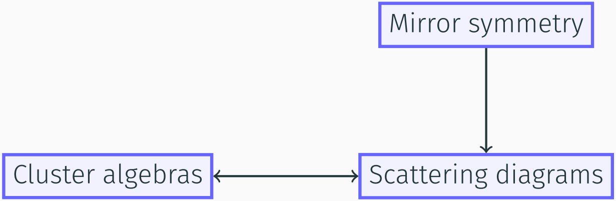
Compactifications and Dualities for Cluster varieties

Man Wai, Mandy, Cheung

22 Dec 2022

IPMU

Outline



[Gross-Hacking-Keel-Kontsevich] the linkage between scattering diagrams, broken lines, theta functions $\leftrightarrow$ cluster algebras developed by Fomin-Zelevinsky

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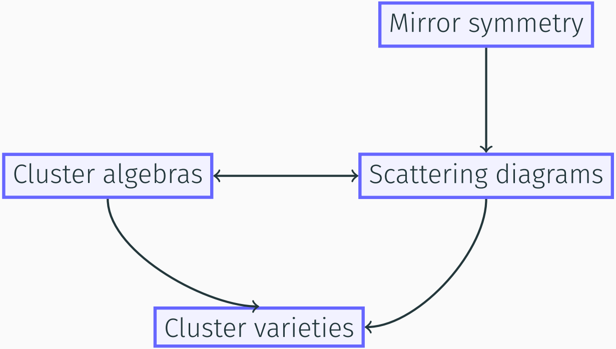

[C-Magee-Najera Chavez] used the tropical structures of the scattering diagrams to give compactification of the cluster varieties

[Bossinger-C-Magee-Najera Chavez] Apply the tropical properties to study Newton–Okounkov bodies.

Cluster algebras [Fomin Zelevinsky]

A seed **s** consists of a set of cluster variables and exchange data (b_{ij}).

Start with initial seed

mutation $\mu_k^{\mathcal{A}} \rightsquigarrow$ new seed with replacing the variable A_k to the new variable A_k' by

$$A_k A_k' = \prod_{b_{ij}>0} A_j^{b_{ij}} + \prod_{b_{ij}<0} A_j^{-b_{ij}}.$$

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Principal coefficients (or X-variables)

Similar procedure only changing the mutation map $\mu_k^{\mathcal{X}}$.

$\rightsquigarrow \mathcal{X}$ cluster algebra

Cluster varieties

What are cluster varieties?

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$$N^\circ \rightsquigarrow T_{N^\circ} \qquad\qquad M \rightsquigarrow T_M$$

$$\mathcal{A} = \bigcup T_{N^\circ}/\mu^{\mathcal{A}} \qquad\qquad \mathcal{X} = \bigcup T_M/\mu^{\mathcal{X}}$$

$\rightsquigarrow$ $\mathcal{A}$ and $\mathcal{X}$ cluster varieties

Compactification (go back to toric geometry)

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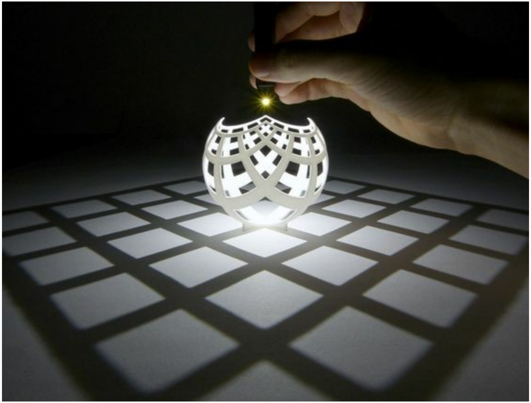
To compactify it:

$$\mathbb{C}^* \subset \mathbb{C} \subset \mathbb{CP}^1.$$



Projective toric varieties

Motivating example: $\mathbb{CP}^2$

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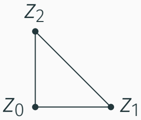

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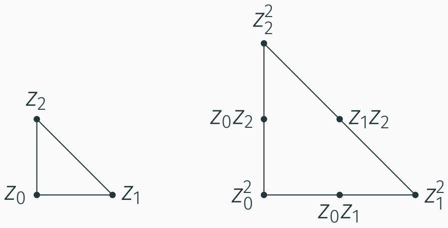

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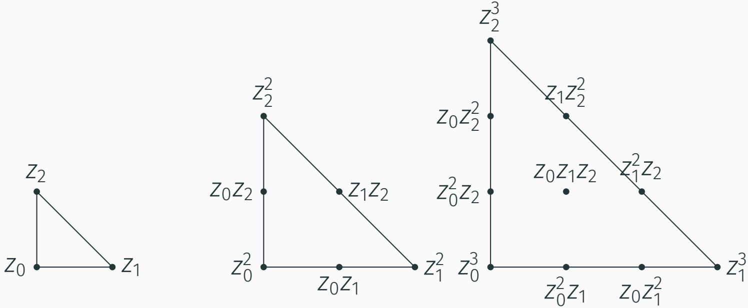

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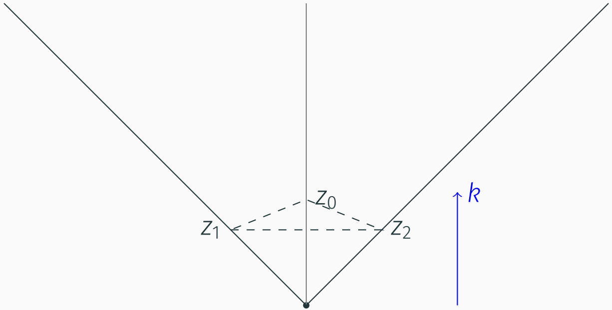

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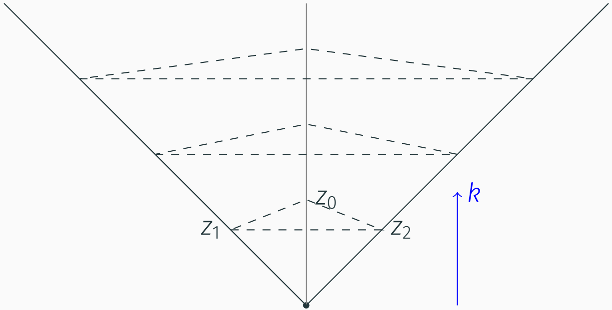

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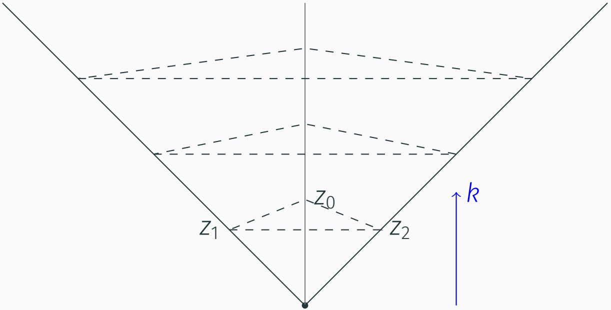

Polytope construction:

Convex lattice polytope Δ in $\mathbb{R}^n$

$\rightsquigarrow$ a graded ring (graded by k)

$$S_\Delta = \langle z^m \rangle_{m \in k\Delta}.$$

$\rightsquigarrow$ projective toric geometry $\mathbb{P}_\Delta = \mathrm{Proj}(S_\Delta)$.

Scattering diagrams

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Wall : $(\mathfrak{d}, f_{\mathfrak{d}})$

- $\mathfrak{d} \subseteq M_{\mathbb{R}}$ support of wall - convex rational polyhedral cone of codim 1, contained in $n^{\perp}$, $n \in N$.
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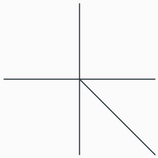
Example: A_2 (Note that $z^{(m_1, m_2)} = A_1^{m_1} A_2^{m_2}$.)

$$1 + z^{(0,1)} = 1 + A_2$$



$$1 + z^{(-1,0)} = 1 + A_1^{-1}$$

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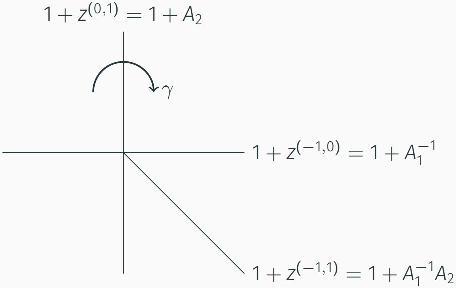

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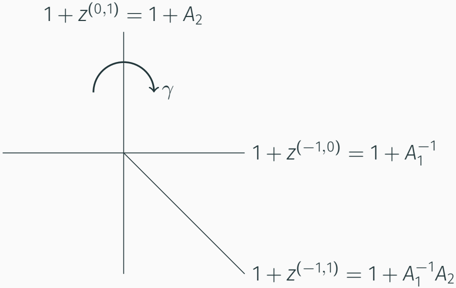
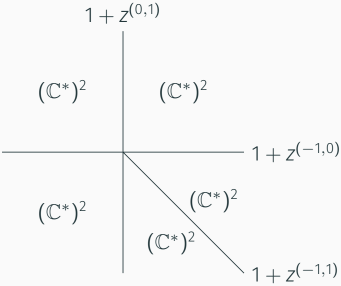


$$z^{(-1,0)} \mapsto z^{(-1,0)}(1 + z^{(0,1)}) = \frac{1 + A_2}{A_1}.$$

Associate each maximal cone of the scattering diagrams with $(\mathbb{C}^*)^n$



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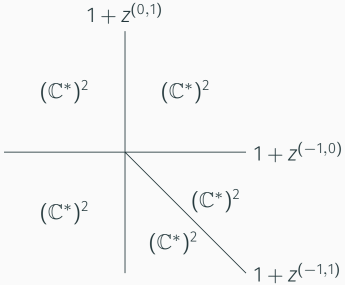

$f_{\mathfrak{d}} \rightsquigarrow$ wall crossing $\rightsquigarrow$ gluing the $(\mathbb{C}^*)^2$'s.

$\Rightarrow \mathcal{A}$-cluster variety of type A_2

Theta functions

Motivating example: functions on $(\mathbb{C}^*)^2$:

$$1 + c_1 z_1^{a_1} z_2^{b_1} + c_2 z_1^{a_2} z_2^{b_2} + \cdots \in \bigoplus_{m_1, m_2 \in \mathbb{Z}} \mathbb{C} z_1^{m_1} z_2^{m_2}.$$

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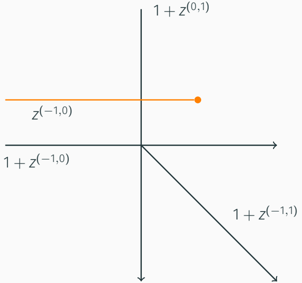

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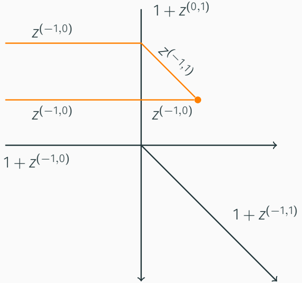

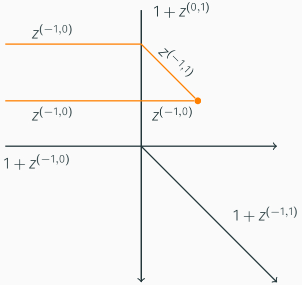

$$\vartheta_{Q,(-1,0)} = z^{(-1,0)} + z^{(-1,1)}.$$

Vector space generated by theta functions as an algebra

Motivating example: $H^0((\mathbb{C}^*)^2, \mathcal{O}) = \bigoplus_{m_1, m_2 \in \mathbb{Z}^2} \mathbb{C} z_1^{m_1} z_2^{m_2}$.

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[Gross-Hacking-Keel-Konsevich]

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where $L = M^\circ$ or N, α_{pq}^r structure constant.

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$\star$ gives **algebra structure** to the vector space generated by theta functions.

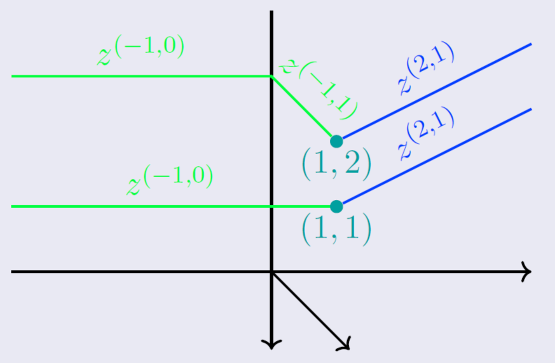
α_{pq}^r are expressed in terms of broken lines:

$$\alpha_{pq}^r := \sum c(\gamma^{(1)})\ c(\gamma^{(2)}),$$

where summing over pairs of broken lines $(\gamma^{(1)}, \gamma^{(2)})$ such that $I(\gamma^{(1)}) = p,\ I(\gamma^{(2)}) = q, \gamma^{(1)}(0) = \gamma^{(2)}(0) = r, F(\gamma^{(1)}) + F(\gamma^{(2)}) = r$

Example:

$$\vartheta_{(-1,0)} \cdot \vartheta_{(2,1)} = \vartheta_{(1,1)} + \vartheta_{(1,2)}.$$



Positive polytope

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Definition

A closed subset $S \subseteq L_{\mathbb{R}}$ is *positive* if
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$\star$ [C-Magee-Nájera Chávez] construct the correspondence between those two broken lines and broken line segments with (scaling of) the endpoints p, q and r.

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$\rightsquigarrow$ get graded ring R

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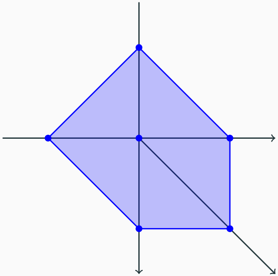
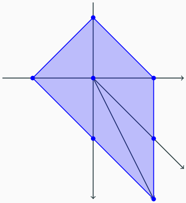
Example: Type A_2



[Gross-Hacking-Keel-Kontsevich]del Pezzo surface of degree 5

Compactification

Type B_2:



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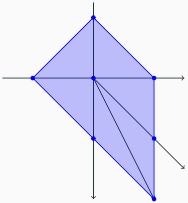

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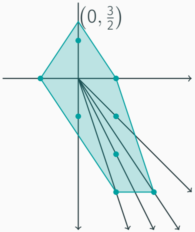
Type G_2



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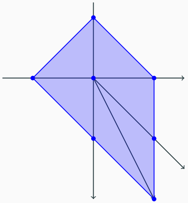

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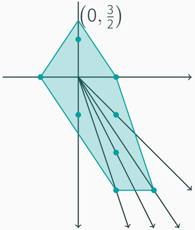

non-integral point coming from bending of broken line!

Any evidence?
Why we care?

Grassmannian

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The NO bodies are positive polytopes.

Non-integral example from NO body calculation: $\mathrm{Gr}_3(\mathbb{C}^6)$.

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Non-integral example from NO body calculation: $\mathrm{Gr}_3(\mathbb{C}^6)$.

[Bossinger-C-Magee-Nájera Chávez] Get the non-integral point from broken line convexity!

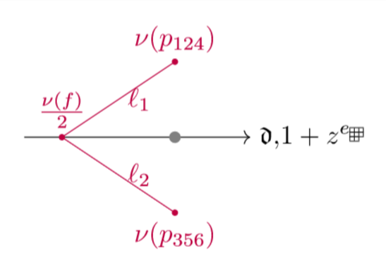

Figure 1: Part of the scattering diagram of $\mathrm{Gr}_3(\mathbb{C}^6)$.

$$\frac{\nu(f)}{2} = \left(\frac{1}{2}, 1, \frac{3}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}\right)$$

Intrinsic NO body

Idea behind the example $\mathrm{Gr}_3(\mathbb{C}^6)$ holds in general context

Intrinsic NO body

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[Bossinger-C-Magee-Nájera Chávez] defines Intrinsic Newton-Okounkov body by considering **broken line convex polytopes** instead of convex polytopes.

$\rightarrow$ 'usual' Newton-Okounkov body without taking closure.

Thank you!