Non-Invertible Symmetries of $\mathcal{N}$=4 SYM and Twisted Compactification

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Motivation

- Recently there has been much interest in the study of novel symmetries.
- Such symmetries may put constraints on RG flows leading to new insights on the dynamics of field theories.
- Here we shall consider a class of such symmetries, dubbed non-invertible symmetries, and shall study their appearance in 4d $\mathcal{N}$=4 super Yang-Mills.
- Apply them to generate new RG flows using twisted compactification, leading to new 3d $\mathcal{N}$=6 SCFTs.

Outline

1. Introduction: non-invertible symmetries
2. Non-invertible symmetries in $\mathcal{N}$=4
3. Twisted compactification
4. Conclusions

Symmetries

- Symmetries play an important role in Physics:
  - Conservation laws, Ward identities.
  - Selection rules.
  - ‘t Hooft anomalies.
- Renewed interest in symmetries recently.

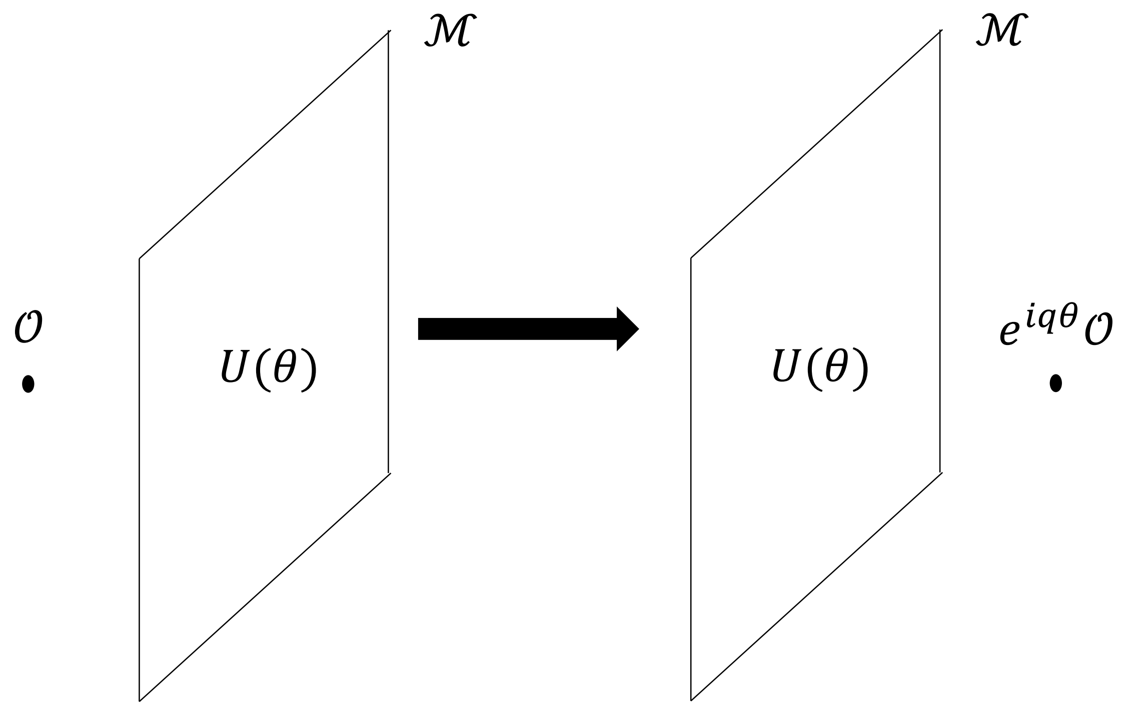
Symmetries = Topological operators

- Symmetries can be associated with topological operators.
- Example: $U(1)$. Have conserved current: $d\ *j = 0$.
- Leads to conserved charge:

  $Q = \int j_0\, d^{d-1}x.$
- Can use this to build an operator:

$$U = e^{i\theta Q} = e^{i\theta \oint_{\mathcal{M}} *j}$$

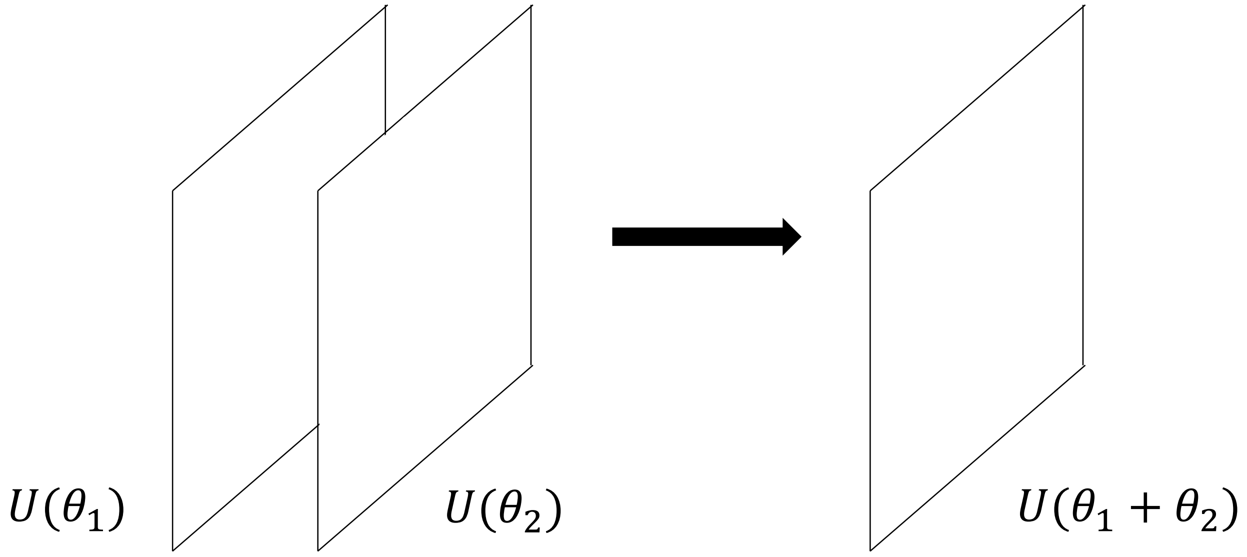
- Current conservation → Operator is topological.
- Works similarly for discrete symmetries.



[Frolich, Fuchs, Runkel, Schweigert, 2009; Kapustin, Seiberg, 2014; Gaiotto, Kapustin, Seiberg, Willett, 2014; …]

Properties of topological operators

- Properties of the topological operators then imply properties of the associated symmetries.



$$U(\theta_1) \otimes U(\theta_2) = U(\theta_1 + \theta_2)$$

- Example: fusion rule → group property.

Generalizations

- The topological operator viewpoint suggests several generalizations of the notion of symmetries.
- Topological operators of higher codimension → higher form symmetries (codimension p+1 operator → p-form symmetry).
- Non-invertible symmetries: symmetries that do not form a group.
- Elements don't necessarily possess an inverse.

$$a \otimes \bar{a} = 1 \oplus \ \dots$$

Types of non-invertible symmetries

- There are various known ways to realize non-invertible symmetries:
  - Gauging non-abelian discrete symmetries [Bhardwaj, Tachikawa, 2017].
  - Gauging a symmetry with a mixed anomaly with another symmetry [Tachikawa, 2017; Kaidi, Ohmori, Zheng, 2021; Choi, Cordova, Hsin, Lam, Shao, 2021].
  - Symmetries under gauging [Kaidi, Ohmori, Zheng, 2021; Choi, Cordova, Hsin, Lam, Shao, 2021].
- Here we shall be mostly interested in the last case.
- Many other references [Roumpedakis, Seifnashri, Shao, 2022; Bhardwaj, Bottini, Schafer-Nameki, Tiwari, 2022; Hayashi, Tanizaki, 2022; Arias-Tamargo, Rodriguez-Gomez, 2022; …]

Gauging of symmetries

- Recall that when we gauge a standard symmetry we sum over all possible holonomies.
- We can formulate the gauging of a symmetry also in the language of topological operators.
- For discrete symmetries, implemented by summing over all possible insertions of topological operators associated with the gauged symmetry.
- Can consider the case when a theory is invariant under the operation of gauging a discrete symmetry.

Gauging of symmetries

- Given a theory with a discrete (abelian) anomaly free 0-form symmetry H, can consider gauging this symmetry.
- In the resulting theory H is no longer a global symmetry and as such, naively we end up with a different theory.
- However, we get a d-2 form symmetry instead [Gaiotto, Kapustin, Seiberg, Willett, 2014]. In particular in d=2 we get a dual 0-form symmetry.
- As such, in d=2, theories can be self-dual under gauging.
- Gauging the dual d-2 form symmetry $\rightarrow$ brings us back to the original theory.
- Works similarly for higher form symmetries. Gauging a p form symmetry, we get a dual d-p-2 form symmetry.

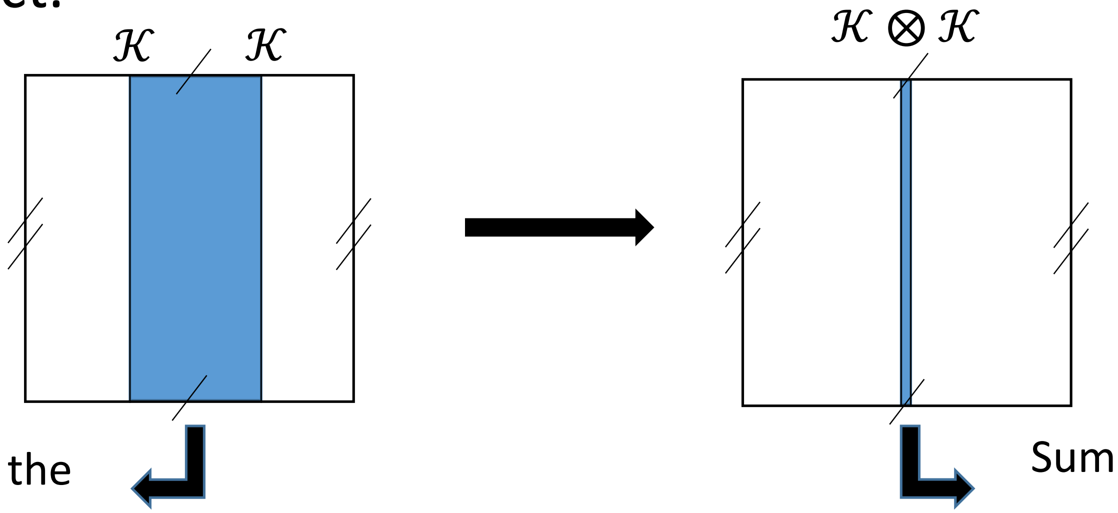
Example: 2d critical Ising model

- Consider the 2d critical Ising model. Has two topological operators:
  - η : $\mathbb{Z}_2$ 0-form symmetry. Spin flip.
  - $\mathcal{K}$: Half-space gauging of the $\mathbb{Z}_2$ 0-form symmetry. Kramers-Wannier duality defect.

$$\eta \otimes \eta = 1$$

$$\eta \otimes \mathcal{K} = \mathcal{K}$$

$$\mathcal{K} \otimes \mathcal{K} = 1 \oplus \eta$$



Sum over holonomies of the $\mathbb{Z}_2$ symmetry

Sum over holonomies: $1 \oplus \eta$

Symmetries under gauging behind 2d

- We have seen that in 2d we can have invariance under gauging of discrete 0-form symmetries. Here we lose the discrete symmetry we gauged but gain a new discrete symmetry.
- No longer expected to hold beyond 2d, as the new symmetry we gain is no longer a 0-form symmetry.
- However, in 4d if we gauge a 1-form symmetry, we gain a new 1-form symmetry.
- In 4d, possible to have self-duality under gauging a discrete 1-form symmetry. Recently, many examples of this type have been discovered [Kaidi, Ohmori, Zheng, 2021; Choi, Cordova, Hsin, Lam, Shao, 2021, 2022; Choi, Lam, Shao, 2022; Cordova, Ohmori, 2022; …].
- Example: $\mathcal{N}$=4 SYM.

$\mathcal{N}$=4 SYM

- $\mathcal{N}$=4 SYM is defined first by a choice of gauge algebra g. Here for simplicity $g = su(2)$.
- This does not fix the theory completely. Still have a choice of the precise group [Aharony, Seiberg, Tachikawa, 2013]:

	$SU(2)$	$SO(3)_+$	$SO(3)_-$
Fundamental Wilson line	Exist $\mathbb{Z}_2$ 1-form	None	None
Fundamental 't Hooft line	None	Exist $\mathbb{Z}_2$ 1-form	None
Fundamental dyonic line	None	None	Exist $\mathbb{Z}_2$ 1-form

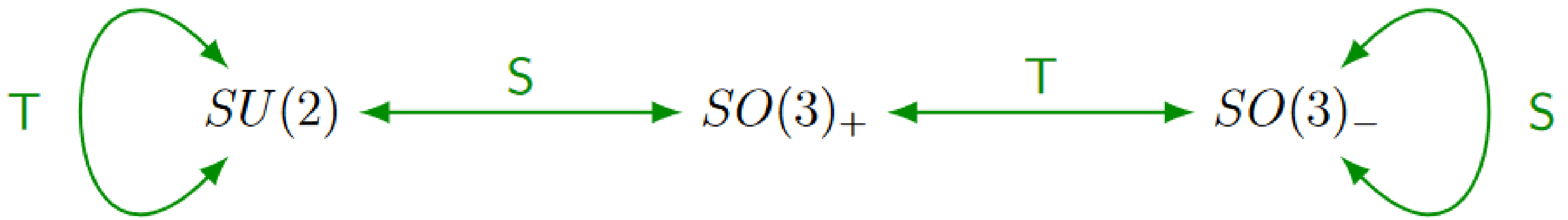
- We can move between the different global structures by gauging the 1-form symmetry. Example: $SU(2) \to SO(3)_+$, by gauging the $\mathbb{Z}_2$ 1-form symmetry.

S-duality

- $\mathcal{N}$=4 SYM possesses the $SL(2,\mathbb{Z})$ duality relating theories with different values of τ_{YM}

$$\text{S: } \tau_{YM} \to -\frac{1}{\tau_{YM}}, \text{ T: } \tau_{YM} \to \tau_{YM}+1$$

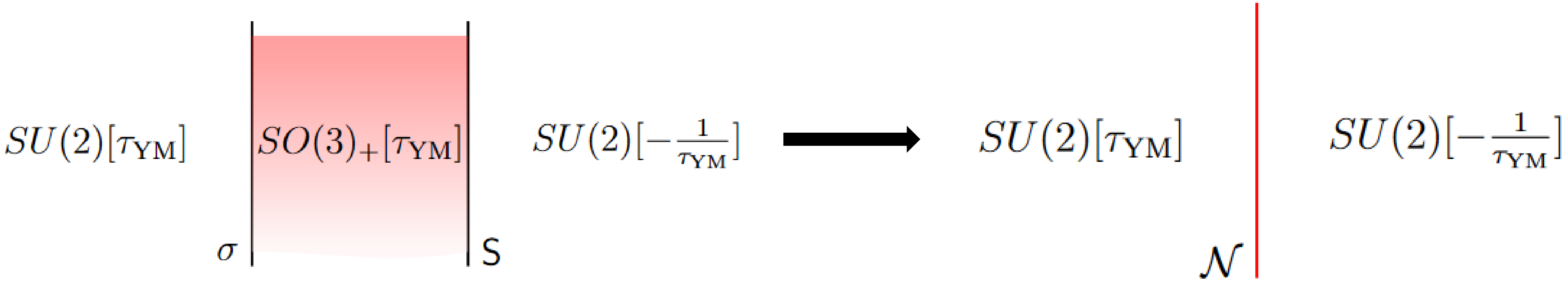
- The different global structures are transferred to one another by S-duality [Aharony, Seiberg, Tachikawa, 2013]:



- The case of $SO(3)_-$ at $\tau_{YM} = i$ is self-dual under S. As such, at that point the S-transformation becomes a symmetry.
- What about $SU(2)$ and $SO(3)_+$ at $\tau_{YM} = i$?

Non-invertible symmetries in $\mathcal{N}$=4 SYM

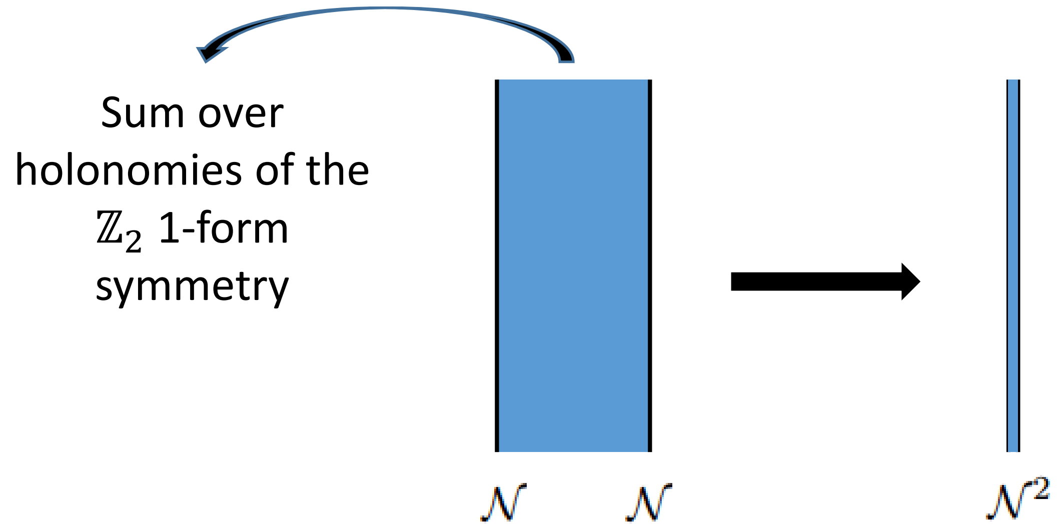
- Can still get a symmetry in these cases by gauging the 1-form symmetry.



σ: half-space gauging of the 1-form symmetry , S: S-duality interface , $\mathcal{N} = \sigma$ S

[Kaidi, Ohmori, Zheng, 2021]

Non-invertible symmetries in $\mathcal{N}$=4 SYM



$$\mathcal{N}^2 \sim \sum_{\Sigma \in H_2(M_3, \mathbb{Z}_2)} L_\Sigma$$

L_Σ top operator of $\mathbb{Z}_2$ 1-form symmetry wrapped on the 2-cycle Σ

- Price → symmetry become non-invertible.

[Kaidi, Ohmori, Zheng, 2021]

Non-invertible symmetries in $\mathcal{N}$=4 SYM

- The invertible S-duality symmetry in the $SO(3)_-$ theory at $\tau_{YM} = i$ becomes a non-invertible symmetry in $SU(2)$ and $SO(3)_+$.
- Are there other such symmetries? Can there be non-invertible symmetries that are not related to an invertible one?
- We can try to systematically search for such symmetries by studying the transformation properties of $\mathcal{N}$=4 SYM under both S-duality and gauging of the 1-form symmetry.

S-duality and the σ and τ operations

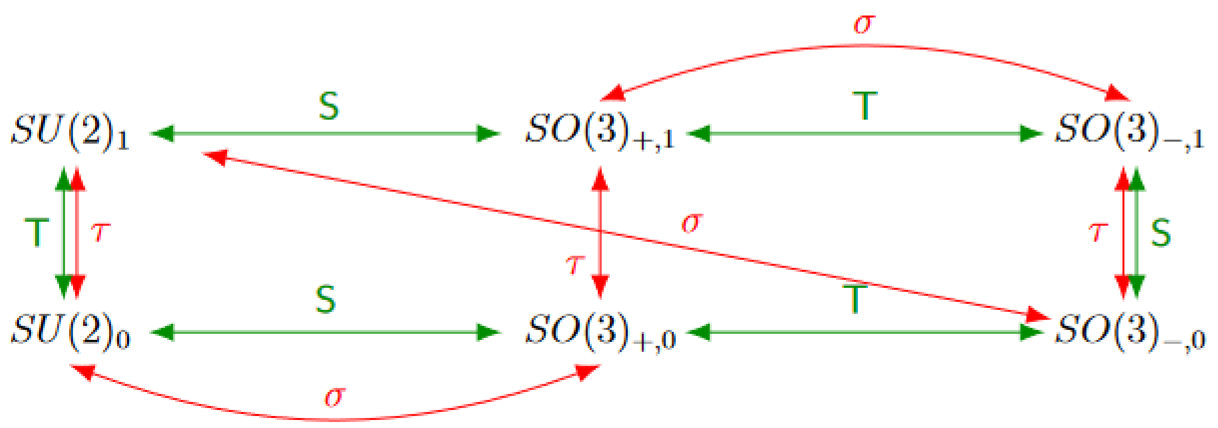
- We can consider two families of operations on $su(2)$ $\mathcal{N}$=4 SYM.
- The S-duality transformations:

  S: $\tau_{YM} \to -\frac{1}{\tau_{YM}}$

  T: $\tau_{YM} \to \tau_{YM} + 1$
- These form $(P)SL(2,\mathbb{Z})$: $S^2 = (ST)^3 = 1$.
- The σ and τ operations:
  - σ: gauging the $\mathbb{Z}_2$ 1-form symmetry.
  - τ: stacking with an invertible phase $\frac{\pi}{2}\int \mathcal{P}(B)$.
- These form $SL(2,\mathbb{Z}_2)$: $\sigma^2 = \tau^2 = (\sigma\tau)^3 = 1$.
  [Witten, 2003; Gaiotto, Kapustin, Seiberg, Willett, 2014]

B: background gauge field for the 1-form symmetry. $\mathcal{P}(B)$: Pontryagin square of B. Analogue of $B \wedge B$ for forms valued in $\mathbb{Z}_2$.

The orbit



- Can determine the orbit under the S, T, σ and τ transformations.
- Limiting τ_{YM} to the fundamental domain, possible symmetries are:
  - $\mathbb{Z}_2$ at $\tau_{YM} = i$, given by the S transformation.
  - $\mathbb{Z}_3$ at $\tau_{YM} = e^{\frac{2\pi i}{3}}$, given by the ST transformation.
- Need to combine with σ and τ to actually form a symmetry.

The possible non-invertible symmetries

Theory	Defect	n-ality
$SU(2)_m$, $SO(3)_{+,m}$	$\tau^m \sigma \mathsf{S} \tau^{-m}$	2
$SO(3)_{-,m}$	$\tau^m \tau \mathsf{S} \tau^{-m}$	1

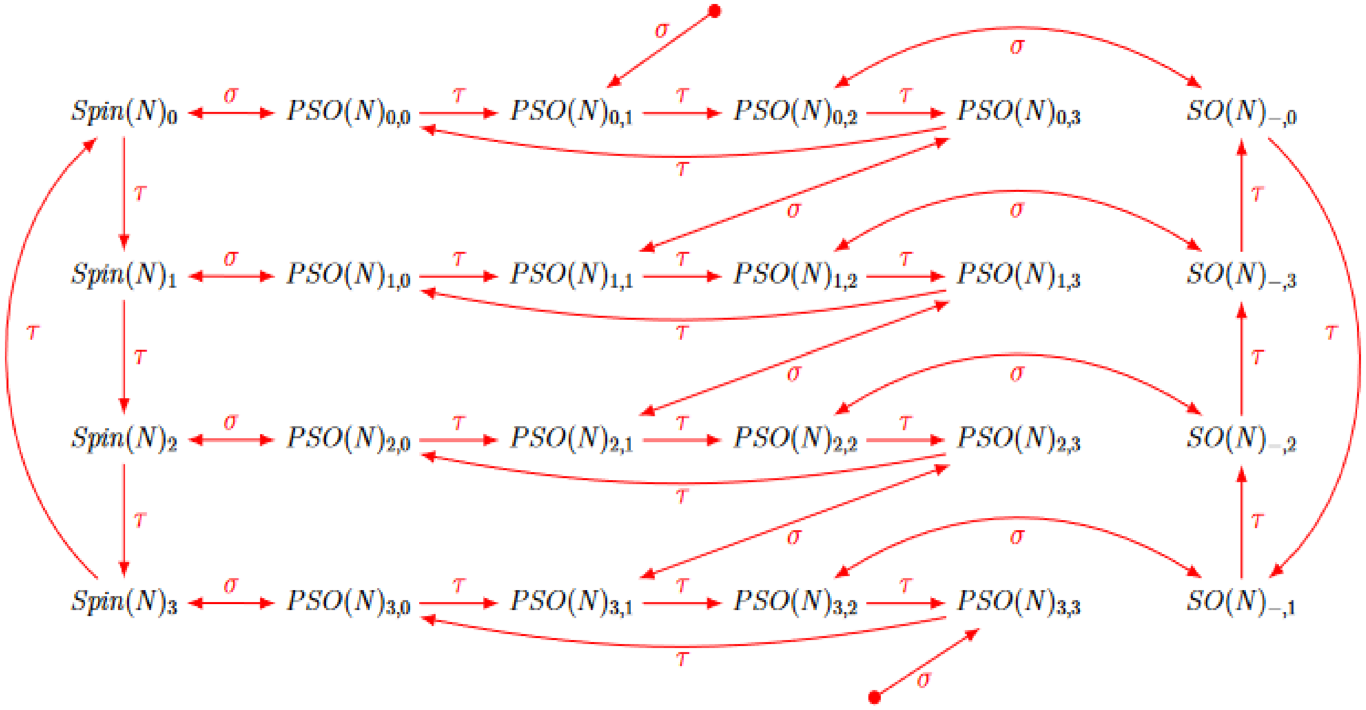
(Non-)invertible symmetries of $\mathfrak{su}(2)$ at $\tau_{\rm YM} = i$.

Theory	Defect	n-ality
$SU(2)_m$, $SO(3)_{-,m}$	$\tau^m \sigma \tau \mathsf{ST} \tau^{-m}$	3
$SO(3)_{+,m}$	$\tau^m \tau \sigma \mathsf{ST} \tau^{-m}$	3

Non-invertible symmetries of $\mathfrak{su}(2)$ at $\tau_{\rm YM} = e^{2\pi i/3}$.

- Can use the orbit to determine the possible non-invertible symmetries.

The possible non-invertible symmetries



Theory	Defect	n-ality
$Spin(N)_m$, $PSO(N)_{0,m}$	$\tau^m \sigma^3 \mathsf{S} \tau^{-m}$	2
$PSO(N)_{2,m}$, $SO(N)_{-,m}$	$\tau^{2+m} \sigma^3 \mathsf{S} \tau^{2-m}$	2
$PSO(N)_{n,m}$, $n = 1,3$	$\tau^m \sigma \tau^n \sigma \tau^{-n} \sigma \mathsf{S} \tau^{-m}$	2

Non-invertible symmetries of $\mathfrak{so}(N)$ with $N \in 4\mathbb{Z}+2$ at $\tau_{\text{YM}} = i$.

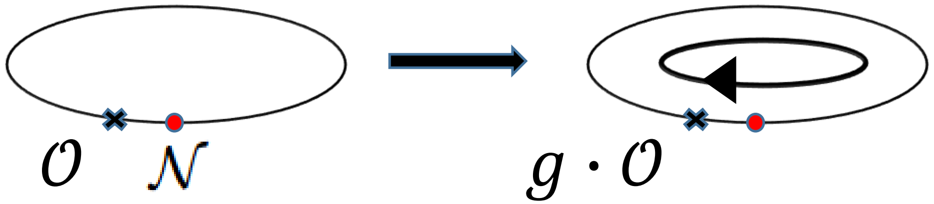
Theory	Defect	n-ality
$Spin(N)_m$, $PSO(N)_{3,m}$	$\tau^m \sigma \tau^{-1} \mathsf{ST} \tau^{-m}$	3
$PSO(N)_{0,m}$	$\tau^m \tau^{-1} \sigma \mathsf{ST} \tau^{-m}$	3
$PSO(N)_{1,m}$, $SO(N)_{-,m}$	$\tau^{m+2} \sigma \tau^{-1} \mathsf{ST} \tau^{-m}$	3
$PSO(N)_{2,m}$	$\tau^{m-1} \sigma \tau^2 \mathsf{ST} \tau^{-m}$	3

Non-invertible symmetries of $\mathfrak{so}(N)$ with $N \in 4\mathbb{Z}+2$ at $\tau_{\text{YM}} = e^{2\pi i/3}$.

- Can similarly do this for other gauge groups (example $SO(N = 4n + 2)$).
- Provides an extensive understanding of non-invertible symmetries in $\mathcal{N}$=4 SYM.

Twisted compactification with non-invertible symmetries

- Given a symmetry we can consider inserting a holonomy around a cycle.
- In the context of circle reduction: twisted compactification. Take $R^d \to R^{d-1} \times S^1$ and enforce $\mathcal{O}(\theta + 2\pi) \to g \cdot \mathcal{O}(\theta)$, where g is the symmetry operation.
- In the language of topological operators: insert the operator localized on the S^1.



Twisted compactification with non-invertible symmetries

- Expect to also be able to perform twisted compactifications with non-invertible symmetries.
- Generically analyzing compactification of strongly interacting theories is difficult.
- $\mathcal{N}$=4 SYM provides a good case study due to ample SUSY.
- However, this only helps us if the compactification preserves SUSY. Want to determine how much SUSY can be preserved by the non-invertible symmetries we discussed.

Preserving supersymmetry

- Consider the 4 operations we mentioned previously. How do these act on the supercharges?
- σ, τ: only act non-trivially on extended operators. Should act trivially on the supercharges.
- S, T: have non-trivial action on the supercharges [Kapustin, Witten, 2006]

$$\tau_{YM} \to \frac{a\,\tau_{YM}+b}{c\,\tau_{YM}+d}\ ,\ Q \to e^{-\frac{i\,\vartheta}{2}}Q\ ,\vartheta = \arg(c\,\tau_{YM} + d)$$

- As such the non-invertible symmetries built from these operations will not preserve any SUSY.

Conjecturing the resulting theories

- Can preserve some SUSY by combining with an $SU(4)$ R-symmetry transformation [Ganor, Hong, 2008].

$$\begin{pmatrix} e^{i\phi_1} & 0 & 0 & 0 \\ 0 & e^{i\phi_2} & 0 & 0 \\ 0 & 0 & e^{i\phi_3} & 0 \\ 0 & 0 & 0 & e^{i\phi_4} \end{pmatrix} \qquad \sum_i \phi_i = 0$$

- Under the combined action: $\mathcal{Q}_i \rightarrow e^{i(\phi_i - \frac{\vartheta}{2})}\mathcal{Q}_i$.
- If we take $\phi_1 = \phi_2 = \phi_3 = \frac{\vartheta}{2}$, can preserve 12 supercharges.
- Here the phases are going to be $\mathbb{Z}_k$ with k related to the order of the transformation (n-ality in the non-invertible case).

Conjecturing the resulting theories

- We expect the resulting 3d theory to be an $\mathcal{N}$=6 SCFT.
- We can get more information by considering what happens to the moduli space.
- In 4d $\mathcal{N}$=4 SCFTs, the moduli space is spanned by vevs of the adjoint scalars. These can be described schematically as the vevs of the independent gauge invariants $u_n = Tr(\Phi^n)$.
- Their number is given by the rank of the gauge group, and their dimensions are given in terms of the dimensions of invariant polynomials of the group.
- For the $\mathcal{N}$=4 SCFTs considered here, the moduli space is freely generated by these basic invariants.

Conjecturing the resulting theories

- Consider the action of the symmetry on the u_n operators.
- These are invariant under S-duality and gauging the 1-form symmetry, but not under the R-symmetry transformation were we have:
$u_n \rightarrow e^{\frac{2\pi i n}{k}} u_n$.
- When performing the twisted compactification, non-invariant u_n operators would be projected out.
- As such we expect to get a 3d $\mathcal{N}$=6 SCFT whose moduli space is freely generated by the subset of the invariant u_n.
- Can use these to formulate conjectures on the identity of these SCFTs. In many cases these appear to be ABJM and ABJ type theories [Aharony, Bergman, Jafferis, Maldacena, 2008; Aharony, Bergman, Jafferis, 2008].

New $\mathcal{N}$=6 SCFTs

- This construction can be used to generate new $\mathcal{N}$=6 SCFTs. Some cases necessitate the use of non-invertible symmetries.
- For example, consider the case of $g = e_7$. Here the 1-form symmetry is $\mathbb{Z}_2$ and the spectrum of possible global structures and relations between them is the same as in the $g = su(2)$ case.
- As such, we have the same spectrum of non-invertible symmetries.
  - $\mathbb{Z}_2$ symmetry at $\tau_{YM} = i$, which is invertible for $G = (E_7/\mathbb{Z}_2)_-$. Expected to give $\mathcal{N}$=6 SCFT with moduli space $\mathbb{C}^8/G_8$.
  - "$\mathbb{Z}_3$" symmetry at $\tau_{YM} = e^{\frac{2\pi i}{3}}$, no variant where it is invertible. Expected to give $\mathcal{N}$=6 SCFT with moduli space $\mathbb{C}^{12}/G_{26}$.

G_8, G_{26}: Two of the exceptional complex reflection groups .

Conclusions

- Recently there as been a renewed interest in novel types of symmetries. Examples include non-invertible symmetries that do not form a group.
- These exist in 4d where they can appear as invariance under gauging 1-form symmetries.
- Appear in $\mathcal{N}$=4 SYM and can be studied systematically.
- Can perform twisted compactification by these symmetries.
- Can be used to realize new $\mathcal{N}$=6 SCFTs.

Open questions

- Other implications of the non-invertible symmetries.
- Can this be generalized to other $\mathcal{N}$=2 SCFTs with a geometric construction?
- General description of the space of twisted compactifications.
- Can we say more about the resulting $\mathcal{N}$=6 SCFTs?

Thank you