Kazhdan–Lusztig Equivalence at the Iwahori Level

Yuchen Fu

RIMS, Kyoto University

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Overview

Kazhdan-Lusztig Equivalence

Theorem (D. Kazhdan and G. Lusztig '94)

If $c \in \mathbb{C} \setminus \mathbb{Q}$, or $c \in \dfrac{m}{n} \in \mathbb{Q}^{<0}$ for $(m, n) = 1$ and m not too small, then there exists a braided monoidal equivalence $\mathrm{KL}_\kappa(G)^\heartsuit \simeq \mathrm{Rep}_q(G)^\heartsuit$.

- $\hat{\mathfrak{g}}_\kappa$: Central extension of $\mathfrak{g}((t))$ given by the 2-cocycle
$$\kappa := \frac{c - h^\vee}{2h^\vee}\mathrm{Kil}_\mathfrak{g}$$
- $\mathrm{KL}_\kappa(G)^\heartsuit$: Abelian category of finitely generated, smooth, $G[[t]]$-integrable $\hat{\mathfrak{g}}_\kappa$-modules at level κ
- $U_q^{\mathrm{Lus}}(\mathfrak{g})$: Lusztig's quantum group specialized at $q := e^{\frac{\pi i}{dc}}$, where d is the lacing number of $\mathfrak{g}$
- $\mathrm{Rep}_q(G)^\heartsuit$: Abelian category of finite dimensional $\check{\Lambda}$-graded $U_q^{\mathrm{Lus}}(\mathfrak{g})$-modules, where $\check{\Lambda}$ is the weight lattice

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- $\hat{\mathfrak{g}}_\kappa\text{-mod}^I$, the derived category of $(\hat{\mathfrak{g}}_\kappa, I)$-Harish-Chandra modules, where I is the Iwahori subgroup;
- $\mathrm{Rep}_q^{\mathrm{mxd}}(G)$, the derived category of "mixed" quantum group representations (coming up!)

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At generic levels both are equivalent to $\mathfrak{g}\text{-mod}^B$. Rational levels are more interesting.

Main Result

Theorem (L. Chen and Y.F.; Conjectured by D. Gaitsgory)

If $c \in \mathbb{C} \setminus \mathbb{Q}$, or $c \in \dfrac{m}{n} \in \mathbb{Q}$ for $(m, n) = 1$ and m not too small, then there exists an equivalence of (DG) categories

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- Renormalization is necessary for both sides; after doing so, neither side is the derived category of its heart. The equivalence is *not* t-exact;
- The proof is independent from the original one by K-L. Comparison with K-L is ongoing work;
- The RHS carries a braided monoidal structure (compatible with $\mathrm{Rep}_q(G)^{\heartsuit}$); consequently it equips LHS with a braided monoidal structure. We do not yet know how to describe it explicitly.

Context: Quantum Geometric Langlands

Recall the (conjectural) unramified global geometric Langlands equivalence:

$$\mathrm{DMod}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}})$$

here Bun_G is the moduli of G-bundles on a smooth complete curve X, and $\mathrm{LocSys}_{\check{G}}$ is the moduli of $\check{G}$-local systems on X.

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Recent works of Arinkin, Gaitsgory, Beraldo and Chen reduce the above to the following "tempered version":

$$\mathrm{DMod}_{\mathrm{temp}}(\mathrm{Bun}_G) \simeq \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$$

which are full subcategories of the two sides of above (the rest comes from parabolic induction from proper Levi). We shall not define what LHS is.

In establishing the tempered version the following[1] is crucial:

Theorem (Geometric Casselman-Shalika Formula)

There exists an equivalence of factorization categories

$$\mathrm{Whit}(\mathrm{Gr}_G) \simeq \mathrm{Rep}(\check{G}).$$

Here $\mathrm{Whit}(\mathrm{Gr}_G) := \mathrm{DMod}(\mathrm{Gr}_G)^{N((t)),\chi}$, where $\chi : N((t)) \to \mathbb{G}_a$ is a non-degenerate character of $N((t))$.

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We shall say what factorization categories are in a minute. Intuitively, these are categories that "can move along the curve" and "can be integrated". And, *very* roughly speaking, we have

$$\mathrm{DMod}_{\mathrm{temp}}(\mathrm{Bun}_G) \simeq \int_X \mathrm{Whit}(\mathrm{Gr}_G) \simeq \int_X \mathrm{Rep}(\check{G}) \simeq \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$$

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Now we move away from the critical level. The usual form of the global quantum Langlands conjecture is

$$\mathrm{DMod}_{\kappa}(\mathrm{Bun}_G) \simeq \mathrm{DMod}_{\check{\kappa}}(\mathrm{Bun}_{\check{G}})$$

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What replaces the Casselman-Shalika formula is the following:

Conjecture (Fundamental Local Equivalence)

There exists an equivalence of factorization categories

$$\mathrm{Whit}_{\kappa}(\mathrm{Gr}_G) \simeq \mathrm{KL}_{\check{\kappa}}(\check{G})$$

This statement is also key to the *categorical local* geometric Langlands corresondence (i.e. what happens to S^1 for the geometric Langlands QFT).

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$$\mathrm{Whit}_{\kappa}(\mathrm{Fl}_G) \simeq \hat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{-mod}^{\check{I}}_{\mathrm{ren}};$$

here Fl_G *is the affine flag variety.*

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Our secret hope is that (2) is extendable to the factorization setting.

Proof Strategy

The following strategy works (only) for $c > 0$. The $c < 0$ case follows formally via categorical duality.

$$
\begin{tikzcd}
\hat{\mathfrak{g}}_\kappa\text{-mod}^I_{\mathrm{ren}} \arrow[r, dashed] \arrow[d, "J^{\mathrm{KM}}_*"', "\simeq"] & \mathrm{Rep}^{\mathrm{mxd}}_q(G)_{\mathrm{ren}} \arrow[d, "\simeq"', "J^{\mathrm{Quant}}_*"] \\
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In general, given a lax monoidal functor $F : C \to D$ between monoidal categories, it automatically factors as

$$C \simeq \mathbf{1}_C\text{-mod}(C) \xrightarrow{F_{\mathrm{enh}}} F(\mathbf{1}_C)\text{-mod}(D) \xrightarrow{\mathrm{oblv}} D;$$

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F_{enh} is more likely to be an equivalence. Our J^{KM}_* and J^{Quant}_* will follow the *factorizable* ($\approx$ braided monoidal) version of this pattern.

Factorization Objects

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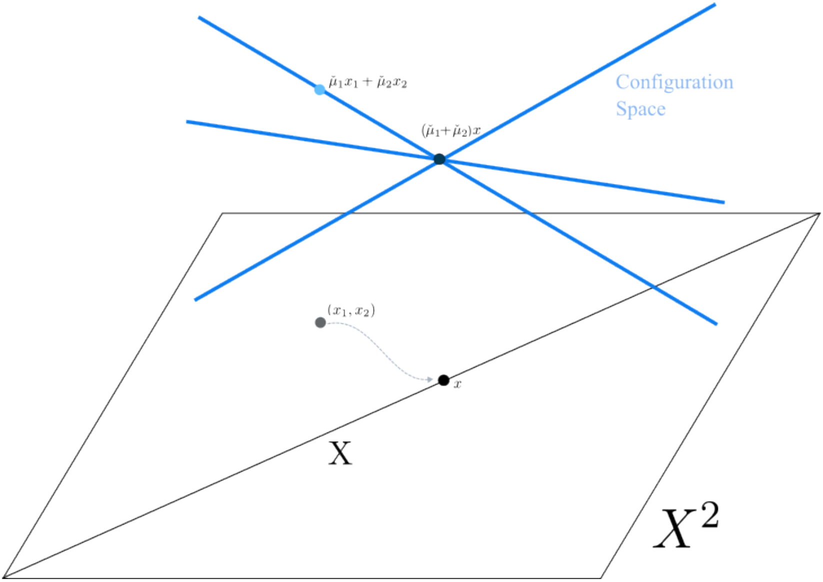
Over main diagonal, the configuration space is $X^{\check{\Lambda}}$; over $X^2 \setminus X$ it is $X^{\check{\Lambda}\times\check{\Lambda}}$. The additional data includes an isomorphism

$$\iota^!_{\check{\lambda}x+\check{\mu}y}(A) \simeq \iota^!_{\check{\lambda}x}(A) \otimes \iota^!_{\check{\mu}y}(A)$$

for all $\check{\lambda}, \check{\mu}, x \neq y$.


$\check{\mu}_1 x_1 + \check{\mu}_2 x_2$
$(\check{\mu}_1 + \check{\mu}_2)x$
Configuration Space
(x_1, x_2)
x
X
X^2

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Then we can associated to it a bialgebra $B(A)$, such that $\iota^!_{\check{\lambda}\cdot x}(A)$ is the $\check{\lambda}$-component of $\mathrm{Ext}_{B(A)^\vee}(\mathbf{1},\mathbf{1})$, and $\iota^*_{\check{\lambda}\cdot x}(A)$ is the $\check{\lambda}$-component of $\mathrm{Tor}_{B(A)}(\mathbf{1},\mathbf{1})$.

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Similarly, a factorization module for A encodes simultaneously an $B(A)$-module structure and an $B(A)$-comodule structure. Together: a Yetter-Drinfeld module structure.

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This story on the abelian level is well understood by the works of Bezrukavnikov, Finkelberg, Schechtman, Kapranov et al, using hyperbolic restriction.

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Mixed Quantum Groups

Recall that both the Lusztig algebra $U_q^{\mathrm{Lus}}(\mathfrak{n})$ and the Kac-De Concini algebra $U_q^{\mathrm{KD}}(\mathfrak{n})$ can be realized as Hopf algebras *internal* to $\mathrm{Rep}_q(T)^{\heartsuit}$.

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The abelian category $\mathrm{Rep}_q^{\mathrm{mxd}}(G)^{\heartsuit}$ consists of $V \in \mathrm{Rep}_q(T)^{\heartsuit}$ with a *locally nilpotent* $U_q^{\mathrm{Lus}}(\mathfrak{n})$ action and a compatible (arbitrary) $U_q^{\mathrm{KD}}(\mathfrak{n}^-)$ action.

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The DG category $\mathrm{Rep}_q^{\mathrm{mxd}}(G)_{\mathrm{ren}}$ is a certain modification (at cohomological level $-\infty$) of $D(\mathrm{Rep}_q^{\mathrm{mxd}}(G)^\heartsuit)$.

Proposition

There exists an $\mathbb{E}_2$-algebra ($\simeq$ topological factorization algebra) Ω^{Quant} and an equivalence of DG categories

$$J_*^{\mathrm{Quant}} : \mathrm{Rep}_q^{\mathrm{mxd}}(G)_{\mathrm{ren}} \simeq \Omega^{\mathrm{Quant}}\text{-}\mathrm{mod}^{\mathbb{E}_2}(\mathrm{Rep}_q(T))$$

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At abelian level, this is analogous to the main result of [BFS06]. We use Koszul duality and [Lur12] instead (thus give a new proof to [BFS06]).

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Remark

$\iota^!_{\check{\lambda}\cdot 0}(J_*^{\mathrm{Quant}}(M))$ is the $\check{\lambda}$-component of $\mathrm{Ext}^\bullet_{U_q^{\mathrm{Lus}}(\mathfrak{n})}(\mathbb{C}, M)$, and
$\iota^*_{\check{\lambda}\cdot 0}(J_*^{\mathrm{Quant}}(M))$ is the $\check{\lambda}$-component of $\mathrm{Tor}^\bullet_{U_q^{\mathrm{KD}}(\mathfrak{n}^-)}(\mathbb{C}, M)$.

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- Exhibit $\mathrm{Rep}_q^{\mathrm{mxd}}(G)_{\mathrm{ren}}$ as the *Hochschild center* (HC) of

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- If A is a commutative algebra, [Fra12] showed that

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- We establish a *categorical* Verdier duality to switch between factorization *cosheaves of categories* (coming from $\mathbb{E}_2$ via Lurie) and factorization *sheaves of categories* (this is the second equivalence).

Lie Algebra Representation via Coherent Sheaves

Let $G_1^\wedge$ denote the formal completion of G at the identity, and $\mathbb{B}G_1^\wedge$ its classifying prestack.

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$$\mathfrak{g}((t))\text{-mod}_{\mathrm{ren}}^{G[[t]]} \simeq \mathrm{IndCoh}^{!}_{\mathrm{ren}}(\mathbb{B}G((t))^\wedge_{G[[t]]}),$$

where renormalization on both sides mean taking the ind-completion of the category of objects induced from finite dimensional *smooth* representations of $G[[t]]$.

To each κ one can assign a *twisting* (an infinitesimal gerbe) on $\mathbb{B}G((t))^{\wedge}_{G[[t]]}$ and use it to twist the IndCoh category. A slight variant of above is

$$\mathrm{KL}_{\kappa}(G)_{\mathrm{ren}} := \mathrm{IndCoh}^{!}_{\mathrm{ren},\kappa}(\mathbb{B}G((t))^{\wedge}_{G[[t]]}).$$

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Proposition ([Ras20])

When restricted to bounded-below objects, the functor

$$\mathrm{KL}_{\kappa}(B)_{\mathrm{ren}} \simeq \mathrm{IndCoh}^{!}_{\mathrm{ren},\kappa}(\mathbb{B}B((t))^{\wedge}_{B[[t]]}) \xrightarrow[\simeq]{\spadesuit} \mathrm{IndCoh}^{*}_{\mathrm{ren},\kappa-\kappa_{\mathrm{crit}}}(\mathbb{B}B((t))^{\wedge}_{B[[t]]})$$

$$\xrightarrow{*\text{-push}} \mathrm{IndCoh}^{*}_{\mathrm{ren},\kappa-\kappa_{\mathrm{crit}}}(\mathbb{B}T((t))^{\wedge}_{T[[t]]}) \simeq \mathrm{KL}_{\kappa-\kappa_{\mathrm{crit}}}(T)_{\mathrm{ren}}$$

coincides with Feigin's semi-infinite cohomology $C_{*}^{\frac{\infty}{2}}(\mathfrak{n}((t)), N[[t]], -)$.

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The behavior as we approach the diagonal encodes the *fusion* structure of $\mathrm{KL}_\kappa(G)$.

Unitality means, for instance, that $\{x\} \hookrightarrow \{x, y\}$ yields a map

$$\mathrm{ins}_{x \rightsquigarrow (x,y)} : \mathrm{KL}_\kappa(G)_{\mathrm{ren}} \to \mathrm{KL}_\kappa(G)_{\mathrm{ren}} \otimes \mathrm{KL}_\kappa(G)_{\mathrm{ren}}$$

given by $M \mapsto \mathbb{V}_\kappa^0 \boxtimes M$, where

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It encodes the *fusion action* of $\mathrm{KL}_\kappa(G)$ on $\hat{\mathfrak{g}}_\kappa\text{-mod}^I$, originally due to Finkelberg.

General yoga of factorization categories gives

$$\hat{\mathfrak{g}}_\kappa\text{-mod}^I_{\mathrm{ren}} \simeq \mathbb{V}^0_\kappa\text{-FactMod}(\mathcal{IKL}_\kappa(G)),$$

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For simplicity we write $C^{\frac{\infty}{2}} := C_*^{\frac{\infty}{2}}(\mathfrak{n}((t)), N[[t]], -)$. The map

$$\hat{\mathfrak{g}}_\kappa\text{-mod}^I_{\mathrm{ren}} \xrightarrow{\mathrm{Res}} \mathrm{KL}_\kappa(B)_{\mathrm{ren}} \xrightarrow{C^{\frac{\infty}{2}}} \mathrm{KL}_{\kappa-\kappa_{\mathrm{crit}}}(T)$$

is a *lax-unital factorizable* functor (this is the analogue of being lax $\mathbb{E}_2$), and thus factors through an "enhanced" map

$$C^{\frac{\infty}{2}}_{\mathrm{enh}} : \hat{\mathfrak{g}}_\kappa\text{-mod}^I_{\mathrm{ren}} \to C^{\frac{\infty}{2}}(\mathbb{V}^0_\kappa)\text{-FactMod}(\mathcal{KL}_\kappa(T)).$$

Proposition ("Torus FLE")

There exists an equivalence of factorizable crystals of categories

$$\mathrm{FLE}_T : \mathcal{KL}_\kappa(T)_{\mathrm{ren}} \simeq \mathrm{DMod}_{\check{\kappa}}(\mathrm{Gr}_{\check{T}});$$

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$$J_*^{\mathrm{KM}} := \mathrm{FLE}_T \circ C_{\mathrm{enh}}^{\frac{\infty}{2}} : \hat{\mathfrak{g}}_\kappa\text{-mod}_{\mathrm{ren}}^I \to \Omega^{\mathrm{KM}}\text{-FactMod}(\mathrm{DMod}_{\check{\kappa}}(\mathrm{Gr}_{\check{T}})).$$

Recall our strategy:

$$\begin{tikzcd}
\hat{\mathfrak{g}}_\kappa\text{-mod}^I_{\mathrm{ren}} \arrow[r, dashed] \arrow[d, "J^{\mathrm{KM}}_*"', "\simeq"] & \mathrm{Rep}^{\mathrm{mxd}}_q(G)_{\mathrm{ren}} \arrow[d, "\simeq"', "J^{\mathrm{Quant}}_*"] \\
\Omega^{\mathrm{KM}}\text{-FactMod}_{\mathrm{alg}} \arrow[r, "\simeq", "\text{Riemann-Hilbert}"'] & \Omega^{\mathrm{Quant}}\text{-FactMod}_{\mathrm{top}}
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One can use direct computation (using e.g. Kashiwara-Tanisaki localization) to achieve this.

Here's the precise meaning in case anyone wants to see:

Proposition

There exists an unique $\check{\Lambda}^{<0}$-graded factorization algebra Ω such that:

- *if $\check{\lambda} \notin \check{\Lambda}^{<0}$, then the !-fiber at $\check{\lambda}x$ is zero;*
- *the !-fiber at every $\check{\lambda}x$ has no negative cohomology;*
- *if $\check{\lambda}$ is a simple negative root, then either the $*$-fiber at $\check{\lambda}x$ is $\mathbb{C}[1]$, or the !-fiber at $\check{\lambda}x$ is $\mathbb{C}[-1]$;*
- *if $\check{\lambda}$ equals $w(\check{\rho}) - \check{\rho}$ for some $\ell(w) = 2$, then the !-fiber at $\check{\lambda}x$ vanishes at H^0 and H^1, and $*$-fiber at $\check{\lambda}x$ vanishes at H^0 and H^{-1};*
- *otherwise, the !-fiber at $\check{\lambda}x$ vanishes at H^0, and $*$-fiber at $\check{\lambda}x$ vanishes at H^0, H^{-1} and H^{-2}.*

Proving J_*^{KM} is an Equivalence

The category Ω^{KM}-$\mathrm{FactMod}_{\mathrm{alg}}$ has a *highest weight* structure: it contains *standard objects* which are compact generators, and *costandard objects* which are their right orthogonals.

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Remark

Our choice is made such that $\mathrm{Ext}_{\hat{\mathfrak{g}}_\kappa\text{-mod}^I_{\mathrm{ren}}}(M^{!,\check{\lambda}}_{\mathrm{KM}}, N)$ gives the $\check{\lambda}$-component of $C^{\frac{\infty}{2}}(N)$.

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Our choice is made such that $\mathrm{Ext}_{\hat{\mathfrak{g}}_\kappa\text{-mod}^I_{\mathrm{ren}}}(M^{!,\check{\lambda}}_{\mathrm{KM}}, N)$ gives the $\check{\lambda}$-component of $C^{\frac{\infty}{2}}(N)$.

It follows from definition that $J^{\mathrm{KM}}_*(M^{*,\check{\lambda}}_{\mathrm{KM}}) \simeq M^{*,\check{\lambda}}_{\mathrm{fact}}$.

Wakimoto modules are the $\frac{\infty}{2}$-analogues of Verma modules.

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To show $M^{!,\check{\lambda}}_{\mathrm{KM}} \mapsto M^{!,\check{\lambda}}_{\mathrm{fact}}$ it suffices to compute the $*$-fiber of $M^{!,\check{\lambda}}_{\mathrm{KM}}$ at every $\check{\mu}x$. This is much less straightforward.

Localization

Fix a collection $\overrightarrow{x}$ of r points on $\mathbb{P}^1$. Set $\mathrm{Bun}_G(\mathbb{P}^1)_{\overrightarrow{x}} := \mathrm{Bun}_G(\mathbb{P}^1) \times_{(\mathrm{pt}/G)^r} (\mathrm{pt}/B)^r$. There exists a *localization* (a.k.a. compactification) functor

$$\mathrm{Loc}_G^{\overrightarrow{x}} : (\hat{\mathfrak{g}}_\kappa\text{-mod}^I)^{\otimes r} \to \mathrm{DMod}_\kappa(\mathrm{Bun}_G(\mathbb{P}^1)_{\overrightarrow{x}}),$$

where the !-fiber at the trivial bundle is given by conformal block of the r modules (placed at $\overrightarrow{x}$) over $\mathbb{P}^1$.

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Work of N. Rozenblyum tells us that there is also a *chiral localization* functor

$$\mathrm{Loc}_{T,\Omega}^{\overrightarrow{x}} : C^{\frac{\infty}{2}}(\mathbb{V}_\kappa^0)\text{-FactMod}_{\overrightarrow{x}}(\mathrm{KL}_{\kappa-\kappa_{\mathrm{crit}}}(T)_{\mathrm{ren}}) \to \mathrm{DMod}_{\kappa-\kappa_{\mathrm{crit}}}(\mathrm{Bun}_T(\mathbb{P}^1));$$

the !-fiber is more interesting here (intuitively, it computes conformal block with $C^{\frac{\infty}{2}}(\mathbb{V}_\kappa^0)$ occupying all points away from $\overrightarrow{x}$).

Let $\mathrm{CT}_* : \mathrm{DMod}_\kappa(\mathrm{Bun}_G(\mathbb{P}^1)_{\overrightarrow{x}}) \to \mathrm{DMod}_{\kappa-\kappa_{\mathrm{crit}}}(\mathrm{Bun}_T(\mathbb{P}^1))$ denote the !-pull-$*$-push along

$$\begin{tikzcd} & \mathrm{Bun}_B(\mathbb{P}^1) \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_G(\mathbb{P}^1)_{\overrightarrow{x}} & & \mathrm{Bun}_T(\mathbb{P}^1) \end{tikzcd}$$

(followed by a κ_{crit} shift).

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(followed by a κ_{crit} shift).

A central result we prove is the commutativity of the following diagram:

$$\begin{tikzcd} (\hat{\mathfrak{g}}_\kappa\text{-mod}^I)^{\otimes r} \arrow[r, "C^{\frac{\infty}{2}}"] \arrow[d, "\mathrm{Loc}_G^{\overrightarrow{x}}"] & C^{\frac{\infty}{2}}(\mathbb{V}_\kappa^0)\text{-}\mathrm{FactMod}_{\overrightarrow{x}}(\mathrm{KL}_{\kappa-\kappa_{\mathrm{crit}}}(T)_{\mathrm{ren}}) \arrow[d, "\mathrm{Loc}_{T,\Omega}^{\overrightarrow{x}}"] \\ \mathrm{DMod}_\kappa(\mathrm{Bun}_G(\mathbb{P}^1)_{\overrightarrow{x}}) \arrow[r, "\mathrm{CT}_*"'] & \mathrm{DMod}_{\kappa-\kappa_{\mathrm{crit}}}(\mathrm{Bun}_T(\mathbb{P}^1)) \end{tikzcd}$$

from which the $*$-fibers can be computed, via contraction principle.

The diagram above is very non-trivial; in particular, it crucially relies on the *unital* factorization structure.

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- The propagation-restriction method ("conformal blocks can be computed relative to any background theory");
- The factorization homology of a *commutative* factorization algebra is the ring of functions of the *space of horizontal sections*;
- Bun_N is a *co-affine stack*, in the sense that

$$\mathrm{Bun}_N(R) \simeq \mathrm{Maps}_{\mathrm{CAlg}}(C^*(\mathrm{Bun}_N), R)$$

  for any connective (derived) commutative algebra R.

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