Special unipotent representations of complex s/s groups

G - complex simple Lie group $SL(n,\mathbb{C})$, $SO(n,\mathbb{C})$, $Sp(2n,\mathbb{C})$

$\mathfrak{g}$ - corr. Lie alg.

Big question (Gelfand)

Describe the set $\hat{G}$ of irred. unitary rep.

$\mathcal{H}$ - Hilbert space, $\rho: G \to \mathcal{U}(\mathcal{H})$.

Idea of orbit method to get some of $\hat{G}$:

$$\mathbb{O} \subset \mathfrak{g}^* \text{ - nilp. orbit} \longrightarrow \text{unitary rep.}$$

sympl. var. $\xrightarrow{\text{quantize}}$ Hilbert space

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g}) / (\underbrace{x \otimes y}_{\deg 2} - \underbrace{y \otimes x}_{\deg 2} - \underbrace{[x,y]}_{\deg 1}) \qquad x, y \in \mathfrak{g}$$

nat. filtr.   nat. grading, $\deg x = 1, x \in \mathfrak{g}$

$$\operatorname{gr} \mathcal{U}(\mathfrak{g}) = \bigoplus_{i \geq 0} \mathcal{F}_i \mathcal{U}(\mathfrak{g}) / \mathcal{F}_{i-1} \mathcal{U}(\mathfrak{g}) \overset{PBW}{\simeq} S(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{g}^*]$$

Quant. orbit:

$$S(\mathfrak{g}) \curvearrowright \mathbb{C}[\mathbb{O}] \curvearrowleft S(\mathfrak{g}) \quad \xrightarrow{\qquad} \quad \mathcal{U}(\mathfrak{g}) \curvearrowright X \curvearrowleft \mathcal{U}(\mathfrak{g})$$

$$\xleftarrow{\ \mathrm{gr}\ }$$

$\mathbb{C}[\mathbb{O}] \curvearrowleft G$, $\quad X \curvearrowleft$ G-int. of $[\mathfrak{g}, \cdot]$

Unipotent rep-s; expectations:

1) X is a quant. of $\mathbb{O}$ for some nilp. orbit $\mathbb{O}$.
some local system on $\mathbb{O}$.

2) $\mathrm{LAnn}_{\mathcal{U}(\mathfrak{g})}(X) = \mathrm{RAnn}_{\mathcal{U}(\mathfrak{g})}(X) \subset \mathcal{U}(\mathfrak{g})$ is maximal

3) X is unitarizable.

Ex: $G = SL_2(\mathbb{C})$, $\quad \mathbb{O} = G\cdot\begin{pmatrix}0&1\\0&0\end{pmatrix} = \mathbb{C}^2 - \{0\} / \mathbb{Z}/2\mathbb{Z}$

$$\mathbb{C}[\mathbb{O}] = \mathbb{C}[x,y]^{\mathbb{Z}/2\mathbb{Z}} = \mathbb{C}[\overset{x^2}{t}, \overset{y^2}{s}, \overset{xy}{u}] / (ts-u^2)$$

Take $X_1 = \mathcal{D}(\mathbb{A}^1)^{\text{even}}$ $\qquad X_2 = \mathcal{D}(\mathbb{A}^1)^{\text{odd}}$

$$\mathcal{D}(\mathbb{A}^1) = T[\underset{\deg 1}{x}, \underset{\deg 1}{\tfrac{\partial}{\partial x}}] / \left(\tfrac{\partial}{\partial x}\cdot x - x\cdot\tfrac{\partial}{\partial x} - 1\right)$$

$$E = \frac{i}{2}x^2 \qquad H = x\frac{d}{dx} + \frac{1}{2} \qquad F = \frac{i}{2}\frac{d^2}{dx^2}$$

does not quantize $\mathbb{O}$,
but local system on $\mathbb{O}$.

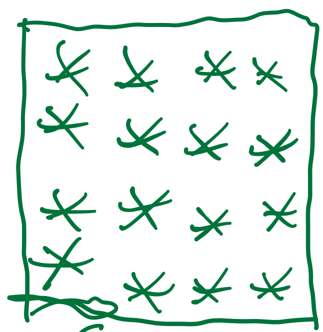
Special unipotent representations (Barbasch-Vogan)

Know $I = L\,\mathrm{Ann}(X) = R\,\mathrm{Ann}(X) =$ max. ideal in $\mathcal{U}(\mathfrak{g})$

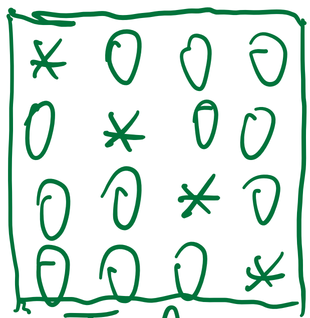
centr. char.

max. ideals in $\mathcal{U}(\mathfrak{g}) \longleftrightarrow$ points in $\mathfrak{h}^*/W$

max. ideals in $Z(\mathcal{U}(\mathfrak{g}))$



$G, \mathfrak{g}$



$T, \mathfrak{h}$

$$Z(\mathcal{U}(\mathfrak{g})) \cong S(\mathfrak{h})^W \simeq \mathbb{C}[\mathfrak{h}^*/W]$$

$$I \subset \mathcal{U}(\mathfrak{g}) \rightsquigarrow \mathcal{J} \subset \mathbb{C}[\mathfrak{g}^*] \rightsquigarrow \underset{\overset{\shortparallel}{V(I)}}{V(\mathcal{J})} = \overline{\mathcal{O}}$$

nilp orbit

Need „good" points of $\mathfrak{h}^*/W$.

Set $G^\vee$ to be Langlands dual of G

G	$G^\vee$
$SL(n)$	$SL(n)$
$SO(2n+1)$	$Sp(2n)$
$Sp(2n)$	$SO(2n+1)$
$SO(2n)$	$SO(2n)$

$\mathcal{O}^\vee \subset \mathfrak{g}^\vee$ - nilpotent orbit

Jacobson-Morozov:

$\varphi : \mathfrak{sl}(2) \longrightarrow \mathfrak{g}^\vee$

$e^\vee = \varphi(e) \in \mathcal{O}^\vee$

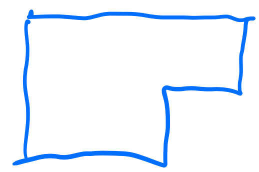
$h^\vee = \varphi(h) \in \mathfrak{h}^\vee \simeq \mathfrak{h}^*$

$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Ex: $\mathcal{O}^\vee = SL(5) \cdot e^\vee$

$$e^\vee = \left(\begin{array}{ccc|cc} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ \hline & & & 0 & 1 \\ & & & & 0 \end{array}\right) \qquad h^\vee = \left(\begin{array}{ccc|cc} 2 & & & & \\ & 0 & & & \\ & & -2 & & \\ \hline & & & 1 & \\ & & & & -1 \end{array}\right)$$

λ



Take „good" point $W \cdot \frac{1}{2} h^\vee$.

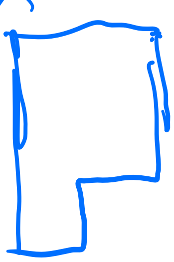
$V(I_{\frac{1}{2}h^\vee}) = \overline{\mathcal{O}}$.

BVLS duality

Define $d : \mathcal{N}/_{G^\vee} \longrightarrow \mathcal{N}/_G$

$d(\mathcal{O}^\vee) = \mathcal{O}$.

λ^t



$$\left(\begin{array}{cc|cc|c} 0 & 1 & & & \\ & 0 & & & \\ \hline & & 0 & 1 & \\ & & & 0 & \\ \hline & & & & 0 \end{array}\right)$$

$\mathcal{O}$ is special if $\mathcal{O} = d(\mathcal{O}^\vee)$ for some $\mathcal{O}^\vee$

Def: Take $\mathbb{O}$ – special.

$$\mathrm{Unip}^S(\mathbb{O}) = \{ X \in HC(G) \text{ irred.}, \ \mathrm{LAnn}(X) = \mathrm{RAnn}(X) = I(\tfrac{1}{2} h^\vee), \ d(\mathbb{O}^\vee) = \mathbb{O} \}$$

Quant. orbit:

$$\begin{tikzcd}
S(\mathfrak{g}) \curvearrowright \mathbb{C}[\mathbb{O}] \curvearrowleft S(\mathfrak{g}) \arrow[r, bend left] & U(\mathfrak{g}) \curvearrowright X \curvearrowleft U(\mathfrak{g}) \arrow[l, bend left, "gr"]
\end{tikzcd}$$

$\mathbb{C}[\mathbb{O}] \curvearrowleft G$ $\qquad$ $HC(G)$ $\qquad$ $X \curvearrowleft G$-int. of $[\mathfrak{g}, \cdot]$

Unipotent rep-s; expectations:

1) X is a quant. of a local system on $\mathbb{O}$. not clear
2) $\mathrm{LAnn}_{U(\mathfrak{g})}(X) = \mathrm{RAnn}_{U(\mathfrak{g})}(X) \subset U(\mathfrak{g})$ is maximal by def.
3) X is unitarizable. requires work, not clear in exc. types
4) $\{X \in HC(G), \mathrm{RAnn}(X) = \mathrm{LAnn}(X) = I_{\frac{1}{2} h^\vee}\}$

$$\updownarrow$$

$\bar{A}(\mathbb{O}^\vee)$ – irreps

Lusztig quotient

known for $\mathbb{O}^\vee$ special (Barbasch, Vogan, Wong)

Alternative approach:
(Ivan Losev, Lucas Mason-Brown, D.M.)

1. $\mathbb{O}^\vee \subset \mathfrak{g}^\vee \xrightarrow{\tilde{d}} \widetilde{\mathbb{O}}$ G-equivariant cover of $\mathbb{O} = d(\mathbb{O}^\vee)$

$$\begin{array}{c} \widetilde{\mathbb{O}} \\ \Gamma \downarrow \\ \mathbb{O} \end{array}$$

$U(\mathfrak{g})$-bimod

2. $\widetilde{\mathbb{O}} \longrightarrow \mathcal{A}(\widetilde{\mathbb{O}})$ – canonical quant. of $\mathbb{C}[\widetilde{\mathbb{O}}]$ ($\Gamma \curvearrowright$, $\curvearrowleft G$)

3. $\mathrm{Unip}(\widetilde{\mathbb{O}}) = \{(\mathcal{A}(\widetilde{\mathbb{O}}) \otimes V)^\Gamma$, V-irrep of $\Gamma\}$

2 + 3 works for any $\widetilde{\mathbb{O}}$ (including non-special)
Gives new def. of unipotent reps.

Unipotent rep-s; expectations:

1) X is a quant. of a local system on $\mathbb{O}$. by def.

2) $\mathrm{LAnn}_{U(\mathfrak{g})}(X) = \mathrm{RAnn}_{U(\mathfrak{g})}(X) \subset U(\mathfrak{g})$ is maximal

$\varphi: U(\mathfrak{g}) \to \mathcal{A}(\widetilde{\mathbb{O}})$

$\mathrm{LAnn}(X) = \mathrm{RAnn}(X) = \mathrm{Ker}\,\varphi \overset{\text{LMBM}}{=} I_{\frac{1}{2}\rho^\vee}$

3) X is unitarizable. requires work, not clear in exc. types

4) $\{X \in HC(G),\ RAnn(X) = LAnn(X) = I_{\frac{1}{2}h^\vee}\}$

$$\Big\updownarrow$$

$\bar{A}(\mathbb{O}^\vee)$ - irreps

Lusztig quotient

$\Gamma \simeq \bar{A}(\mathbb{O}^\vee)$

Lucas Mason-Brown, Shilin Yu, D. M.

$X = e^\vee + \ker([f^\vee, \cdot])$ Slodowy slice

$Y = \operatorname{Spec}(\mathbb{C}[\tilde{\mathbb{O}}])$

Expectations:

① X and Y are symplectic dual

② Many properties of special unipotent reps can be described by the geometry of the Slodowy slice!