The category of 2D rational CFT's

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Main goal of this talk is to show the evolution of the mathematical language of 2D rational CFT's as our understanding of the subject deepens.

$\langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle \rightsquigarrow$ operator product expansion (OPE) $\rightsquigarrow$ operator algebras $\rightsquigarrow$ chiral algebras + chiral vertex operators $\rightsquigarrow$ vertex operator algebras (VOA) + intertwining operators $\rightsquigarrow$ modular tensor categories (MTC) $\rightsquigarrow$ algebras in MTC $\rightsquigarrow$ fusion categories + module categories $\rightsquigarrow$ centers and internal homs $\rightsquigarrow$ enriched fusion categories $\rightsquigarrow$ the category of 2D rational CFT's $\hookrightarrow \bullet/\Sigma_*^3\mathbb{C} \rightsquigarrow$ the category of quantum liquids $\mathcal{QL}^n \simeq \bullet/\Sigma_*^{n+1}\mathbb{C}$.

History of CFT's before 1984

CFT was originated from the study of critical phenomena and strong interaction. The systematic study of CFTs was initiated in late 1960s, focusing mostly on formal properties of these theories. Kadanoff (1969), Wilson (1969), Mack and Salam (1969), Polyakov (1970), Ferrara-Grillo-Gatto (1971-1975), Migdal (1971), Parisi (1972), Polyakov (1974), Mack (1977c), and Dobrev-Mack-Petkova-Petrova (1977)...

1. Operator product expansion: $\phi_1(x)\phi_j(y) = \sum_k C_{ij}^k(x-y)\phi_k(y)$. Polyakov (called correlation joining), later and independently by Kadanoff (1969) and Wilson (1969).
2. Conformal bootstrap: Polyakov (1970), Ferrara et al. (1973b) and Polyakov (1974).

$$\begin{aligned} \langle \phi_1(x_1)\phi_2(x_2)\rangle &\propto \begin{cases} x_{12}^{-2\Delta_1} & \text{if } \Delta_1 = \Delta_2; \\ 0 & \text{if } \Delta_1 \neq \Delta_2. \end{cases} \\ \langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle &\propto (x_{12}x_{23}x_{13})^{-D}. \end{aligned}$$

A.M. Polyakov, Conformal symmetry of critical fluctuations; ZhETF Pis. Red. 12, 538-541 (1970); Engl. transl.: JETP Lett. 12, 381-383 (1970).

We show in the present paper that the correlation functions at the transition point are invariant against transformation of a conformal group that includes a change of scale as a particular case. This circumstance makes it possible to calculate in explicit form any three-point correlators and greatly limit the possible form of multipoint correlators.

$$G_{III} = \mathrm{const}\; r_{12}^{(\Delta_c - \Delta_a - \Delta_b)}\, r_{13}^{(\Delta_b - \Delta_c - \Delta_a)}\, r_{23}^{(\Delta_a - \Delta_b - \Delta_c)}. \tag{10}$$

Formula (10) is confirmed in the flat Ising model, where an expression for the correlator <εσσ> is known, where ε is the energy density and σ is the magnetic moment [6, 7]. For a four-point function, analogous arguments yield the result

$$G_{IV} = r_{12}^{(\Delta_b + \Delta_d}\, r_{24}^{\Delta_a + \Delta_c)}\, r_{12}^{(-\Delta_a - \Delta_b}\, r_{23}^{-\Delta_b - \Delta_c)}\, r_{34}^{(-\Delta_c - \Delta_d}\, r_{41}^{-\Delta_d - \Delta_a)} \times \\ \times F\left(\frac{r_{13} r_{24}}{r_{12} r_{34}}, \frac{r_{14} r_{23}}{r_{12} r_{34}}\right), \tag{11}$$

3 Structure Constants and Operator Algebra:

Non-Hamiltonian approach to conformal quantum field theory

A. M. Polyakov

L. D. Landau Theoretical Physics Institute, USSR Academy of Sciences

(Submitted July 9, 1973)
Zh. Eksp. Teor. Fiz. **66**, 23–42 (January 1974)

The completeness requirement for the set of operators appearing in field theory at short distances is formulated, and replaces the S-matrix unitarity condition in the usual theory. Explicit expressions are obtained for the contribution of an intermediate state with given symmetry in the Wightman function. Together with the "locality" condition, the completeness condition leads to a system of algebraic equations for the anomalous dimensions and coupling constants; these equations can be regarded as sum rules for these quantities. The approximate solutions found for these equations in a space of $4-\epsilon$ dimensions give results equivalent to those of the Hamiltonian approach.

$$\varphi(x)\varphi(0)=\sum_{jn_s} C^{(jn)}_{\alpha_1\ldots\alpha_j|\mu_1\ldots\mu_s}(x)\,\partial_{\mu_1}\ldots\partial_{\mu_s} O^{(jn)}_{\alpha_1\ldots\alpha_j}(0). \tag{2.1}$$

Here, C is a c-number function of x, whose form is fixed, to within a few constants, by the conformal symmetry.

other values of x. The dynamical equations for the C-function appear as the requirement that the quantity

$$\langle\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\rangle$$

possess crossing symmetry after substitution in it of (2.2) for the different pairs of operators $\varphi(x_i)$.

We remark that knowledge of the function C is sufficient for the determination of the four-point, and even the n-point, Wightman functions, since successive use of the relation (2.2) reduces them to three-point functions, the explicit form of which is known[2]. The character of

4. The appearance of the Virasoro algebra in conformally invariant two-dimensional field theory was pointed out by F. Mansouri and Y. Nambu, Phys. Lett. 39B, 375 (1972), and by S. Ferrara, A. F. Gatto, and R. Grillo, Nuovo Cimento 12A, 959 (1972).

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5. The representation theory of conformal symmetries:

- I. M. Gel'fand and D. B. Fuks, "Cohomologies of the Lie algebra of the vector fields on the circle," Funkts. Anal. Prilozhen.,2, No. 4, 92–93 (1968). M. A. Virasoro, Phys. Rev. D I, 2933 (1970)
- B. L. Feigin and D. B. Fuks, "Invariant differential operators on the line", Funkts. Anal. Prilozhen.,13, No. 4, 91–92 (1979).
- V. G. Kac, "Contravariant form for infinite-dimensional Lie algebras and superalgebras," in: W. Beiglböck, A. Böhm, and E. Takasugi (eds.), Group Theoretical Methods in Physics, Lecture Notes in Physics, Vol. 94, Springer-Verlag, Berlin-New York (1979).
- Feigin, B.L., Fuks, D.B. Invariant skew-symmetric differential operators on the line and Verma modules over the Virasoro algebra. Funct Anal Its Appl 16, 114-126 (1982)

CFT's: 1984-1989 (physics)

Belavin, A A, Alexander M. Polyakov, and A. B. Zamolodchikov (1984), "Infinite conformal symmetry in two-dimensional quantum field theory," Nucl. Phys. B241, 333-380.

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2. Among all the fields forming the operator algebra, there are some special kind called *primary fields*.

$$\phi_n(z, \bar{z}) \to \frac{d\xi}{dz}^{\Delta_n} \frac{d\bar{\xi}}{d\bar{z}}^{\bar{\Delta}_n} \phi(\xi, \bar{\xi}).$$

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3. A complete set of the fields consists of conformal families (or conformal descendants), and splits into representations of two pieces of Virasoro algebras: $\oplus_i V_i \otimes_{\mathbb{C}} \overline{V}_i$.

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5 Notion of "conformal block".

$$A^{lk}_{nm}(p|x,\bar{x}) = \mathfrak{F}^{lk}_{nm}(p|x)\bar{\mathfrak{F}}^{lk}_{nm}(p|\bar{x}), \quad (4.15)$$

where, for instance, the function $\mathfrak{F}$ is given by the power series

$$\mathfrak{F}^{lk}_{nm}(p|x) = x^{\Delta_p - \Delta_n - \Delta_m} \sum_{\{k\}} \beta^{p\{k\}}_{nm} x^{\Sigma k_i} \frac{\langle k|\phi_l(1,1) L_{-k_1} \dots L_{-k_N}|p\rangle}{\langle k|\phi_l(1,1)|p\rangle} \quad (4.16)$$

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7 Minimal models from the representation theory of the Virasoro algebra.

From 1984 to 1989, the development of rational CFT's is under a phase transition. Many physicists had made fundamental contributions to CFT. Knizhnik, Zamolodchikov, Friedan, Qiu, Shenker, Witten, Cardy, Ishibashi, Matsuo, Ooguri, Vafa, Alvarez-Gaume, Verlinde, ..., Moore, Seiberg

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Instead of giving the details of the phase transition, we jump to the end of this transition period. This period was culminated in Moore and Seiberg's work: Commun. Math. Phys. 123, 177-254 (1989).

Classical and Quantum Conformal Field Theory

Gregory Moore and Nathan Seiberg*

Institute for Advanced Study, Princeton, NJ 08540, USA

Main results:

- Introduced the notion of a "chiral vertex operator" (i.e. intertwining operator):

dependence on z. That is, given three representations i, j, k a chiral vertex operator of type $\begin{pmatrix} i \\ jk \end{pmatrix}$, can be thought of as a linear transformation

$$\begin{pmatrix} i \\ jk \end{pmatrix}_z : (\mathscr{H}_i)^\vee \otimes \mathscr{H}_j \otimes \mathscr{H}_k \to C\,. \tag{2.6}$$

- Summarized the data of a RCFT as a modular tensor category:

Data:

1. A finite index set I and a one to one map of I to itself written $i \mapsto i^\vee$.
2. Vector spaces: $V_{jk}^i i, j, k \in I$, with $\dim V_{jk}^i = N_{jk}^i < \infty$.
3. Isomorphisms:

$$\Theta_{jk}^i(\pm): V_{jk}^i \cong V_{ji^\vee}^{k^\vee},$$

$$\Omega_{jk}^i(\pm): V_{jk}^i \cong V_{kj}^i,$$

$$F\begin{bmatrix} j_1 & j_2 \\ i_1 & k_2 \end{bmatrix}: \bigoplus_r V_{j_1 r}^{i_1} \otimes V_{j_2 k_2}^r \cong \bigoplus_s V_{s k_2}^{i_1} \otimes V_{j_1 j_2}^s, \tag{4.17}$$

$$S(j): \bigoplus_i V_{ji}^i \cong \bigoplus_i V_{ji}^i,$$

$$T: \bigoplus_i V_{ji}^i \cong \bigoplus_i V_{ji}^i.$$

Conditions:

1. $(i^\vee)^\vee = i$.
2. $V_{0j}^i \cong \delta_{ij} C$, $V_{ij}^0 \cong \delta_{ij^\vee} C$, $V_{jk}^i \cong V_{ji^\vee}^{k^\vee}$, $(V_{jk}^i)^\vee \cong V_{j^\vee k^\vee}^{i^\vee}$.
3. $\Omega^2(+) = \Omega_{jk}^i(+)\Omega_{kj}^i(+)$ is multiplication by a phase. Similarly, the action of T on V_{ji}^i is a diagonal matrix of phases independent of the external index j.
4. The identities:

$$F(\Omega(\varepsilon) \otimes 1)F = (1 \otimes \Omega(\varepsilon))F(1 \otimes \Omega(\varepsilon)), \tag{4.18a}$$

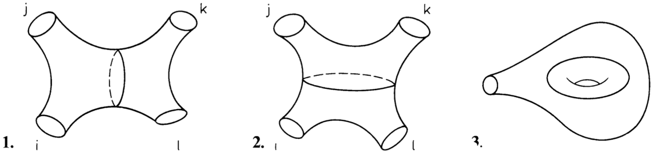
$$F_{23}F_{12}F_{23} = P_{23}F_{13}F_{12}, \tag{4.18b}$$

$$S^2(j) = \bigoplus_i \Theta_{ji}^i(-), \tag{4.18c}$$

$$S(j)TS(j) = T^{-1}S(j)T^{-1}, \tag{4.18d}$$

$$(S \otimes 1)(F(1 \otimes \Theta(-)\Theta(+))F^{-1})(S^{-1} \otimes 1) = FPF^{-1}(1 \otimes \Omega(-)). \tag{4.18e}$$

- “Proved” Verlinder formula by assuming modular tensor category:
- “Proved” the completeness of sewing constraints in genus 0 and 1.



1.

2.

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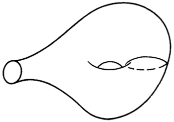


4.

Fig. 1. One sewing of the four-point function on the sphere

Fig. 2. Another sewing of the four-point function on the sphere

Fig. 3. One sewing of the one-point function on the torus

Fig. 4. Another sewing of the one-point function on the torus

- Categorical language was used in Appendix, including rigid abelian tensor categories, internal homs, etc. They did not use internal homs.

Summary and timeline:

1. Gel'fand-Fuks-Virasoro algebra: Gel'fand-Fuks:1968, Virasoro:1970.
2. OPE: Polyakov (called correlation joining), Kadanoff (1969) and Wilson (1969).
3. Conformal bootstrap: Polyakov (1970), Ferrara et al. (1973b), Polyakov (1974).
4. Appearance of Virasoro algebra in CFT's: Mansouri-Nambu:1972, Ferrara-Gatto-Grillo:1972.
5. Operator algebra: Polyakov (1974)
6. Representation theory of Virasoro algebra: Feigin-Fuks:1979, Kac:1979, Feigin-Fuks:1982
7. Chiral algebra and minimal models: Belavin-Polyakov-Zamolodchikov:1984, Friedan-Qiu-Shenker:1984
8. Operator and algebraic-geometric formulations of CFT's: Polyakov, Ishibashi, Matsuo, Ooguri, Vafa, Segal, Alvarez-Gaume, Moore, Gomez, Atiyha, Friedan-Shenker, Beilinson, Schechtman, ...1984-1989
9. Chiral vertex operators and modular tensor categories: Moore-Seiberg: 1989

CFT's: 1984-2020 (mathematics)

Four different approaches toward 2D rational CFT's:

1. Conformal nets: Longo:1994, Longo-Rehren:1995, Rehren:2000, Ocneanu, Böckenhauer-Evans-Kawahigashi:98-01, Xu, Müger, Longo-Rehren:2004,2009.
2. FRS-formalism based on 3D TQFT's: Felder-Fröhlich-Fuchs-Schweigert:2002, Fuchs-Runkel-Schweigert:2002-2007
3. Chiral algebras and chiral homology: Knizhnik, Friedan-Shenker, Beilinson-Schechtman:88, Tsuchiya-Kanie:88, Tsuchiya-Ueno-Yamada:89, Beilinson-Drinfeld:90's.
4. Vertex operator algebras (VOA):1984-2008: Frenkel-Lepowsky-Meurman:1984, Borcherds:1986, Frenkel-Lepowsky-Meurman:1988, Huang:1990, Zhu:1990, Frenkel-Zhu:92, Frenkel-Huang-Leopwsky:93, Huang-Leopwsky:90-94, Li:96, Dong-Li-Mason:97-98, Huang:05-08, K.-Huang:04-06, K.:07-08

Birth of VOA in 1980s

1. In 1978, J. McKay observed that 196884 = 196883 + 1, where

$$j(q) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots$$

and 196883 is the dimension of the smallest irreducible representation of the Monster group $\mathbf{M}$.

2. $21493760 = 1 + 196883 + 21296876$;

3. In 1979, **Monstrous Moonshine Conjecture**, J.H. Conway, S.P. Norton, "Monstrous moonshine", Bull. London Math. Soc. , 11 (1979) 308–339:

   There is an infinite-dimensional graded vector space $V^\natural = V_{-1} \oplus V_1 \oplus V_2 \oplus \cdots$, with the following properties: each V_k carries a finite-dimensional representation of $\mathbf{M}$; write χ_k for its character. For each $g \in \mathbf{M}$, define the Thompson–McKay series $T_g(z) = \sum_{k=1}^{\infty} \chi_k(g) q^k$ for $q = \exp(2\pi i z)$. The T_g is a generator ("Hauptmodul") of the field of modular functions for some genus-0 group $G_g \leq SL_2(\mathbb{R})$.

4. In 1982, R.L. Griess, "The friendly giant" Invent. Math. , 69 (1982) pp. 1–102

5 The construction of moonshine module $V^\natural = V_{-1} \oplus V_1 \oplus V_2 \oplus \cdots$:

- I. B. Frenkel, J. Lepowsky, and A. Meurman. A moonshine module for the monster. Proceedings of a Conference November 10-17, 1983, pages 231–273, Springer, New York, 1985. Publications of the Mathematical Sciences Research Institute
- I. B. Frenkel, J. Lepowsky, and A. Meurman, A natural representation of the Fischer-Griess monster with the modular function J as character, Proc. Natl. Acad. Sci. USA 81 (1984) 3256-3260..
- Relation to 2D CFT was soon noticed by audiences attending their talks.

6 Introduction of the notion of a vertex algebra based on FLM:1984: Borcherds, Vertex algebras, Kac-Moody algebras and the monster, Proc.Nat. Acad. Sci. U.S.A. 83 (1986), 3068-3071;

We will list some identities satisfied by the operators u_n and show how to construct Lie algebras from them. u, v, and w denote elements of V, and 1 is the unit of V.

For any even lattice R the operators u_n on V satisfy the following relations.

(i) $u_n(w) = 0$ for n sufficiently large (depending on u, w). This ensures convergence of the following formulae.

(ii) $1_n(w) = 0$ if $n \neq -1$, w if $n = -1$.

(iii) $u_n(1) = D^{(-n-1)}(u)$.

(iv) $u_n(v) = \sum_{i\geq 0}(-1)^{i+n+1}D^{(i)}(v_{n+i}(u))$.

(v) $(u_m(v))_n(w) = \sum_{i\geq 0}(-1)^i\binom{m}{i}(u_{m-i}(v_{n+i}(w)) - (-1)^m v_{m+n-i}(u_i(w)))$.

(The binomial coefficient $\binom{m}{i}$ is equal to $m(m-1)\ldots(m-i+1)/i!$ if $i \geq 0$ and 0 otherwise.)

R.E. Borcherds: (1) "Generalized Kac–Moody algebras" J. Algebra , 115 (1988) pp. 501-512; (2). "Monstrous moonshine and monstrous Lie superalgebras" Invent. Math., 109 (1992) pp. 405-444.

$\rightsquigarrow$ Fields Medal, 1998.

7 Introduction of the notion of a vertex operator algebra (VOA):

Frenkel-Lepowsky-Meurman, "Vertex operator algebras and the Monster", Pure and Appl. Math., 134, Academic Press, New York, 1988..

(Jacobi identity) *For every* $\psi, \varphi \in \mathcal{F}$,

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)\mathcal{V}(\psi, z_1)\mathcal{V}(\varphi, z_2) - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right)\mathcal{V}(\varphi, z_2)\mathcal{V}(\psi, z_1)$$
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\mathcal{V}(\mathcal{V}(\psi, z_0)\varphi, z_2)$$

Definition of a VOA: Frenkel-Lepowsky-Meurman:1988, Dong's Lemma, Li:1996, Kac:1997, Li-Lepowsky:2004

1. $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$, $\mathbf{1} \in V_{(0)}, \omega \in V_{(2)}$, $\dim V_{(n)} < \infty$ and $V_{(n)} = 0, n << 0$.
2. for $u, v \in V$, $u_n \in \mathrm{End}(V)$ s.t. $u_n v = 0$ for $n >> 0$.

$$Y : V \otimes V \to V((x)),$$
$$u \otimes v \mapsto Y(u,x)v = \sum_{n\in\mathbb{Z}} u_n v x^{-n-1}$$

3. $Y(\mathbf{1}, x) = \mathrm{id}_V$ and $\lim_{x\to 0} Y(u,x)\mathbf{1} = u$.
4. $\exists N \in \mathbb{N}$ for each pair of u, v such that $(x_1 - x_2)^N [Y(u,x_1), Y(v,x_2)] = 0$.
5. $Y(\omega, x) = \sum_{n\in\mathbb{Z}} L(n) x^{-n-2}$ where $L(n), n \in \mathbb{Z}$ are generators of Virasoro algebra.
   - $L(0)$ is the grading operator.
   - $[L(-1), Y(u,x)] = Y(L(-1)u, x) = \frac{d}{dx} Y(u,x)$.

Mathematics-Physics Dictionary:

1. $Y(\phi, z) = \phi(z)$;
2. Associativity in VOA = OPE: for $|z_1| > |z_2| > |z_1 - z_2| > 0$,

$$Y(\phi, z_1)Y(\psi, z_2)v = Y(Y(\phi, z_1 - z_2)\psi, z_2)v.$$

$$\phi(z_1)\psi(z_2) \sim \frac{(\phi_k\psi)(z_2)}{(z_1 - z_2)^{k+1}} + \frac{(\phi_{k-1}\psi)(z_2)}{(z_1 - z_2)^k} + \cdots$$

3. Commutativity in VOA = Locality:

$$\langle v', Y(\phi, z_1)Y(\psi, z_2)v\rangle \sim \langle v', Y(\psi, z_2)Y(\phi, z_1)v\rangle$$

$$\phi(z_1)\psi(z_2) \sim \psi(z_2)\phi(z_1).$$

Algebraic theory of VOA's and representations

1. Chiral vertex operator = Intertwining Operator Frenkel-Huang-Lepowsky:1989,1993:

Definition

Let V be a VOA. (W_i, Y_i) for $i = 1, 2, 3$ are V-modules. An intertwining operator of type $(W_1, W_2 | W_3)$ is a linear map $W_1 \otimes_{\mathbb{C}} W_2 \to W_3\{x\}$, i.e.

$$\begin{aligned} W_1 &\to (\mathrm{hom}(W_2, W_3))\{x\} \\ w &\mapsto \mathcal{Y}(w, x) = \sum_{n \in \mathbb{C}} w_n x^{-n-1} \end{aligned}$$

such that $(w_{(1)})_n w_{(2)} = 0$ for n sufficiently large and

$$\begin{aligned} x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) Y_3(v, x_1) \mathcal{Y}(w_{(1)}, x_2) - x_0^{-1} \delta\left(\frac{x_2 - x_1}{-x_0}\right) \mathcal{Y}(w_{(1)}, x_2) Y_2(v, x_1) w_{(2)} \\ = x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) \mathcal{Y}(Y_1(v, x_0) w_{(1)}, x_2) w_{(2)}. \end{aligned}$$

2. Zhu's thesis (1990): Y.-C. Zhu, Modular invariance of vertex operator algebras, J. Amer. Math. Soc. 9 (1996) 237-302

(1) Zhu's algebra: The quotient space $A(V) = V/O(V)$, where $O(V)$ is a subspace of V spanned by $\mathrm{Res}_z(Y(a,z)z^{-2}(z+1)^{\mathrm{deg} a}b)$, has an associative algebraic structure defined by $a * b := \mathrm{Res}_z(Y(a,z)z^{-1}(z+1)^{\mathrm{deg} a}b)$.

(2) Proof of modularity of partition functions of modules over a rational VOA, i.e. $\chi_{M_i}(q) = \mathrm{Tr}_{M_i}(q^{L_0 - \frac{c}{24}})$ span a representation of $\mathrm{SL}(2,\mathbb{Z})$.

$$Tr|_{M_i} Y(e^{2\pi z_1 L_0} a_1, e^{2\pi z_1 L_0}) \cdots Y(e^{2\pi z_1 L_0} a_n, e^{2\pi z_n L_0}) q^{L_0 - \frac{c}{24}}.$$

No intertwining operators!

3. Zhu's thesis led to the classification of modules over various VOA's.

(1) I. B. Frenkel and Y. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, Duke Math. J. 66 (1992), 123–168.

(2) W. Wang, Rationality of Virasoro vertex operator algebras, International Mathematics Research Notices 7 (1993), 197–211.

(3) V. Kac and W. Wang, Vertex operator superalgebras and their representations, Contemp. Math. Amer. Math. Soc. 175 (1994), 161-191.

(4) W. Wang, Classification of irreducible modules of W3 algebra with $c = -2$, Comm. Math. Phys. 195 (1998), 113-128.

3. Tensor category theory of the modules over a rational VOA, Huang-Lepowsky:1991,1995

- Y.-Z. Huang, J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, I, II, Selecta Math. (N.S.) 1 (1995) 699-756, 757-786.
- Y.-Z. Huang, J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, III, J. Pure Appl. Alg. 100 (1995) 141-171.
- Y.-Z. Huang, A theory of tensor products for module categories for a vertex operator algebra, IV, J. Pure Appl. Alg. 100 (1995) 173-216.

$$(W_1 \boxtimes W_2)' := W_1 \boxdot_{P(z)} W_2 = \sum_{W \in \mathcal{W}_{P(z)}} W = \bigcup_{W \in \mathcal{W}_{P(z)}} W \subset (W_1 \otimes W_2)^*,$$

$$\{\text{Intertwining Operators} : W_1 \otimes_{\mathbb{C}} W_2 \to W_3\{x\}\} \simeq \mathrm{hom}_{\mathrm{Mod}_V}(W_1 \boxtimes W_2, W_3).$$

4. There are many important algebraic studies of VOA: Dong's Lemma, Li's local systems, generalized Zhu's algebras, regularity, twisted modules, etc.: Dong, Li, Mason, Abe, Buhl, Miyamoto, Adamović, Yamada, Tanabe, Nagatomo, Tsuchiya, ...

Since these works are not directly related to what I will talk about on 2D rational CFT's, we won't expand the discussion here.

5. Huang's proof of Verlinde formula and modular tensor category

- Y.-Z. Huang, Differential equations and intertwining operators, Commun. Contemp. Math. 7 (2005) 375-400.
- Y.-Z. Huang, Differential equations, duality and modular invariance, Commun. Contemp. Math. 7 (2005) 649-706.
- Huang, Y.-Z.: Vertex operator algebras and the verlinde conjecture. Commun. Contemp. Math. 10(1) (2008) 103-154.
- Y.-Z. Huang, Rigidity and modularity of vertex tensor categories, Commun. Contemp. Math. 10 (2008) 871-911

Theorem (Huang's Theorem)

Let V be a simple VOA satisfying the following conditions: (1) $V_{(n)} = 0$ for $n < 0$, $V_{(0)} = \mathbb{C}\mathbb{1}$ and V' is isomorphic to V as a V-module; (2) Every $\mathbb{N}$-gradable weak V-module is completely reducible; (3) V is C_2-cofinite. Then Mod_V is a modular tensor category.

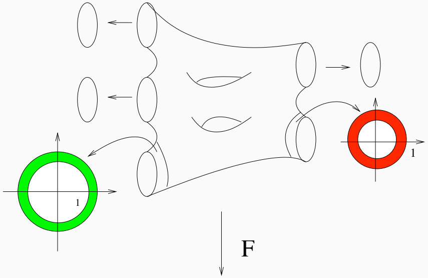
6. VOA extensions of a rational VOA as algebras in MTC: Kirillov-Ostrik:2001, Huang-Kirillov-Lepowsky:2003, Fuchs-Schweigert:2003

Theorem (Huang-Kirillov-Lepowsky:2003)

For a rational VOA V, a VOA extension A of V is equivalent to a commutative algebra A in Mod_V.

Geometric theory of VOA's and CFT's

1. Igor Frenkel had considered the geometric meaning of VOA and had initiated a program of studying CFT based on the mathematical theory of VOA before the appearance of Segal's definition of 2D CFT.
2. Segal's definition of a 2D CFT (also Kontsevich) $\mathcal{F}: \mathrm{Bord}^2 \to \mathrm{TV}$.



$$H_{\mathrm{cl}}^{\otimes 3} \longrightarrow H_{\mathrm{cl}}^{\otimes 2}$$

Two disadvantages of Kontsevich-Segal's definition:

- The quantum fields $\phi(x), \psi(x)$ in physics usually are associated to a point in the space-time. They do not live in Kontsevich-Segal's definition in a direct way. This suggest to replace Riemann surfaces with parametrized boundaries by Riemann surfaces with parametrized punctures.

  Vafa, Conformal theories and punctured surfaces, Phys. Lett. B 199, 2 (1987) 195-202.
- Replace Hilbert spaces by graded vector spaces ($\mathrm{Vect}_{\mathbb{C}}^{gr}$). Huang's Rutgers thesis:1990.

3. Huang's thesis (1990): A VOA is equivalent to an algebra over the complex-analytic version of 2-disk operad: Huang's Rutgers thesis:1990; Huang, Geometric interpretation of vertex operator algebras. Proceedings of the National Academic Society USA, 88 :9964–9968, 1991. Y.-Z. Huang, Two-dimensional conformal geometry and vertex operator algebras, Progress in Mathematics, Vol. 148, Birkhäuser, Boston, 1997.

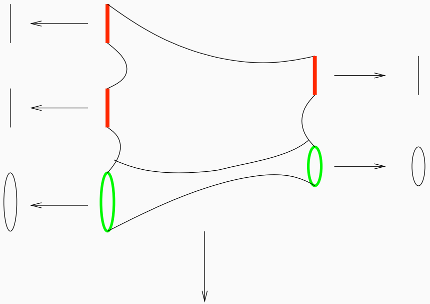
Theorem (Huang:1990,1991,1997)

A VOA is nothing but a complex-analytic E_2-algebra (2-sphere with punctures).

4. From 2-sphere with punctures to 2-sphere with holes via a completion procedure:

- Y.-Z. Huang, A functional-analytic theory of vertex (operator) algebras, I, Commun. Math. Phys. 204 (1999), 61-84.
- Y.-Z. Huang, A functional-analytic theory of vertex (operator) algebras, II, Commun. Math. Phys. 242 (2003), 425-444.

5. Mathematical definition of a boundary-bulk CFT (or open-closed CFT): Y.-Z. Huang, Riemann surfaces with boundaries and the theory of vertex operator algebras, Vertex operator algebras in mathematics and physics (Toronto, ON, 2000), 109-125, Fields Inst. Commun., 39, Amer. Math. Soc., Providence, RI, 2003.



$$H_{op}^{\otimes 2} \otimes H_{cl} \longrightarrow H_{op} \otimes H_{cl}$$

6. Boundary CFT and open-string VOA: Huang-K.:math/0308248.

- $Y_{op}(\phi, r) = \phi(r) = \sum_{n\in\mathbb{R}} \phi_n r^{-n-1}$.
- $$\phi(r_1)\psi(r_2) \sim \frac{(\phi_k\psi)(r_2)}{(r_1-r_2)^{k+1}} + \frac{(\phi_{k-1}\psi)(r_2)}{(r_1-r_2)^{k}} + \cdots .$$
  where $k \in \mathbb{R}$ is not necessarily an integer!
- No commutativity!: $\phi(r_1)\psi(r_2) \nsim \psi(r_2)\phi(r_1)$.

Theorem (Huang-K.:math/0308248)

An open-string VOA is equivalent to a "real-analytic E_1-algebra" (2-disk with only punctures on the boundary).

7. Introduction of full field algebra for bulk CFT (or closed CFT): Huang-K.:math/0511328; K.:math/0603065

- An example of full field algebra: $V_{cl} = V_L \otimes_{\mathbb{C}} \overline{V}_R$ where V_L, V_R are VOA's;
- Full field algebra over $V_L \otimes_{\mathbb{C}} \overline{V}_R$: $V_L \otimes_{\mathbb{C}} \overline{V}_R \hookrightarrow V_{cl}$;
- $Y_{cl}(u; z, \bar{z})v = \sum_{m,n} u_{m,n} z^{-m-1} \bar{z}^{-n-1}$;
- Associativity and commutativity hold.

Theorem (K.:math/0603065)

(1). A full field algebra is a "real-analytic E_2-algebra" (2-sphere with punctures).
(2). When both V_L and V_R are rational, a full field algebra over $V_L \otimes_{\mathbb{C}} \overline{V}_R$ is equivalent to a commutative algebra in $\mathrm{Mod}_{V_L} \boxtimes \overline{\mathrm{Mod}_{V_R}}$.

8. Open-closed field algebra: K.:math/0610293.

- A full field algebra (V_{cl}, Y_{cl}), an open-string VOA (V_{op}, Y_{op}) and $Y_{cl-op}(u; z, \bar{z}) = \sum_{m,n} u_{m,n} z^{-m-1} \bar{z}^{-n-1}$.
- Associativity I & II:

$$\langle v', Y_{cl-op}(u; z, \zeta) Y_{op}(v_1, r) v_2 \rangle = \langle v', Y_{op}(Y_{cl-op}(u; z-r, \zeta - r) v_1, r) v_2 \rangle.$$

$$\langle w', Y_{cl-op}(u; z_1, \zeta_1) Y_{cl-op}(v; z_2, \zeta_2) \rangle = \langle w', Y_{cl-op}(Y_{cl}(u_1; z_1 - z_2, \zeta_1 - \zeta_2) u_2; z_2, \zeta_2) v_2 \rangle.$$

- Commutiativity I & II:
- V-symmetric boundary condition: If $V = \langle T_{\mu\nu} \rangle$, then it is called conformal-symmetric boundary condition. An open-closed field algebra satisfies a V-symmetric boundary condition is called an open-closed field algebra over V.

Theorem (K.:math/0610293)

If V is rational, an open-closed field algebra over V is an algebra over Swiss-cheese partial operad (2-disk with punctures on the boundary and in the interior).

Unification of algebraic and geometric approach: after Huang's theorem on MTC

Combining the following development:

1. Huang's Theorem: If V is a rational VOA, then Mod_V is a modular tensor category.
2. Kirillov-Ostrik:2001, Huang-Kirillov-Lepowsky:2015: For a rational VOA V, a VOA extension A of V is equivalent to a commutative algebra A in Mod_V.
3. Open-string VOA, full field algebra, open-closed field algebra, boundary-states: Huang-K.:2004-2010, K.:2007-2008,

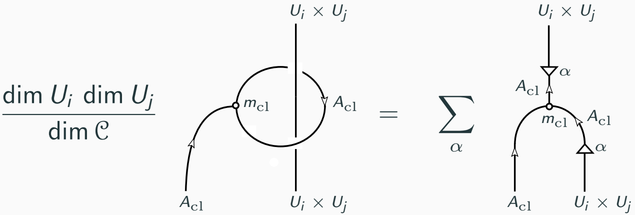
We obtain a categorical characterization/classification of open-closed CFT over a rational V (i.e. satisfying a V-symmetric boundary condition).

(preceded by Fuchs-Runkel-Schweigert:2002-2006: simple special Frobenius algebra A in Mod_V + state sum construction of all structure constants.)

Theorem K.:math/0612255,K.-Runkel:0807.3356: An open-closed CFT over a rational VOA V is a triple $(A_{\mathrm{op}}|A_{\mathrm{cl}}, \iota_{\mathrm{cl-op}})$, where

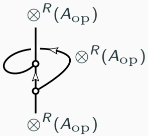
1. A_{cl}: a commutative symmetric Frobenius algebra in $\mathfrak{Z}_1(\mathrm{Mod}_V) = \mathrm{Mod}_V \boxtimes \overline{\mathrm{Mod}_V}$,
2. A_{op}: a symmetric Frobenius algebra in Mod_V,
3. $\iota_{\mathrm{cl-op}} : A_{\mathrm{cl}} \to Z(A_{\mathrm{op}}) \hookrightarrow \otimes^R(A_{\mathrm{op}})$: an algebra homomorphism.

satisfying (1) modular invariance condition:



(2). Cardy condition:

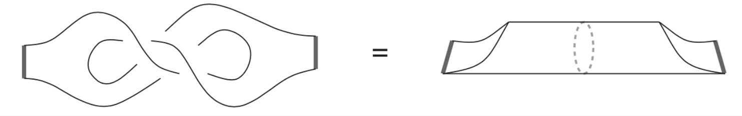
$$\iota_{\mathrm{cl-op}} \circ \iota^*_{\mathrm{cl-op}} \;=$$



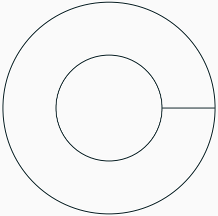
where $\iota^*_{\mathrm{cl-op}}$ is defined using the self-duality of Frobenius algebras.

Remark: further but not significant simplification was available, e.g. replacing the modular invariant condition by the Lagrangian condition: $(\dim A_{\mathrm{cl}})^2 = \dim \mathfrak{Z}_1(\mathrm{Mod}_V)$. K.-Runkel:2009

Cardy condition



A weaker veresion of Cardy condition:



In physics literature, the complete Cardy condition has never been written down explicitly. Only a weaker version is commonly used in literature. The completion version can be found in K.:math/0612255

Definition 3.4. *The open-closed field algebra over V given in (2.12) and equipped with nondegenerate bilinear forms $(\cdot,\cdot)_{op}$ and $(\cdot,\cdot)_{cl}$ is said to satisfy* **Cardy condition** *if the left hand sides of the following formula, $\forall z_1, z_2 \in \mathbb{H}$, $v_1, v_2 \in V_{op}$,*

$$Tr_{V_{op}} \left(Y_{op}(\mathcal{U}(q_{s_1})v_1, q_{s_1}) Y_{op}(\mathcal{U}(q_{s_2})e^{-2\pi i L(0)} v_2, q_{s_2}) q_\tau^{L(0)-c/24} \right)$$
$$= \left((T_1^L \otimes T_1^R)^* \iota^*_{cl-op}(z_1, \bar{z}_1)(T_2 v_1),\ q_{-\frac{1}{\tau}}^{-c/24} (T_3^L \otimes T_3^R)^* \iota^*_{cl-op}(z_2, \bar{z}_2)(T_4 v_2) \right)_{cl} \tag{3.30}$$

converge absolutely when $1 > |q_{s_1}| > |q_{s_2}| > |q_\tau| > 0$, and the right hand side of (3.30) converge absolutely for all $s_1, s_2 \in \mathbb{H}$ satisfying $Re\, s_1 = 0$, $Re\, s_2 = \frac{1}{2}$. Moreover, Eq. (3.30) holds when $1 > |q_{s_1}| > |q_{s_2}| > |q_\tau| > 0$.

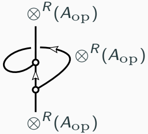
$\rightsquigarrow$ Cardy condition in terms of intertwining operators:

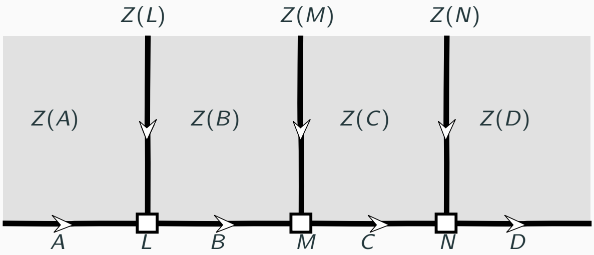
Theorem 3.10. *The Cardy condition can be rewritten as follows:*

$$\left(\theta_{W_{r_R(i)}} \circ \sigma_{123}(\mathcal{Y}_{\iota'_{cl-op}}) \circ (\varphi_{op} \otimes \mathrm{id}_{W'_{r_L(i)}})\right) \otimes \left(\Omega_0(\sigma_{132}(\mathcal{Y}_{\iota_{cl-op}})) \circ (\varphi_{op} \otimes \mathrm{id}_{W_{r_R(i)}})\right) = S^{-1}\left(Y_{op}^{f} \otimes \left(\Omega_{-1}(Y_{op}^{f}) \circ (\theta_{V_{op}} \otimes \mathrm{id}_{V_{op}})\right)\right). \tag{3.73}$$

$\rightsquigarrow$ Cardy condition in categorical language:

$$\iota_{\mathrm{cl-op}} \circ \iota^{*}_{\mathrm{cl-op}} \quad =$$





From boundary-bulk duality to the center functor (i.e. holography is functorial):

1. Bulk CFT is the center of a boundary CFT Fjelstad-Fuchs-Runkel-Schweigert:hep-th/0612306, Davydov:0908.1250;
2. A and B are Morita equivalent iff $Z(A) \simeq Z(B)$ K.-Runkel:0708.1897;
3. $\mathrm{BrPic}(A) \simeq \mathrm{Aut}(Z(A))$ Davydov-K.-Runkel:1004.4725;

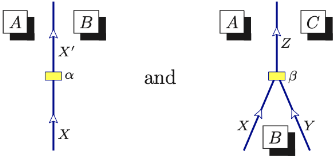
2D RCFT's (2009-now)

After 2009,

- Some people from VOA and CFT community shifted to the study of irrational CFT's (or logarithmic CFT's). Huang, Milas, Fuchs, Runkel, Schweigert, ... ??
- VOA community: W-algebras, conjectures on rationality, orbiford theory, higher genus theory, etc. Huang, Dong, Arakawa, ..., Gui ??
- Conformal net community: Bischoff, Longo, Kawahigashi, Rehren, Carpi, Weiner, Xu, ... ??
- The theory of defects: Fröhlich-Fuchs-Runkel-Schweigert:06, Davydov-K.-Runkel:2011,2013, Carqueville-Runkel:2012-2022, Bartels-Douglas-Henriques:2014-2019, K.-Zheng:2017-2021, K.-Yuan-Zheng:2021.
- We see a significant increase in the applications of 2D CFT in the study of topological phases: Wen, Wu, Nayak, Ludwig, Qi, Levin, Barkeshli, Wang, Jian, Cheng, Zhang, Ryu, Chen, You, Xu, Kong, Wan, Ji, Else, K., Zheng, Yin, Shao, Chang, ..., DMRG, tensor network, ... ??

Fröhlich-Fuchs-Runkel-Schweigert, Duality and defects in rational conformal field theory, Nucl. Phys. B 763 (2007) 354-430. hep-th/0607247 (based on the so-called FRS-formulation of 2D RCFT):

Defects can also be joined. The junction is labelled by an element of the relevant morphism space of bimodules. For example, when joining two A-B-defects X and X', or an A-B-defect X and a B-C-defect Y to an A-C-defect Z, according to



(2.24)

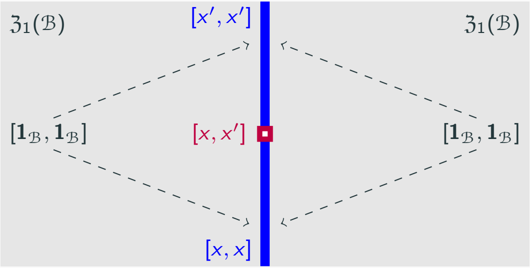
the junctions get labelled by morphisms $\alpha \in \mathrm{Hom}_{A|B}(X', X)$ and $\beta \in \mathrm{Hom}_{A|C}(Z, X \otimes_B Y)$, respectively.[6] Note also that a junction linking an A-A-defect X to the invisible defect A is

Partition function of a torus with two parallel defect line labeled by X and Y with opposite orientations and 2-cells are all labeled by $A \in \mathrm{Mod}_V$ can be determined by the formula:

$$Z(A)_{ij}^{X|Y} = \dim \hom_{A|A}(i \otimes^+ X \otimes^- j, Y).$$

The mathematical language of a RCFT was further simplified by using a powerful categorical language: internal homs. Davydov:0908.1250, Davydov-K.-Runkel:1307.5956

(1) $\mathcal{C}$ a fusion category, $\hom_{\mathcal{C}}(x,y) \in \mathrm{Vec}$, $\mathcal{C} = {}^{\mathrm{Vec}}\mathcal{C}$;
(2) Replacing $\hom_{\mathcal{C}}(x,y)$ by internal homs $[x,y] = y \otimes x^* \in \mathcal{C} \rightsquigarrow$, we obtain ${}^{\mathcal{C}}\mathcal{C}$.
(3) For a left $\mathcal{C}$-module category $\mathcal{M} = \mathrm{RMod}_A(\mathcal{C})$, replacing $\hom_{\mathcal{M}}(x,y)$ by internal homs $[x,y] = (x \otimes_A y^*)^*$. $\rightsquigarrow$, we obtain ${}^{\mathcal{C}}\mathcal{M}$.



Ignoring VOA, all the rest data (bulk + defects $x, x' \in \mathcal{B}$, $[x,x'] \in \mathfrak{Z}_1(\mathcal{B})$) can be reduced to $\mathcal{B}$ (or an enriched fusion category ${}^{\mathfrak{Z}_1(\mathcal{B})}\mathcal{B}$), which is called the topological skeleton of the RCFT.

It seems natural to generalize the MTC $\mathcal{B} = \mathrm{Mod}_V$ to any fusion category $\mathcal{S}$ (or ${}^{\mathfrak{Z}_1(\mathcal{S})}\mathcal{S}$).

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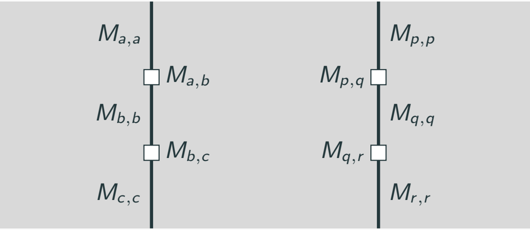


The topological skeleton of an anomalous/anomaly-free 2D quantum liquid (2D rational CFT's, 1D SPT/SET orders and symmetry-breaking orders) form an enriched fusion categories ${}^{\mathcal{P}}\mathcal{S}$, which is anomaly-free iff $\mathcal{P} \simeq \mathfrak{Z}_1(\mathcal{S})$. K.-Zheng:1705.01087,1905.04924,1912.01760, K.-Wen-Zheng:2108.08835

1. Objects: sectors of states or labels for topological defect lines (TDL): $a, b, c, \cdots \in \mathcal{S}$.
2. Morphisms: $\hom_{{}^{\mathcal{P}}\mathcal{S}}(a, b) = M_{a,b} = [a, b] \in \mathcal{P}$ denotes the space of non-local operators mapping a sector of states a to another sector b, or equivalently, the 0D domain wall between TDL's labeled by a and b.

Example: K.-Wen-Zheng:2108.08835

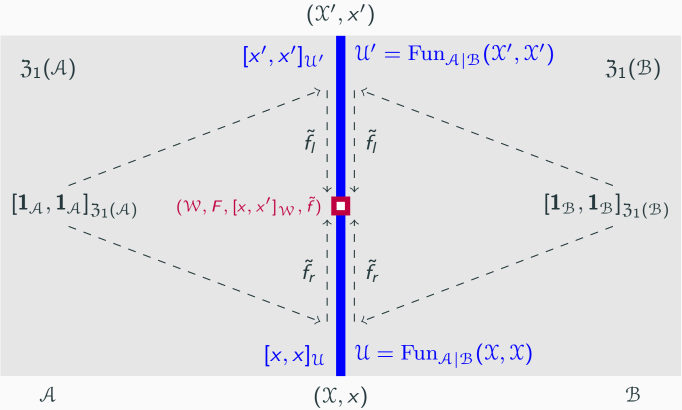
- 1+1D Ising chain:
  1. the symmetric phase: ${}^{\mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2))}\mathrm{Rep}(\mathbb{Z}_2)$, i.e. a trivial SPT order;
  2. the symmetry-broken phase: ${}^{\mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2))}\mathrm{Vec}_{\mathbb{Z}_2}$.
- 1+1D Kitaev chain:
  1. the trivial fermionic SPT order: ${}^{\mathfrak{Z}_1(\mathrm{sVec})}\mathrm{sVec}$;
  2. a non-trivial fermionic SPT order: ${}^{\mathfrak{Z}_1(\mathrm{sVec})}_{e\leftrightarrow m}\mathrm{sVec}$.

Theorem (Chen-Gu-Wen:1008.3745, Schuch-Peŕez-García-Cirac:1010.3732, K-Zheng:2011.02859,K-Wen-Zheng:2108.08835)

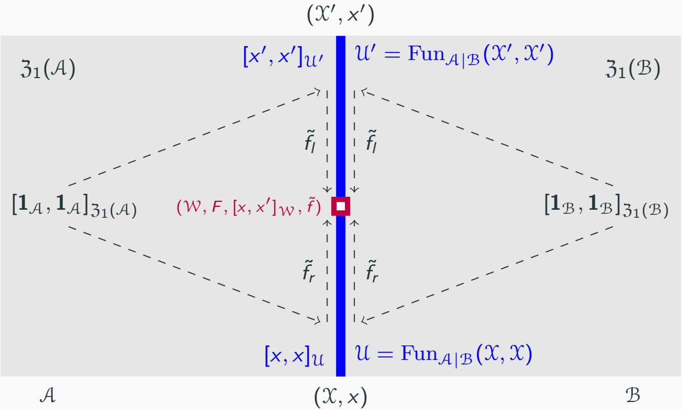
1+1D gapped quantum liquids with a bosonic finite onsite symmetry G are classified by a pair (H,ω), where H is a subgroup of G and $\omega \in H^2(H, U(1))$.

Theorem (K-Zheng:2011.02859,K-Wen-Zheng:2108.08835)

The topological skeleton associated to each phase is given by ${}^{\mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2))}(\mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2))_{A_{(H,\omega)}}$. The same classification and topological skeleton works for a fermionic finite onsite symmetry (G, z) but with different physical meanings.



$\rightsquigarrow$ The category $\mathcal{QL}^2_{\mathrm{sk}}$ of the topological skeletons of all 2D quantum liquids: objects are multi-fusion 1-categories $\mathcal{A}, \mathcal{B}$, 1-morphisms $(\mathcal{X}, x)$, ... K.-Yuan-Zheng:1912.13168



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$\rightsquigarrow$ $\mathcal{QL}^2_{\mathrm{sk}} \simeq \bullet/\Sigma^3\mathbb{C} = \bullet/3\mathrm{Vec}.$ K.-Zheng:2011.02859, 2201.05726

$\mathcal{QL}^2_{\mathrm{sk}} \simeq \bullet/\Sigma^3\mathbb{C} = \bullet/3\mathrm{Vec}$. First, recall that $3\mathrm{Vec}$ consists of separable 2-categories, functors, natural transformations and modifications.

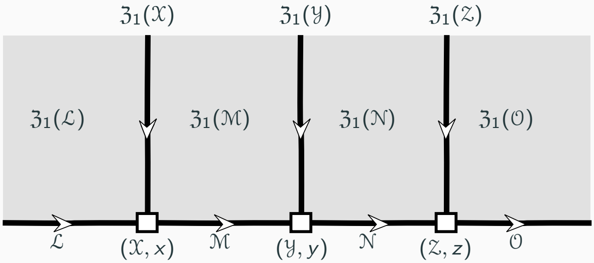
1. an object in $\bullet/3\mathrm{Vec}$ is a 1-morphism $\bullet \xrightarrow{a} A$, which defines an object $a \in A$, $\mathcal{A} = \mathrm{hom}_A(a, a)$ is a multi-fusion category. If A is indecomposable, then $A \simeq \Sigma\mathcal{A} \simeq \mathrm{RMod}_{\mathcal{A}}(2\mathrm{Vec})$.
2. a 1-morphism in $\bullet/3\mathrm{Vec}$ is a pair $(F, f) : a \to b$, where $F \in \mathrm{Fun}(A, B) \simeq \mathrm{BMod}_{\mathcal{A}|\mathcal{B}}(2\mathrm{Vec})$ and $f : Fa \to b$ determines an object $f^R \in F$.
3. ...

Generalize to all quantum liquids (topological orders, SPT/SET orders, symmetry-breaking orders, CFT-like gapless phases)

1. Theory of defects: Fröhlich-Fuchs-Runkel-Schweigert:06 $\rightsquigarrow$ Kitaev-K.:2011
2. The categories of topological orders: K.-Wen:2014, K.-Wen-Zheng:2015
3. Condensation completion: Carqueville-Runkel:2012, Douglas-Reutter:2018, Gaiotto-Johnson-Freyd:2019, Johnson-Freyd:2020, K.-Lan-Wen-Zhang-Zheng:2020
4. Multi-fusion n-categories: Douglas-Reutter:2018, Johnson-Freyd:2020, K.-Zheng:2020
5. Classification theory of SPT/SET orders: Barkeshli-Bonderson-Cheng-Wang:2014, Lan-K.-Wen:2017-2018, Lan-Wen:2018, K.-Lan-Wen-Zhang-Zheng:2020, Johnson-Freyd:2020, K.-Zheng:2021
6. Rational CFT's, gapless boundaries of 2+1D topological orders and topological Wick rotation: K.-Zheng:2018,2020,2021

Combining 1-6 $\rightsquigarrow$ the category of nD quantum liquids (topological orders, SPT/SET orders, symmetry-breaking orders and CFT's). K.-Zheng:2011.02859, 2201.05726

$$\mathcal{QL}^n_{\mathrm{sk}} \simeq \bullet/\Sigma^{n+1}_*\mathbb{C}.$$



From boundary-bulk duality to the center functor (i.e. holography is functorial):

1. Center functor in 2D rational CFT's Davydov-K.-Runkel:1307.5956
2. Center functor in topological orders: K.-Wen-Zheng:1502.01690, K.-Zheng:1507.00503, K.-Yuan-Zheng:1912.13168, K.-Zheng:2107.03858 .

The functoriality of center demands all defects!

Thank you !