proj}\,R$ ($\because$ rep. thy of R is trivial)
- R is Gorenstein $\Leftrightarrow$ $\underset{\text{inj}}{\omega_R} \simeq \underset{\text{proj}}{R}$ $\Leftrightarrow$ $\mathrm{CM}\,R$ is Frobenius

stable cat

$\rightsquigarrow$ $\underline{\mathrm{CM}}\,R$: triangulated

Thm (Buchweitz, 1986, Published 2022 !!)
$\exists$ canonical equiv $\underline{\mathrm{CM}}\,R \xrightarrow{\sim} \dfrac{D^b(\mathrm{mod}\,R)}{\mathrm{per}\,R} =: D_{sg}(R)$

singularity cat

Thm (Herzog, Knörrer, Buchweitz–Greuel–Schreyer ... late 1980s)
R : complete Gorenstein local ring containing $R/m = \mathbb{C}$. $\dim R = d$
Then R is rep. finite iff $R \simeq k[[x, y, z_2, \cdots, z_d]]/(f)$
with f one of the following :

(A_n) $x^{n+1} + y^2 + z_2^2 + \cdots + z_d^2$ $(n \geq 1)$

(D_n) $x^{n-1} + xy^2 + z_2^2 + \cdots + z_d^2$ $(n \geq 4)$

(E_6) $x^4 + y^3 + z_2^2 + \cdots + z_d^2$

(E_7) $x^3y + y^3 + z_2^2 + \cdots + z_d^2$

(E_8) $x^5 + y^3 + z_2^2 + \cdots + z_d^2$

Q Relations :

Thm (Auslander)

give certain grading on R

$R = k[x, y, z]/(f)$: simple sing. of dim = 2

Q : corresponding Dynkin quiver.

$\Rightarrow$ $\exists$ commutative diagram

$$\begin{tikzcd} \underline{CM}^{\mathbb{Z}} R \arrow[r, "\sim"] \arrow[d] & D^b(\mathrm{mod}\, kQ) \arrow[d] \\ \underline{CM}\, R \arrow[r, "\sim"] & \mathcal{C}_1(kQ) \end{tikzcd}$$

← "1-cluster category"

Goal : Give methods to construct such diagrams.

§ Cluster category

A : fin. dim, $d \geq 1$ $\rightsquigarrow$ $\mathcal{C}_d(A)$

Properties

- triangulated cat
- d-Calabi-Yau, and has a d-cluster tilting obj

⇕

([d] is Serre)

Thm (Auslander-Reiten duality) $d = \dim R$.

R : comm. Gor. isolated sing. ($\Leftrightarrow$ $\forall p \in \mathrm{Spec}\, R \setminus \{m\}$, R_p : regular)

$\Rightarrow$ $\underline{CM}\, R$ is $(d-1)$-Calabi-Yau.

Def (Iyama, 2007) $\mathcal{T}$: triang. cat, $d \geq 1$

$M \in \mathcal{T}$ is **d-cluster tilting** if

$$\underset{\{\text{direct summand of } M^{\oplus n}\}}{\overset{\text{add } M}{\parallel}} = \{ X \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(M, X[i]) = 0 \text{ for } 0 < i < d \}$$

(⊂) $\mathrm{Ext}^{1\sim d-1}_{\mathcal{T}}(M, M) = 0$ (⊃) certain maximality

Rem

- (d = 1) ⇝ $M \in \mathcal{T}$ is 1-CT ⟺ add $M = \mathcal{T}$
  $\mathcal{T}$ is "rep. finite"
- (d = 2) 2-CT ∈ 2-CY is essential in categorification of cluster alg
- ($\mathcal{T} = \underline{\mathrm{CM}}\, R$) d-CT is analogue of NCCR

Thm (Iyama, 2007)

$S = k[[x_0, x_1, \ldots, x_d]]$, $G \subset SL_{d+1}(k)$: finite, s.t. $R = S^G$: iso. sing

$\Rightarrow$ $S \in \underline{\mathrm{CM}}\, R$: d-cluster tilting

(cf. (d = 1) R : simple sing. of dim = 2, $\underline{\mathrm{CM}}\, R$ = add S
: R is rep. finite)

Def (Buan–Marsh–Reineke–Reiten–Todorov, 2006)

Q : finite acyclic quiver. Its **d-cluster category** is

$$\mathcal{C}_d(kQ) = D^b(\mathrm{mod}\, kQ) \Big/ \underbrace{D(kQ)[-d]}_{\text{Serre} \circ [-d]} \overset{L}{\underset{kQ}{\otimes}} -$$

Thm (1) (Keller, 2005) $\mathcal{C}_d(kQ)$ is d-CY triang cat.

(2) (BMRRT) $kQ \in \mathcal{C}_d(kQ)$ is d-cluster tilting object.

Rem This generalizes to $\mathcal{C}_d(A) \xhookleftarrow[\text{"triang. hull"}]{} D^b(\mathrm{mod}\, A) / \text{Serre} \circ [-d]$

∘ Morita-type theorem = recognition theorem.

Thm $\mathcal{T}$ = algebraic d-CY triang. cat, $T \in \mathcal{T}$: d-cluster tilting

(1) [Keller–Reiten, 2008] $d=2$, Assume $\mathrm{End}_{\mathcal{T}}(T) = kQ$

$$\Rightarrow \quad \mathcal{T} \simeq \mathcal{C}_2(kQ)$$

(2) [Keller–Murfet–Van den Bergh, 2011] $d=3$, Assume $\mathrm{End}_{\mathcal{T}}(T) = k$

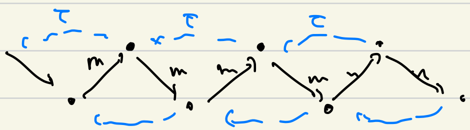
Put $m = \dim \mathrm{Hom}_{\mathcal{T}}(T, T[-1])$, $Q = \bullet \overset{\rightarrow}{\underset{\rightarrow}{\vdots}} \bullet$ (m arrows)

$$\Rightarrow \quad \mathcal{T} \simeq \mathcal{C}_3^{(1/2)}(kQ) = \frac{D^b(\mathrm{mod}\, kQ)}{F}$$

$\mathbb{Z}/2\mathbb{Z}$-quotient of $\mathcal{C}_3(kQ)$

$F^2 = \mathrm{Serre} \circ [-3]$

$F = \tau^{1/2} \circ [-1]$

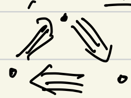


$G = \langle \frac{1}{3}(1,1,1) \rangle$

Cor (KR) Let $R = k[[x,y,z]]^{(3)} = S^G$

$$\Rightarrow \quad \underline{\mathrm{CM}}\, R \simeq \mathcal{C}_2(\bullet \rightrightarrows\!\!\!\!\rightarrow \bullet)$$

(**proof** $S \in \underline{\mathrm{CM}}\, R$: 2-CT ∈ 2-CY, $\mathrm{End}_R(S)$ can be computed from McKay quiver of G :



□)

Thm 1 $\mathcal{T}$: alg. d-CY triang. cat with $T \in \mathcal{T}$: d-CT

Suppose $\mathrm{End}_{\mathcal{T}}(T \oplus T[-1] \oplus \cdots \oplus T[-(d-2)]) = kQ$ for Q : non-Dynkin

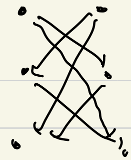
$$\Rightarrow \quad \mathcal{T} \simeq \mathcal{C}_d^{(1/(d-1))}(kQ) = D^b(\mathrm{mod}\, kQ)/F$$

$F = \tau^{1/(d-1)} \circ [-1]$

$(F^{d-1} = \mathrm{Serre} \circ [-d])$

Rem. This is a common generalization

Cor $R = k[[x_0, x_1, x_2]] * k[[y_0, y_1, y_2]]$: 5-dim'l
Segre

$\Rightarrow$ $\underline{\mathrm{CM}}\, R \simeq \mathcal{C}_4^{(1/3)}(kQ)$, Q :  $\tilde{A}_5$-quiver with triple arrows

§ Beyond hereditary case (joint with Osamu Iyama)

We need more structures than just triang. cats = enhancements

differential graded

$\boxed{\text{enh. of } \underline{\mathrm{CM}}\, R}$

Recall : $\underline{\mathrm{CM}}\, R \simeq D^b(\mathrm{mod}\, R)/\mathrm{per}\, R$

dg quotient gives an enh. $\mathcal{C}^{-,b}(\mathrm{proj}\, R)_{dg} / C^b(\mathrm{proj}\, R)_{dg} =: \mathcal{C}$
(or $\mathcal{C} = \mathcal{C}_{ac}(\mathrm{proj}\, R)_{dg}$)

Thm 2 Let R: comm. Gorenstein iso. sing of $\dim = d$

Matlis dual $\underline{D}\mathcal{C} \simeq \mathcal{C}[d-1]$ in $D(\mathcal{C}^{op} \otimes_R \mathcal{C})$

$D\mathcal{C}(M,N) \underset{\text{qis}}{\simeq} \mathcal{C}(N,M)[d-1]$ functorially ($\forall M, N \in \underline{\mathrm{CM}}\, R$)

$\boxed{\text{enh. of } \mathcal{C}_d(A)}$ $\mathcal{C}_d(A) \hookrightarrow D^b(\mathrm{mod}\, A)/\mathrm{Serre} \circ [-d]$

$\mathcal{C}_d(A)_{dg}$ is the dg orbit category of $D^b(A)/\mathrm{Serre} \circ [-d]$

✲ $\mathcal{C}_d(A)_{dg}$ has a natural $\mathbb{Z}$-grading ($\mathbb{Z}^2$-graded $\mathcal{A}_j^i$ + differential $d: \mathcal{A}_j^i \to \mathcal{A}_j^{i+1}$)

Thm 3 $\mathcal{B}$: $\mathbb{Z}$-graded dg category

Assume $\begin{cases} \bullet\ D\mathcal{B} \simeq \mathcal{B}(-1)[d-1] \text{ in } D^{\mathbb{Z}}(\mathcal{B}^e) \\ \bullet\ \mathrm{per}^{\mathbb{Z}}\mathcal{B} \simeq \mathrm{thick}\{\mathcal{B}(-,B) \mid B \in \mathcal{B}\} \end{cases}$ without degree shifts

If $\mathcal{B}_0 \overset{\sim}{\leftrightarrow} A$ derived Morita equiv.,

then $\mathcal{B} \simeq \mathcal{C}_d(A)_{dg}$

$\rightsquigarrow$ Consequently, $\operatorname{per} \mathcal{B} \simeq \mathcal{C}_d(A)$

Combining $\rightsquigarrow$,

$\boxed{\begin{array}{l} \underline{\text{Thm 4}} \quad R = \bigoplus_{i \geq 0} R_i : \text{ graded, comm. Gor. iso. sing, dim} = d \\ \qquad\qquad \text{with Gorenstein parameter } a \neq 0 \\ \text{Suppose } \quad \underline{\mathrm{CM}}^{\mathbb{Z}} R \simeq D^b(\operatorname{mod} A) \quad \text{for } \exists A \\ \text{Then} \quad \begin{array}{ccc} \underline{\mathrm{CM}}^{\mathbb{Z}} R & \simeq & D^b(\operatorname{mod} A) \\ \downarrow & & \downarrow \\ \underline{\mathrm{CM}}^{\mathbb{Z}/a\mathbb{Z}} R & \simeq & \mathcal{C}_{d-1}(A) \\ \downarrow & & \downarrow \\ \underline{\mathrm{CM}}\, R & \simeq & \mathcal{C}_{d-1}^{(1/a)}(A) \end{array} \end{array}}$

<u>Example</u> $S = k[x_1, \cdots, x_d]$, $n \mid d$, $R = S^{(n)}$. $GP = \frac{d}{n} =: a$

($\Rightarrow$ R: Gorenstein)

<u>Thm</u> (1) $T := S \oplus \Omega S(1) \oplus \cdots \oplus \Omega^{a-1} S(a-1) \in \underline{\mathrm{CM}}^{\mathbb{Z}} R$

is tilting

($\Longleftrightarrow$ the summands of T form a full strong. exc. collection)

$\Rightarrow$ gl-dim $= d-a-1$

(2) $A := \underline{\operatorname{End}}_R^{\mathbb{Z}}(T)$ is $(d-a-1)$-representation infinite

($\Longleftrightarrow$ the exc. coll is geometric)

$$\rightsquigarrow \quad \begin{array}{ccc} \underline{\mathrm{CM}}^{\mathbb{Z}} R & \simeq & D^b(\operatorname{mod} A) \\ \downarrow & & \downarrow \\ \underline{\mathrm{CM}}^{\mathbb{Z}/a\mathbb{Z}} R & \simeq & \mathcal{C}_{d-1}(A) \\ \downarrow & & \downarrow \\ \underline{\mathrm{CM}}\, R & \simeq & \mathcal{C}_{d-1}^{(1/a)}(A) \end{array} \qquad \cong \text{ "} D^b(\operatorname{coh}_{nc} \mathbb{P}^{d-a-1}) \text{"}$$

<u>Ex</u> $R = k[x_1, \cdots x_6]^{(2)}$ $\left\{ \begin{array}{l} d = 6 \\ n = 2 \end{array} \right.$ $a = 3$

A: 2-rep. infinite (gl. dim $= 2$)

not. hered!

Rem We do not need "R : iso. sing" in Thm 2:

$\boxed{\begin{array}{l} \textbf{Thm 2'} \quad R : \text{comm. Gor. ring with } \dim R < \infty \\ \qquad \mathcal{C} : \text{enh. of } \underline{\mathrm{CM}}\, R \\ \Rightarrow \quad \mathcal{C}^* \simeq \mathcal{C}[-1] \quad \text{in } D(\mathcal{C}^{op} \otimes_R \mathcal{C}), \\ \qquad \text{where } (-)^* : \mathrm{RHom}_R(-, R) \end{array}}$

When R : iso. sing, then each cohomology of $\mathcal{C}$ is f.l.,

so $\quad (-)^* = D \circ [-d] \quad$ for $\mathcal{C}$.