$D_5^{(1)}$ and $D_6^{(1)}$-Geometric Crystals

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Definition

A ***Lie Algebra*** $\mathbf{L}$ is a vector space over the field $\mathbb{C}$ together with an operation (called the ***bracket***), $[\ ,\ ] : \mathbf{L} \times \mathbf{L} \to \mathbf{L}$ such that for all $x, y, z \in \mathbf{L}$ and $a, b \in \mathbb{C}$,

1. $[ax + by, z] = a[x, z] + b[y, z]$ and $[x, ay + bz] = a[x, y] + b[x, z]$,
2. $[x, x] = 0$,
3. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (Jacobi identity).

Definition

- A Lie algebra $\mathbf{L}$ is ***simple*** if $[\mathbf{L}, \mathbf{L}] \neq \{0\}$, and its only ideals are $\{0\}$ and itself.
- A Lie algebra $\mathbf{L}$ is ***semisimple*** if it is a direct sum of simple Lie algebras.

Generalized Cartan matrix (GCM)

Definition

An $n \times n$ integral matrix $A = (a_{ij})$ is a ***GCM*** if $a_{ii} = 2, a_{ij} \leq 0$ if $i \neq j$, and $a_{ij} = 0$ if and only if $a_{ji} = 0$.

Definition

1. A GCM A is ***indecomposable*** if it is not equivalent to a matrix in block form.
2. A GCM A is ***symmetrizable*** if there exists a nonsingular diagonal matrix D such that DA is symmetric.

An indecomposable symmetrizable GCM A is of ***affine*** type if there exists $u > 0$ such that $Au = 0$.

Definition

The ***Cartan datum*** associated with the symmetrizable GCM $A = (a_{ij})_{i,j \in I}$ is a quintuple $(A, \Pi, \check{\Pi}, P, \check{P})$ where

$\check{P} = \text{span}_{\mathbb{Z}}\{\{\check{\alpha}_1, \ldots, \check{\alpha}_n\} \cup \{d_s | s = 1, \ldots, |I| - \text{rank } A\}\}$ is a free abelian group of rank $2|I| - \text{rank } A$ called the ***coweight lattice,***

Define $\mathfrak{t} = \mathbb{C} \otimes_{\mathbb{Z}} \check{P}$ to be the complex extension of $\check{P}$ called the ***Cartan subalgebra,***

$P = \{\lambda \in \mathfrak{t}^* | \lambda(\check{P}) \subset \mathbb{Z}\}$ is called the ***weight lattice,***

$\check{\Pi} = \{\check{\alpha}_1, \ldots, \check{\alpha}_n\} \subset \mathfrak{t}$ is called the set of ***simple coroots,***

$\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset \mathfrak{t}^*$ is called the set of ***simple roots*** which satisfy $\alpha_j(\check{\alpha}_i) = a_{ij}$ and $\alpha_j(d_s) = \delta_{sj}$.

Definition

The ***Kac-Moody algebra*** $\mathfrak{g} = \mathfrak{g}(A)$ associated with the Cartan datum $(A, \Pi, \check{\Pi}, P, \check{P})$ is a Lie algebra with generators e_i, f_i $(i \in I)$ and $h \in \check{P}$ satisfying the following relations.

1. $[h, h'] = 0$ for $h, h' \in \check{P}$,
2. $[e_i, f_j] = \delta_{ij}\check{\alpha}_i$,
3. $[h, e_i] = \alpha_i(h)e_i$ for $h \in \check{P}$,
4. $[h, f_i] = -\alpha_i(h)f_i$ for $h \in \check{P}$,
5. $(\mathrm{ad}\ e_i)^{1-a_{ij}}(e_j) = 0$ for $i \neq j$,
6. $(\mathrm{ad}\ f_i)^{1-a_{ij}}(f_j) = 0$ for $i \neq j$.

An ***affine Lie algebra*** is a Kac-Moody algebra for which the GCM A is of affine type.

Definition

The ***quantum group*** $U_q(\mathfrak{g})$ associated with $(A, \Pi, \check{\Pi}, P, \check{P})$ is the associative algebra over $\mathbb{C}(q)$ with 1 generated by e_i, f_i $(i \in I)$ and q^h $(h \in \check{P})$ satisfying the following relations.

1. $q^0 = 1,\ q^h q^{h'} = q^{h+h'}$ for $h, h' \in \check{P}$,
2. $e_i f_j - f_j e_i = \delta_{ij} \frac{q^{d_i \check{\alpha}_i} - q^{-d_i \check{\alpha}_i}}{q^{d_i} - q^{-d_i}}$ for $i, j \in I$,
3. $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$ for $h \in \check{P},\ i \in I$,
4. $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$ for $h \in \check{P},\ i \in I$,
5. $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ for $i \neq j$,
6. $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ for $i \neq j$.

The quantum group associated with the affine Cartan datum $(A, \Pi, \check{\Pi}, P, \check{P})$ is called a ***quantum affine algebra***, also denoted by $U_q(\mathfrak{g})$.

Let $\lambda \in P^+ = \{\mu \in P \mid \mu(\check{\alpha}_i) \in \mathbb{Z}_{\geq 0}$ for all $i \in I\}$ of a level $l = \lambda(\mathbf{c})$ ($\mathbf{c}$ = the canonical central element) and $V^q(\lambda)$ be the irreducible integrable highest weight $U_q(\mathfrak{g})$-module.

Note: As $q \to 1$, $V^1(\lambda)$ is an irreducible integrable highest weight $\mathfrak{g}$-module.

Definition

[K, 1990] A ***crystal base*** of V is a pair (L, B) such that

1. L is a crystal lattice for V,
2. B is a $\mathbb{C}$-basis of $L/qL \cong \mathbb{C} \otimes_{\mathbf{A}_0} L$ where $\mathbf{A}_0$ is the ring of rational functions regular at $q = 0$,
3. $B = \sqcup_{\lambda \in P} B_\lambda$ where $B_\lambda = B \cap (L_\lambda / qL_\lambda)$,
4. $\tilde{e}_i B \subset B \cup \{0\}, \tilde{f}_i B \subset B \cup \{0\}$ for all $i \in I$,
5. $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$ for any $b, b' \in B$ and $i \in I$.

A ***perfect crystal*** is indeed a crystal for certain finite dimensional module called Kirillov-Reshetikhin module (KR-module for short) of the quantum affine algebra $U_q(\mathfrak{g})$.

The KR-modules are parametrized by two integers (i, l) where $i \in I \setminus \{0\}$ and l any positive integer.

Let $\{B^{i,l}\}_{l \geq 1}$ be a family of perfect crystals. If it satisfies certain conditions, there exists a ***limit*** $B^{i,\infty}$ of $\{B^{i,l}\}_{l \geq 1}$. In such a case the family $\{B^{i,l}\}_{l \geq 1}$ is called a ***coherent family*** of perfect crystals.

Definition

[BK, 2000] The ***geometric crystal*** $\mathcal{V}(\mathfrak{g})$ for the simply laced affine Lie algebra $\mathfrak{g}$ is a quadruple

$$(X, \{e_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$$

where X is a variety, $e_i : \mathbb{C}^\times \times X \longrightarrow X$ $((c,x) \mapsto e_i^c(x))$ are rational $\mathbb{C}^\times$-actions and $\gamma_i, \varepsilon_i : X \longrightarrow \mathbb{C}$ $(i \in I)$ are rational functions satisfying the following:

1. $\{1\} \times X \subset dom(e_i)$ for any $i \in I$
2. $\gamma_j(e_i^c(x)) = c^{a_{ij}}\gamma_j(x)$
3. $\{e_i\}_{i\in I}$ satisfy the following relations
$$\begin{aligned} e_i^{c_1}e_j^{c_2} &= e_j^{c_2}e_i^{c_1} && \text{if } a_{ij} = a_{ji} = 0,\\ e_i^{c_1}e_j^{c_1c_2}e_i^{c_2} &= e_j^{c_2}e_i^{c_1c_2}e_j^{c_1} && \text{if } a_{ij} = a_{ji} = -1,\end{aligned}$$
4. $\varepsilon_i(e_i^c(x)) = c^{-1}\varepsilon_i(x)$ and $\varepsilon_i(e_j^c(x)) = \varepsilon_i(x)$ if $a_{ij} = a_{ji} = 0$.

Remarkable relation between positive geometric crystal and algebraic crystal:

ultra-discretization functor $\mathcal{UD}$

Applying this functor, positive rational functions are transferred to piecewise linear functions by the simple correspondence:

$$x \times y \longmapsto x + y, \qquad \frac{x}{y} \longmapsto x - y, \qquad x + y \longmapsto max\{x, y\}.$$

Conjecture

It is conjectured in [KNO, 2008] that for each $k \in I \setminus \{0\}$, the affine Lie algebra $\mathfrak{g}$ has a positive geometric crystal whose ultra-discretization is isomorphic to the limit of certain coherent family of perfect crystals for the Langlands dual $\mathfrak{g}^L$ of $\mathfrak{g}$.

It has been shown that this conjecture is true for

- $k = 1$ and $\mathfrak{g} = B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}, D_{n+1}^{(2)}, D_4^{(3)}, G_2^{(1)}$
- $1 \leq k \leq n$ and $\mathfrak{g} = A_n^{(1)}$
- $k = 5$ and $\mathfrak{g} = D_5^{(1)}$, $k = 6$ and $\mathfrak{g} = D_6^{(1)}$

Outline

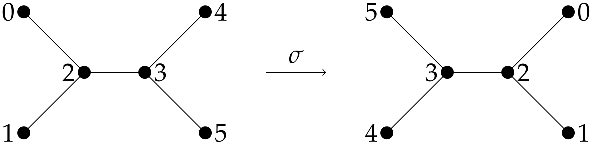
1. Section V : Affine Geometric Crystal $\mathcal{V}(D_5^{(1)})$
2. Section VI : Ultra-discretization of $\mathcal{V}(D_5^{(1)})$
3. Section VII : Affine Geometric Crystal $\mathcal{V}(D_6^{(1)})$
4. Section VIII : Ultra-discretization of $\mathcal{V}(D_6^{(1)})$

$\mathfrak{g} = D_5^{(1)}$ with index set $I = \{0, 1, 2, 3, 4, 5\}$

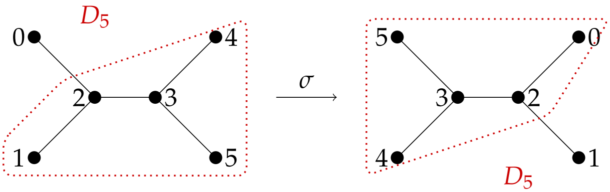
GCM

$$A = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{bmatrix}$$

Dynkin diagram



$$\sigma : 0 \mapsto 5, 1 \mapsto 4, 2 \mapsto 3, 3 \mapsto 2, 4 \mapsto 0, 5 \mapsto 1$$



Let $I_0 = \{1,2,3,4,5\}$ and $I_1 = \{0,2,3,4,5\}$.

Let $\mathfrak{g}_j$ (resp. $\sigma(\mathfrak{g})_j$)) be the subalgebra of $\mathfrak{g}$ (resp. $\sigma(\mathfrak{g})$) with index set I_j.

Then $\mathfrak{g}_0$ as well as $\sigma(\mathfrak{g})_1$ are isomorphic to D_5.

$\{\alpha_0, \alpha_1, \ldots, \alpha_5\}$ is the set of simple roots,
$\{\check{\alpha}_0, \check{\alpha}_1, \ldots, \check{\alpha}_5\}$ is the set of simple coroots,
$\{\Lambda_0, \Lambda_1, \ldots, \Lambda_5\}$ is the set of fundamental weights.

- The canonical central element is $\mathbf{c} = \check{\alpha}_0 + \check{\alpha}_1 + 2\check{\alpha}_2 + 2\check{\alpha}_3 + \check{\alpha}_4 + \check{\alpha}_5$.
- The null root is $\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5$.
- The classical weight lattice is $P_{cl} = \oplus_{j=0}^{5} \mathbb{Z}\Lambda_j$.
- The weight lattice is $P = P_{cl} \oplus \mathbb{Z}\delta$.

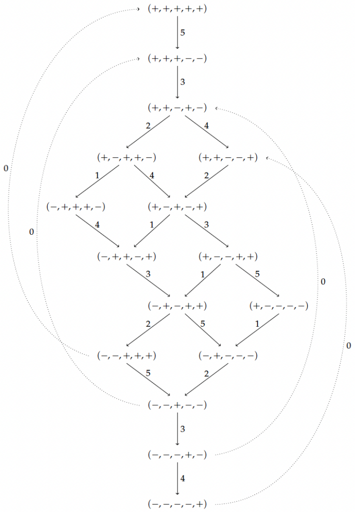
Let $W(\varpi_5)$ be the level 0 fundamental $U_q'(\mathfrak{g})$-module associated with the level 0 weight $\varpi_5 = \Lambda_5 - \Lambda_0$.

The fundamental $\mathfrak{g}$-module $W(\varpi_5)$ is a 16-dimensional module with the basis

$$\{(i_1, i_2, i_3, i_4, i_5) | i_k \in \{+, -\},\ i_1 i_2 i_3 i_4 i_5 = +\}.$$

The actions of f_k on these basis vectors is given by

$$f_k(i_1, i_2, i_3, i_4, i_5) = \begin{cases} (+, +, i_3, i_4, i_5) & \text{if } k = 0,\ (i_1, i_2) = (-, -) \\ (i_1, i_2, i_3, -, -) & \text{if } k = 5,\ (i_4, i_5) = (+, +) \\ (i_1, \ldots, \underset{k}{-}, \underset{k+1}{+}, \ldots, i_5) & \text{if } k \neq 0,\ k \neq 5,\ (i_k, i_{k+1}) = (+, -) \\ 0 & \text{otherwise.} \end{cases}$$



$(+,+,+,+,+)$ is a $\mathfrak{g}_0$ highest weight vector with weight $\varpi_5 = \Lambda_5 - \Lambda_0$,
$(-,+,+,+,-)$ is a $\sigma(\mathfrak{g})_1$ highest weight vector with weight $\check{\varpi}_5 := \Lambda_4 - \Lambda_1$.

Denote $\mathfrak{t}_{cl}^* = \mathfrak{t}^*/\mathbb{C}\delta$, $(\mathfrak{t}_{cl}^*)_0 = \{\lambda \in \mathfrak{t}_{cl}^* \mid \langle \mathbf{c}, \lambda \rangle = 0\}$.

For $\xi \in (\mathfrak{t}_{\mathrm{cl}}^*)_0$, let $t(\xi)$ be the translation as in [K, 2002].

Define simple reflections $s_k(\lambda) := \lambda - \lambda(\check{\alpha}_k)\alpha_k$, $k \in I$ and let $W = \langle s_k \mid k \in I \rangle$ be the Weyl group for $D_5^{(1)}$.

Proposition

$$t(\varpi_5) = \sigma s_4 s_3 s_2 s_5 s_3 s_4 s_1 s_2 s_3 s_5 = \sigma w_1,$$
$$t(\check{\varpi}_5) = \sigma s_5 s_3 s_2 s_4 s_3 s_5 s_0 s_2 s_3 s_4 = \sigma w_2.$$

Associated with Weyl group elements $w_1, w_2 \in W$, we define algebraic varieties $\mathcal{V}_1, \mathcal{V}_2 \subset W(\varpi_5)$ as follows.

$$\mathcal{V}_1 = \{V_1(x) := Y_4(x_4^{(2)})Y_3(x_3^{(3)})Y_2(x_2^{(2)})Y_5(x_5^{(2)})Y_3(x_3^{(2)})Y_4(x_4^{(1)})Y_1(x_1^{(1)}) \\ Y_2(x_2^{(1)})Y_3(x_3^{(1)})Y_5(x_5^{(1)})(+,+,+,+,+) | x_m^{(l)} \in \mathbb{C}^\times\},$$

$$\mathcal{V}_2 = \{V_2(y) := Y_5(y_5^{(2)})Y_3(y_3^{(3)})Y_2(y_2^{(2)})Y_4(y_4^{(2)})Y_3(y_3^{(2)})Y_5(y_5^{(1)})Y_0(y_0^{(1)}) \\ Y_2(y_2^{(1)})Y_3(y_3^{(1)})Y_4(y_4^{(1)})(-,+,+,+,-) | y_m^{(l)} \in \mathbb{C}^\times\},$$

where
$x = (x_4^{(2)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_3^{(2)}, x_4^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_5^{(1)})$,
$y = (y_5^{(2)}, y_3^{(3)}, y_2^{(2)}, y_4^{(2)}, y_3^{(2)}, y_5^{(1)}, y_0^{(1)}, y_2^{(1)}, y_3^{(1)}, y_4^{(1)})$.

From the explicit actions of f_k's on $W(\varpi_5)$, we have $f_k^2 = 0$, for all $k \in I$, hence

$$Y_k(c) = (1 + \frac{f_k}{c})\check{\alpha}_k(c) = (1 + \frac{f_k}{c})c^{\check{\alpha}_k} \text{ for all } k \in I.$$

$$V_1(x) = x_5^{(2)}x_5^{(1)}(+,+,+,+,+) + \Big(x_3^{(3)}x_5^{(1)} + \frac{x_3^{(3)}x_3^{(2)}x_3^{(1)}}{x_5^{(2)}}\Big)(+,+,+,-,-) + \Big(x_4^{(2)}x_5^{(1)} + \frac{x_4^{(2)}x_3^{(2)}x_3^{(1)}}{x_5^{(2)}} + \frac{x_4^{(2)}x_2^{(2)}x_3^{(1)}}{x_3^{(3)}} + \frac{x_4^{(2)}x_2^{(2)}x_4^{(1)}x_2^{(1)}}{x_3^{(3)}x_3^{(2)}}\Big)(+,+,-,+,-) + \Big(x_5^{(1)} + \frac{x_3^{(2)}x_3^{(1)}}{x_5^{(2)}} + \frac{x_2^{(2)}x_3^{(1)}}{x_3^{(3)}} + \frac{x_2^{(2)}x_4^{(1)}x_2^{(1)}}{x_3^{(3)}x_3^{(2)}} + \frac{x_2^{(2)}x_2^{(1)}}{x_4^{(2)}}\Big)(+,+,-,-,+) + \Big(x_4^{(2)}x_3^{(1)} + \frac{x_4^{(2)}x_4^{(1)}x_2^{(1)}}{x_3^{(2)}} + \frac{x_4^{(2)}x_4^{(1)}x_1^{(1)}}{x_2^{(2)}}\Big)(+,-,+,+,-) + \Big(x_3^{(1)} + \frac{x_4^{(1)}x_2^{(1)}}{x_3^{(2)}} + \frac{x_4^{(1)}x_1^{(1)}}{x_2^{(2)}} + \frac{x_3^{(3)}x_2^{(1)}}{x_4^{(2)}} + \frac{x_3^{(3)}x_3^{(2)}x_1^{(1)}}{x_4^{(2)}x_2^{(2)}}\Big)(+,-,+,-,+) + \Big(x_2^{(1)} + \frac{x_3^{(2)}x_1^{(1)}}{x_2^{(2)}} + \frac{x_5^{(2)}x_1^{(1)}}{x_3^{(3)}}\Big)(+,-,-,+,+) + x_1^{(1)}(+,-,-,-,-) + x_4^{(2)}x_4^{(1)}(-,+,+,+,-) + \Big(x_4^{(1)} + \frac{x_3^{(3)}x_3^{(2)}}{x_4^{(2)}}\Big)(-,+,+,-,+) + \Big(x_3^{(2)} + \frac{x_2^{(2)}x_5^{(2)}}{x_3^{(3)}}\Big)(-,+,-,+,+) + x_5^{(2)}(-,-,+,+,+) + x_2^{(2)}(-,+,-,-,-) + x_3^{(3)}(-,-,+,-,-) + x_4^{(2)}(-,-,-,+,-) + (-,-,-,-,+).$$

$$V_2(y) = y_4^{(2)}y_4^{(1)}(-,+,+,+,-) + \left(y_3^{(3)}y_4^{(1)} + \frac{y_3^{(3)}y_3^{(2)}y_3^{(1)}}{y_4^{(2)}}\right)(-,+,+,-,+) + \left(y_5^{(2)}y_4^{(1)} + \frac{y_5^{(2)}y_3^{(2)}y_3^{(1)}}{y_4^{(2)}} + \frac{y_5^{(2)}y_2^{(2)}y_3^{(1)}}{y_3^{(3)}} + \frac{y_5^{(2)}y_2^{(2)}y_5^{(1)}y_2^{(1)}}{y_3^{(3)}y_3^{(2)}}\right)(-,+,-,+,+) + \left(y_4^{(1)} + \frac{y_3^{(2)}y_3^{(1)}}{y_4^{(2)}} + \frac{y_2^{(2)}y_3^{(1)}}{y_3^{(3)}} + \frac{y_2^{(2)}y_5^{(1)}y_2^{(1)}}{y_3^{(3)}y_3^{(2)}} + \frac{y_2^{(2)}y_2^{(1)}}{y_5^{(2)}}\right)(-,+,-,-,-) + \left(y_5^{(2)}y_3^{(1)} + \frac{y_5^{(2)}y_5^{(1)}y_2^{(1)}}{y_3^{(2)}} + \frac{y_5^{(2)}y_5^{(1)}y_0^{(1)}}{y_2^{(2)}}\right)(-,-,+,+,+) + \left(y_3^{(1)} + \frac{y_5^{(1)}y_2^{(1)}}{y_3^{(2)}} + \frac{y_5^{(1)}y_0^{(1)}}{y_2^{(2)}} + \frac{y_3^{(3)}y_2^{(1)}}{y_5^{(2)}} + \frac{y_3^{(3)}y_3^{(2)}y_0^{(1)}}{y_5^{(2)}y_2^{(2)}}\right)(-,-,+,-,-) + \left(y_2^{(1)} + \frac{y_3^{(2)}y_0^{(1)}}{y_2^{(2)}} + \frac{y_4^{(2)}y_0^{(1)}}{y_3^{(3)}}\right)(-,-,-,+,-) + y_0^{(1)}(-,-,-,-,+) + y_5^{(2)}y_5^{(1)}(+,+,+,+,+) + \left(y_5^{(1)} + \frac{y_3^{(3)}y_3^{(2)}}{y_5^{(2)}}\right)(+,+,+,-,-) + \left(y_3^{(2)} + \frac{y_2^{(2)}y_4^{(2)}}{y_3^{(3)}}\right)(+,+,-,+,-) + y_4^{(2)}(+,-,+,+,-) + y_2^{(2)}(+,+,-,-,+) + y_3^{(3)}(+,-,+,-,+) + y_5^{(2)}(+,-,-,+,+) + (+,-,-,-,-).$$

Now for a given x we solve the equation

$$V_2(y) = a(x)\sigma(V_1(x))$$

where $a(x)$ is a rational function in x and the action of σ on $V_1(x)$ is induced by its action on $W(\varpi_5)$.

We define the map

$$\begin{aligned}\bar{\sigma} \colon \mathcal{V}_1 &\to \mathcal{V}_2 \\ V_1(x) &\mapsto V_2(y).\end{aligned}$$

Proposition

The map $\bar{\sigma} : \mathcal{V}_1 \to \mathcal{V}_2$ is a bi-positive birational isomorphism with the inverse positive rational map

$$\begin{aligned}\bar{\sigma}^{-1} \colon \mathcal{V}_2 &\to \mathcal{V}_1 \\ V_2(y) &\mapsto V_1(x).\end{aligned}$$

It is known that $\mathcal{V}_1 = \{V_1(x), e_k^c, \gamma_k, \varepsilon_k \mid k \in I_0 = \{1,2,\ldots,5\}\}$ $\left(\text{resp. } \mathcal{V}_2 = \{V_2(y), \bar{e}_k{}^c, \bar{\gamma}_k, \bar{\varepsilon}_k \mid k \in I_1 = \{0,2,\ldots,5\}\}\right)$ has the structure of a $\mathfrak{g}_0$ (resp. $\sigma(\mathfrak{g})_1$) positive geometric crystal.

In order to give $\mathcal{V}_1$ a $\mathfrak{g} = D_5^{(1)}$-geometric crystal structure, we need to define the actions of e_0^c, γ_0, and ε_0 on $V_1(x)$ and prove that they satisfy the following relations:

1. $\gamma_0(e_k^c(V_1(x))) = c^{a_{k0}}\gamma_0(V_1(x))$ for all $k \in I$
2. $\gamma_k(e_0^c(V_1(x))) = c^{a_{0k}}\gamma_k(V_1(x))$ for all $k \in I$
3. $\varepsilon_0(e_0^c(V_1(x))) = c^{-1}\varepsilon_0(V_1(x))$
4. $e_0^{c_1}e_k^{c_2} = e_k^{c_2}e_0^{c_1}$ for all $k \in \{1,3,4,5\}$
5. $e_0^{c_1}e_2^{c_1c_2}e_0^{c_2} = e_2^{c_2}e_0^{c_1c_2}e_2^{c_1}$

We use the $\sigma(\mathfrak{g})_1$-geometric crystal structure on $\mathcal{V}_2$ to define the action of e_0^c, γ_0, and ε_0 on $V_1(x)$ as follows.

$$e_0^c(V_1(x)) := \bar{\sigma}^{-1} \circ \overline{e_{\sigma(0)}}^c \circ \bar{\sigma}(V_1(x)) = \bar{\sigma}^{-1} \circ \bar{e}_5^c(V_2(y)), \tag{1}$$

$$\gamma_0(V_1(x)) := \bar{\sigma}^{-1} \circ \overline{\gamma_{\sigma(0)}} \circ \bar{\sigma}(V_1(x)) = \bar{\sigma}^{-1} \circ \overline{\gamma_5}(V_2(y)), \tag{2}$$

$$\varepsilon_0(V_1(x)) := \bar{\sigma}^{-1} \circ \overline{\varepsilon_{\sigma(0)}} \circ \bar{\sigma}(V_1(x)) = \bar{\sigma}^{-1} \circ \overline{\varepsilon_5}(V_2(y)). \tag{3}$$

Set $B = \frac{x_2^{(2)} x_3^{(1)}}{x_3^{(3)}} + \frac{x_3^{(2)} x_3^{(1)}}{x_5^{(2)}}, C = \frac{x_2^{(2)} x_2^{(1)}}{x_4^{(2)}} + \frac{x_2^{(2)} x_4^{(1)} x_2^{(1)}}{x_3^{(3)} x_3^{(2)}}$ and $A = B + C$.

Theorem

The algebraic variety $\mathcal{V}(D_5^{(1)}) = \{V_1(x), e_k^c, \gamma_k, \varepsilon_k \mid k \in I\}$ is a positive geometric crystal for the affine Lie algebra $\mathfrak{g} = D_5^{(1)}$ with the e_0^c, γ_0, and ε_0 actions on $V_1(x)$ given by:

$$\gamma_0(V_1(x)) = \frac{1}{x_2^{(2)} x_2^{(1)}}, \qquad \varepsilon_0(V_1(x)) = x_5^{(1)} + A,$$

$$e_0^c(V_1(x)) = V_1(x') = V_1(x_4^{(2)'}, x_3^{(3)'}, \ldots, x_5^{(1)'}) \text{ where}$$

$$x_5^{(2)'} = x_5^{(2)} \cdot \frac{cx_5^{(1)} + A}{c(x_5^{(1)} + A)}, \quad x_5^{(1)'} = x_5^{(1)} \cdot \frac{x_5^{(1)} + A}{cx_5^{(1)} + A},$$

$$x_4^{(2)'} = x_4^{(2)} \cdot \frac{c(x_5^{(1)} + B + \frac{x_2^{(2)}x_4^{(1)}x_2^{(1)}}{x_3^{(3)}x_3^{(2)}}) + \frac{x_2^{(2)}x_2^{(1)}}{x_4^{(2)}}}{c(x_5^{(1)} + A)},$$

$$x_4^{(1)'} = x_4^{(1)} \cdot \frac{x_5^{(1)} + A}{c(x_5^{(1)} + B + \frac{x_2^{(2)}x_4^{(1)}x_2^{(1)}}{x_3^{(3)}x_3^{(2)}}) + \frac{x_2^{(2)}x_2^{(1)}}{x_4^{(2)}}},$$

$$x_3^{(3)'} = x_3^{(3)} \cdot \frac{c(x_5^{(1)} + \frac{x_3^{(2)}x_3^{(1)}}{x_5^{(2)}}) + \frac{x_2^{(2)}x_3^{(1)}}{x_3^{(3)}} + C}{c(x_5^{(1)} + A)},$$

$$x_3^{(2)'} = x_3^{(2)} \cdot \frac{c(x_5^{(1)} + B) + C}{c(c(x_5^{(1)} + \frac{x_3^{(2)}x_3^{(1)}}{x_5^{(2)}}) + \frac{x_2^{(2)}x_3^{(1)}}{x_3^{(3)}} + C)},$$

$$x_3^{(1)'} = x_3^{(1)} \cdot \frac{x_5^{(1)} + A}{c(x_5^{(1)} + B) + C}, \quad x_2^{(2)'} = \frac{x_2^{(2)}}{c}, \quad x_2^{(1)'} = \frac{x_2^{(1)}}{c}, \quad x_1^{(1)'} = \frac{x_1^{(1)}}{c}.$$

Parameterizing the $D_5^{(1)}$-perfect crystals $\{B^{5,l}\}_{l\geq 1}$ of level l given in $[(KMN)^2, 1992]$, we obtained that the set

$$B^{5,l} = \left\{ b = (b_{ij})_{\substack{i \leq j \leq i+4, \\ 1 \leq i \leq 5}} \,\middle|\, \begin{array}{l} b_{ij} \in \mathbb{Z}_{\geq 0},\ \sum_{j=i}^{i+4} b_{ij} = l,\ 1 \leq i \leq 5, \\ \sum_{j=i}^{5-t} b_{ij} = \sum_{j=i+t}^{4+t} b_{i+t,j},\ 1 \leq i, t \leq 4, \\ \sum_{j=i}^{t} b_{ij} \geq \sum_{j=i+1}^{t+1} b_{i+1,j},\ 1 \leq i \leq t \leq 4 \end{array} \right\}$$

equipped with suitable maps $\tilde{e}_k, \tilde{f}_k : B^{5,l} \longrightarrow B^{5,l} \cup \{0\}$, $\varepsilon_k, \varphi_k : B^{5,l} \longrightarrow \mathbb{Z}$, $0 \leq k \leq 5$ and wt : $B^{5,l} \longrightarrow P_{cl}$ is a coherent family of perfect crystals with limit $B^{5,\infty}$ given as:

$$B^{5,\infty} = \left\{ b = (b_{ij})_{\substack{i \le j \le i+4, \\ 1 \le i \le 5}} \;\middle|\; \begin{array}{l} b_{ij} \in \mathbb{Z},\ \sum_{j=i}^{i+4} b_{ij} = 0,\ 1 \le i \le 5, \\ \sum_{j=i}^{5-t} b_{ij} = \sum_{j=i+t}^{4+t} b_{i+t,j},\ 1 \le i,t \le 4 \end{array} \right\}.$$

containing the special vector $b^\infty = \mathbf{0}$ (i.e. $(b^\infty)_{ij} = 0$ for $i \le j \le i+4,\ 1 \le i \le 5$).

Apply the ultra-discretization functor $\mathcal{UD}$ to the positive geometric crystal $\mathcal{V}(D_5^{(1)}) = \{V_1(x), e_k^c, \gamma_k, \varepsilon_k \mid k \in I\}$ constructed in Section V.

$\mathcal{X} = \mathcal{UD}(\mathcal{V}(D_5^{(1)}))$ with maps

$\tilde{e}_k, \tilde{f}_k : \mathcal{X} \longrightarrow \mathcal{X} \cup \{0\}$, $\varepsilon_k, \varphi_k : \mathcal{X} \longrightarrow \mathbb{Z}$, $0 \leq k \leq 5$

and $\mathrm{wt} : \mathcal{X} \longrightarrow P_{cl}$ is a Kashiwara's crystal where for $x \in \mathcal{X}$,

$$\tilde{e}_k(x) = \mathcal{UD}(e_k^c)(x)|_{c=1}, \ \tilde{f}_k(x) = \mathcal{UD}(e_k^c)(x)|_{c=-1},$$

$$\mathrm{wt}(x) = \sum_{k=0}^{5} \mathrm{wt}_k(x)\Lambda_k \text{ where } \mathrm{wt}_k(x) = \mathcal{UD}(\gamma_k)(x),$$

$$\varepsilon_k(x) = \mathcal{UD}(\varepsilon_k)(x), \ \varphi_k(x) = \mathrm{wt}_k(x) + \varepsilon_k(x).$$

Theorem

The map

$$\begin{aligned} \Omega: \quad B^{5,\infty} \quad &\rightarrow \quad \mathcal{X}, \\ b = (b_{ij})_{i \le j \le i+4,\ 1 \le i \le 5} \quad &\mapsto \quad x = (x_4^{(2)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_3^{(2)}, \\ & \qquad x_4^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_5^{(1)}) \end{aligned}$$

defined by

$$x_m^{(l)} = \begin{cases} \sum_{j=m-l+1}^{m} b_{m-l+1,j}, & \text{for} \quad m = 1,2,3 \\ \sum_{j=m-2l+1}^{m} b_{m-2l+1,j}, & \text{for} \quad m = 4 \\ \sum_{j=m-2l+1}^{m-1} b_{m-2l+1,j}, & \text{for} \quad m = 5. \end{cases}$$

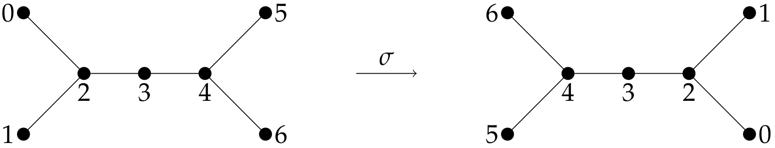
is an isomorphism of crystals.

$$\mathfrak{g} = D_6^{(1)} \text{ with index set } I = \{0, 1, 2, 3, 4, 5, 6\}$$

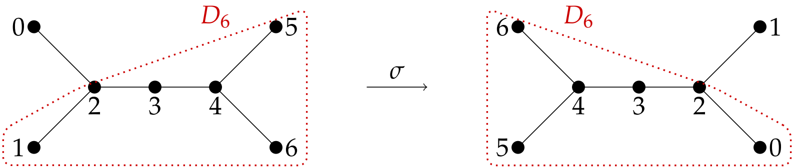
GCM

$$A = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 \end{bmatrix}$$

Dynkin diagram

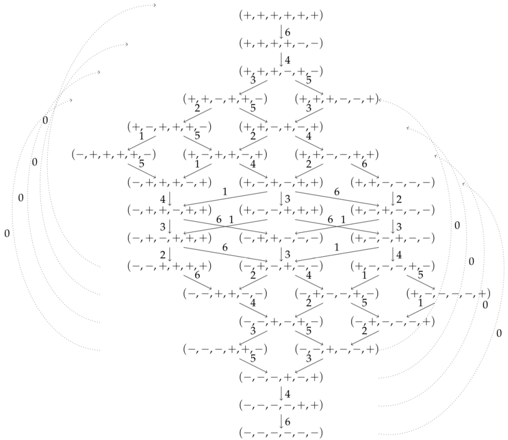


$$\sigma : 0 \mapsto 6, 1 \mapsto 5, 2 \mapsto 4, 3 \mapsto 3, 4 \mapsto 2, 5 \mapsto 1, 6 \mapsto 0$$



Let $I_0 = \{1, 2, 3, 4, 5, 6\}$ and $I_1 = \{0, 2, 3, 4, 5, 6\}$.

Then $\mathfrak{g}_0$ as well as $\sigma(\mathfrak{g})_1$ are isomorphic to D_6.



$(+,+,+,+,+,+)$ is a $\mathfrak{g}_0$ highest weight vector with weight $\varpi_6 = \Lambda_6 - \Lambda_0$,

$(-,+,+,+,+,-)$ is a $\sigma(\mathfrak{g})_1$ highest weight vector with weight $\check{\varpi}_6 := \Lambda_5 - \Lambda_1$.

Associated with Weyl group elements $w_1, w_2 \in W$, we define algebraic varieties $\mathcal{V}_1, \mathcal{V}_2 \subset W(\varpi_6)$ as follows.

$$\begin{aligned}\mathcal{V}_1 = \{V_1(x) := {} & Y_6(x_6^{(3)})Y_4(x_4^{(4)})Y_3(x_3^{(3)})Y_2(x_2^{(2)})Y_5(x_5^{(2)})Y_4(x_4^{(3)})Y_3(x_3^{(2)})\\ & Y_6(x_6^{(2)})Y_4(x_4^{(2)})Y_5(x_5^{(1)})Y_1(x_1^{(1)})Y_2(x_2^{(1)})Y_3(x_3^{(1)})Y_4(x_4^{(1)})\\ & Y_6(x_6^{(1)})(+,+,+,+,+,+) | x_m^{(l)} \in \mathbb{C}^\times\},\\ \mathcal{V}_2 = \{V_2(y) := {} & Y_5(y_5^{(3)})Y_4(y_4^{(4)})Y_3(y_3^{(3)})Y_2(y_2^{(2)})Y_6(y_6^{(2)})Y_4(y_4^{(3)})Y_3(y_3^{(2)})\\ & Y_5(y_5^{(2)})Y_4(y_4^{(2)})Y_6(y_6^{(1)})Y_0(y_0^{(1)})Y_2(y_2^{(1)})Y_3(y_3^{(1)})Y_4(y_4^{(1)})\\ & Y_5(y_5^{(1)})(-,+,+,+,+,-) | y_m^{(l)} \in \mathbb{C}^\times\},\end{aligned}$$

where
$x = (x_6^{(3)}, x_4^{(4)}, x_3^{(3)}, x_2^{(2)}, x_5^{(2)}, x_4^{(3)}, x_3^{(2)}, x_6^{(2)}, x_4^{(2)}, x_5^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_6^{(1)})$,
$y = (y_5^{(3)}, y_4^{(4)}, y_3^{(3)}, y_2^{(2)}, y_6^{(2)}, y_4^{(3)}, y_3^{(2)}, y_5^{(2)}, y_4^{(2)}, y_6^{(1)}, y_0^{(1)}, y_2^{(1)}, y_3^{(1)}, y_4^{(1)}, y_5^{(1)})$.

We use the $\sigma(\mathfrak{g})_1$-geometric crystal structure on $\mathcal{V}_2$ to define the action of e_0^c, γ_0, and ε_0 on $V_1(x)$ as follows.

$$e_0^c(V_1(x)) := \bar{\sigma}^{-1} \circ \overline{e_{\sigma(0)}}^c \circ \bar{\sigma}(V_1(x)) = \bar{\sigma}^{-1} \circ \bar{e}_6^c(V_2(y)), \tag{4}$$

$$\gamma_0(V_1(x)) := \bar{\sigma}^{-1} \circ \overline{\gamma_{\sigma(0)}} \circ \bar{\sigma}(V_1(x)) = \bar{\sigma}^{-1} \circ \overline{\gamma_6}(V_2(y)), \tag{5}$$

$$\varepsilon_0(V_1(x)) := \bar{\sigma}^{-1} \circ \overline{\varepsilon_{\sigma(0)}} \circ \bar{\sigma}(V_1(x)) = \bar{\sigma}^{-1} \circ \overline{\varepsilon_6}(V_2(y)). \tag{6}$$

Theorem

Together with the actions of e_0^c, γ_0 and ε_0 on $V_1(x)$ given in (4), (5), (6), we obtain a positive affine geometric crystal

$$\mathcal{V}(D_6^{(1)}) = \{V_1(x), e_k^c, \gamma_k, \varepsilon_k \mid k \in I\}$$

for the affine Lie algebra $\mathfrak{g} = D_6^{(1)}$.

For a positive integer l,

$$B^{6,l} = \left\{ b = (b_{ij})_{\substack{i \le j \le i+5, \\ 1 \le i \le 6}} \;\middle|\; \begin{array}{l} b_{ij} \in \mathbb{Z}_{\ge 0},\ \sum_{j=i}^{i+5} b_{ij} = l,\ 1 \le i \le 6, \\ \sum_{j=i}^{6-t} b_{ij} = \sum_{j=i+t}^{5+t} b_{i+t,j},\ 1 \le i,t \le 5, \\ \sum_{j=i}^{t} b_{ij} \ge \sum_{j=i+1}^{t+1} b_{i+1,j},\ 1 \le i \le t \le 5 \end{array} \right\},$$

$$B^{6,\infty} = \left\{ b = (b_{ij})_{\substack{i \le j \le i+5, \\ 1 \le i \le 6}} \;\middle|\; \begin{array}{l} b_{ij} \in \mathbb{Z},\ \sum_{j=i}^{i+5} b_{ij} = 0,\ 1 \le i \le 6, \\ \sum_{j=i}^{6-t} b_{ij} = \sum_{j=i+t}^{5+t} b_{i+t,j},\ 1 \le i,t \le 5 \end{array} \right\}.$$

Theorem

The map

$$\begin{array}{rccc}\Omega: & B^{6,\infty} & \to & \mathcal{X}, \\ & b=(b_{ij})_{i\le j\le i+5,\ 1\le i\le 6} & \mapsto & x=(x_6^{(3)},x_4^{(4)},x_3^{(3)},x_2^{(2)},x_5^{(2)}, \\ & & & x_4^{(3)},x_3^{(2)},x_6^{(2)},x_4^{(2)},x_5^{(1)}, \\ & & & x_1^{(1)},x_2^{(1)},x_3^{(1)},x_4^{(1)},x_6^{(1)})\end{array}$$

defined by

$$x_m^{(l)}=\begin{cases}\sum_{j=m-l+1}^{m} b_{m-l+1,j}, & \text{for} \quad m=1,2,3,4\\ \sum_{j=m-2l+1}^{m} b_{m-2l+1,j}, & \text{for} \quad m=5\\ \sum_{j=m-2l+1}^{m-1} b_{m-2l+1,j}, & \text{for} \quad m=6.\end{cases}$$

is an isomorphism of crystals.

Thank you!

ขอบคุณค่ะ