Affine Grassmannians in non-archimedean geometry and deformation theory

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Outline

Affine Grassmannian

- G: connected reductive group over $\mathbb{C}$ (or over a field k)
- The quotient

$$\mathrm{Gr}_G(\mathbb{C}) := G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$$

is an infinite dimensional algebraic variety over $\mathbb{C}$, which is called the affine Grassmannian of G.

$\mathbb{C}[[t]] := \{\sum_{n\geq 0} a_n t^n \mid a_n \in \mathbb{C}\}$
$\mathbb{C}((t)) := \{\sum_{n\in\mathbb{Z}} a_n t^n \mid a_n \in \mathbb{C}, a_n = 0\,(\text{small enough } n)\}$.

Example: $G = \mathrm{GL}_n$

$$\mathrm{Gr}_G(\mathbb{C}) \xrightarrow{\sim} \{V \subset \mathbb{C}((t))^n \mid V \text{ is a } \mathbb{C}[[t]]\text{-lattice}\}$$

defined by $g \in \mathrm{GL}_n(\mathbb{C}((t))) \mapsto g(\Lambda_{\mathrm{std}}) \subset \mathbb{C}((t))^n$, where $\Lambda_{\mathrm{std}} := \mathbb{C}[[t]]^n \subset \mathbb{C}((t))^n$ is the standard lattice.

Affine Grassmannians play important roles in several fields of mathematics (and mathematical physics).

(1) Langlands duality, geometric Satake equivalence, geometric Langlands correspondence (if k is a finite field).
(2) (Representation theory of affine Kac-Moody algebras.)

$$G \rightsquigarrow \widehat{G} : \text{Langlands dual group over } \mathbb{C}$$

Geometric Satake equivalence (Mirković–Vilonen '07)

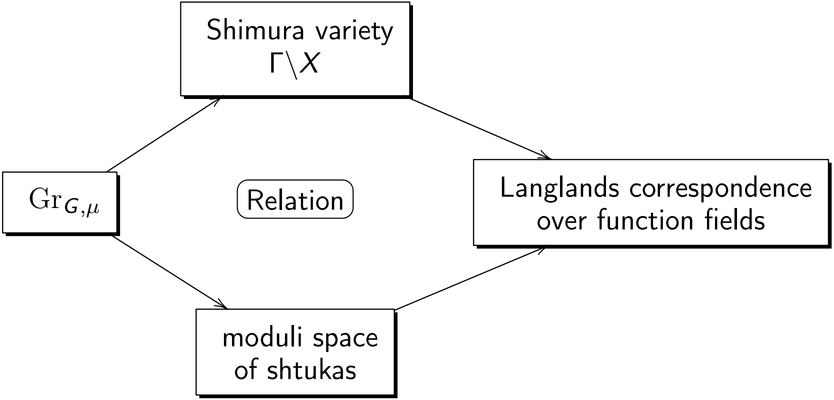
- $\mathrm{Rep}_{\mathbb{C}}(\widehat{G})$: category of finite dim representations of $\widehat{G}$.
- $\mathrm{Sat}(\mathrm{Gr}_G)$: category of "equivariant perverse sheaves" on Gr_G, called the Satake category.

There exists an equivalence of (symmetric monoidal) categories:

$$\mathrm{Sat}(\mathrm{Gr}_G) \xrightarrow{\sim} \mathrm{Rep}_{\mathbb{C}}(\widehat{G}).$$

(3) Closed subvarieties in Gr_G are important:

- For example, Schubert cells $\mathrm{Gr}_G = \coprod_\mu \mathrm{Gr}_{G,\mu}$.
  Here $\mu\colon \mathbb{C}^\times \to G$ are the dominant cocharacters.
- $\mathrm{Gr}_{G,\mu} \rightsquigarrow \mathrm{IC}_\mu \in \mathrm{Sat}(\mathrm{Gr}_G)$: simple objects



Remark

- (Recall $\mathrm{Gr}_G(\mathbb{C}) = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$.)
- X: Riemann surface (resp. a curve over a field k).
  Then the ring $\widehat{\mathcal{O}}_{X,x}$ of formal Taylor series around a point $x \in X$ can be identified with $\mathbb{C}[[t]]$ (resp. $k[[t]]$).
  $\Rightarrow \mathrm{Gr}_G$ is related to X.
  $\Rightarrow \mathrm{Gr}_G$ is related to $\pi_1(X)$.

Let p be a prime number. If k is the finite field

$$k = \mathbb{Z}/p\mathbb{Z} := \{0, 1, 2, \ldots, p-1\}$$

and X is a curve over k, then $\pi_1(X)$ is related to the Galois group of the function field of X (the field of meromorphic functions).

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Today

- $k = \mathbb{Q}_p$: the field of p-adic numbers.
- non-archimedean geometry = "analytic geometry over $\mathbb{Q}_p$"
- G: reductive group over $\mathbb{Q}_p$

Note: $\mathbb{Q}_p$ has the metric such that $p^n \to 0$ when $n \to \infty$.

Remark

In the non-archimedean setting,

$$G(\mathbb{Q}_p((t)))/G(\mathbb{Q}_p[[t]])$$

is not very suitable for applications, such as the local Langlands correspondence for G over $\mathbb{Q}_p$.

Problem: In the local Langlands correspondence, we are interested in the Galois group of $\mathbb{Q}_p$, rather than $\pi_1(X)$ for a curve X over $\mathbb{Q}_p$.

Affine Grassmannian in non-archimedean geometry

We should consider a reductive group G over $\mathbb{Z}_p$ and

$$\mathrm{Gr}_G := G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$$

where $\mathbb{Z}_p \subset \mathbb{Q}_p$ is the ring of p-adic integers.

Example: $G = \mathrm{GL}_n$

$$\mathrm{Gr}_G \xrightarrow{\sim} \{V \subset \mathbb{Q}_p^n \mid V \text{ is a } \mathbb{Z}_p\text{-lattice}\}$$

defined by $g \in \mathrm{GL}_n(\mathbb{Q}_p) \mapsto g(\Lambda_{\mathrm{std}}) \subset \mathbb{Q}_p^n$, where $\Lambda_{\mathrm{std}} := \mathbb{Z}_p^n \subset \mathbb{Q}_p^n$ is the standard lattice.

Remark

Gr_G has a natural geometric structure which can be described by using perfect algebras (explained later).

Geometric Satake equivalence (Zhu '17, Fargues–Scholze '21)

- $\mathrm{Rep}_{\mathbb{C}}(\widehat{G})$: category of finite dim representations of $\widehat{G}$.
- $\mathrm{Sat}(\mathrm{Gr}_G)$: category of "equivariant perverse sheaves" on Gr_G.

There exists an equivalence of (symmetric monoidal) categories:

$$\mathrm{Sat}(\mathrm{Gr}_G) \xrightarrow{\sim} \mathrm{Rep}_{\mathbb{C}}(\widehat{G}).$$

This theorem plays an important role in the geometrization of the local Langlands correspondence (Fargues, Fargues–Scholze).

Schubert cell

We also have Schubert cells $\mathrm{Gr}_G = \bigsqcup_\mu \mathrm{Gr}_{G,\mu}$ in this setting. Schubert cells $\mathrm{Gr}_{G,\mu}$ are related to moduli spaces of "non-archimedean" shtukas.

- Assume that $\mu\colon \mathbb{G}_m \to G$ is minuscule (i.e. the weights of the adjoint action of $\mathbb{G}_m$ of the Lie algebra $\mathrm{Lie}G$ are contained in $\{-1, 0, 1\}$).
- $\mathrm{Gr}_{G,\mu}$ is related to the non-archimedean analogue of Shimura variety, called (integral) local Shimura variety $\mathcal{M}_{G,\mu}$.

Theorem (I.)

Let $x \in \mathcal{M}_{G,\mu}$ be a point. We can attach a point $y \in \mathrm{Gr}_{G,\mu}$ to x. Then the tangent space of $\mathcal{M}_{G,\mu}$ at x is isomorphic to the tangent space of $\mathrm{Gr}_{G,\mu}$ at y.

Key: Establish a "new" deformation theory for non-archimedean shtukas.

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Perfect algebra

- R: ring such that $p = 0$ in R (e.g. $R = \mathbb{Z}/p\mathbb{Z}$).
- $\varphi\colon R \to R$, $x \mapsto x^p$ defines a ring homomorphism, called the Frobenius.
- We say that R is perfect if φ is bijective (e.g. $\mathbb{Z}/p\mathbb{Z}$ is perfect).

The ring $W(R)$

To a perfect algebra R, we can associate a ring

$$W(R),$$

called the ring of Witt vectors of R, in which p is a non-zero divisor (i.e. $W(R) \hookrightarrow W(R)[1/p]$).

Example

$W(\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}_p$ and $W(\mathbb{Z}/p\mathbb{Z})[1/p] = \mathbb{Q}_p$

$\mathrm{Gr}_G(R)$

We define

$$\mathrm{Gr}_G(R) := G(W(R)[1/p])/G(W(R)).$$

This enables us to consider Gr_G as a moduli space, defined over the category of perfect algebras.

On the other hand:

"Classical" deformation theory

Comparison of

- varieties (algebras, modules, etc) over $R[\epsilon]/\epsilon^2$ and
- varieties over R.

Problem: $R[\epsilon]/\epsilon^2$ is not perfect ($\epsilon \mapsto \epsilon^p = 0$).

Idea

Establish a new deformation theory in the category of “prisms”, introduced by Bhatt–Scholze.

Properties

(1) The category of prisms contains
- perfect algebras R, and
- important class of $R[\epsilon]/\epsilon^2$.

(2) We can define Gr_G for prisms.

Thank you very much for your attention!