Charged particle motion near magnetized black holes

arXiv:1008.2985

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MOTIVATIONS

There are indications that magnetic field plays an important role in astrophysical black holes

Mechanism of energy transfer from accretion disk to jets;

Jets collimation

Simple (toy) model as a first step in study of MHD effects in black hole vicinity

Black holes are formed when large mass M is in a compact region $r_g = 2GM/c^2$

Black hole candidates are selected by observing their masses:

(i) Stellar mass black holes in binary systems $(M > 3M_{\otimes})$

(ii) Supermassive black holes

$M \sim 10^6 - 10^9 M_{\otimes}$

Black hole identification is based on GR effects

Important characteristics:

(i) Position of the innermost stable circular orbit

(ii) Period of Keplerian motion

(iii) Energy released by a particle before its fall into BH

These parameters are specific for GR and,
in particular, depend on the rotation parameter a/M.

Examples:

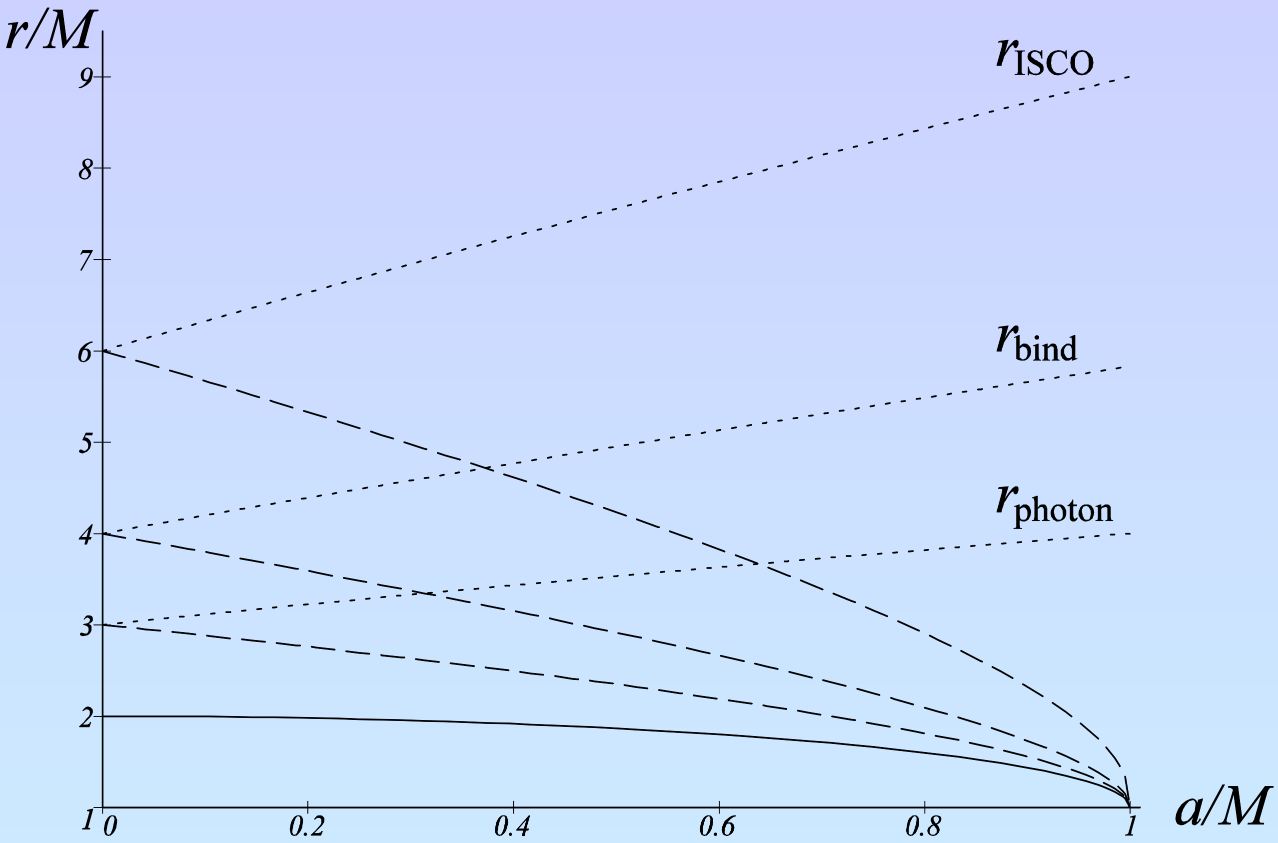
Iron X-ray line broadening: Fe Kα line at 6.4keV. Fluorescence generated by irradiation of weakly-ionized disk by hard X-rays. Broadening is due motion and gravitational field.

Measuring BH spin: Study of spectrum of accretion disk. Inner part of the disk gives considerable contribution to the flux. Rotation parameters for 7 SMBHs were estimated.

a/M=0.85-0.97 (LMC X-1) and
a/M=0.98-1.0 (GRS 1915+10)

[McClintock, Narayan et al]

Radius of the innermost stable circular orbit in Kerr spacetime



Rotation parameter $a / M = J / M^2$

	$a=0$	$a=M$	
		$\mathcal{L}>0$	$\mathcal{L}<0$
$\mathcal{E}$	$\sqrt{8/9}$	$\sqrt{1/3}$	$\sqrt{25/27}$
$1-\mathcal{E}$	0.0572	0.4236	0.0377
$\lvert\mathcal{L}\rvert/M$	$2\sqrt{3}$	$2/\sqrt{3}$	$22/3\sqrt{3}$

Magnetic field in the BH vicinity acting on a charged particle can change these characteristics significantly

Magnetic field in the vicinity of black holes

- Original magnetic field of the collapsed progenitor star.
- The dynamo mechanism in the accretion disc of a black hole.
- Numerical simulations of jet formation in strong magnetic and gravitational fields, see, e.g., S. Koide, et al., Science **295**, 1688 (2002)
- Extraction of the rotational energy form black holes: The Blandford-Znajek and Penrose mechanism.

Observational evidences:

M. de Kool, et al., Publ. Astron. Soc. Aust. **16**, 225 (1999)

J. M. Miller, et. al., Nature **441**, 953 (2006)

Weak or Strong?

Gravitational effect of a magnetic field:

Spacetime curvature $\sim GB^2/c^4$ generated by magnetic field B

is comparable with at the horizon curvature $\sim r_g^{-1}$ if

$$\frac{GB^2}{c^4} \sim \frac{1}{r_g^2} \sim \frac{c^4}{G^2M^2} \quad \Rightarrow \quad B \sim \frac{c^4}{G^{3/2}M_\otimes}\left(\frac{M_\otimes}{M}\right) \times 10^{19}(M_\otimes/M)\,\mathrm{G}.$$

Charged particle motion:

$$m\frac{du^a}{d\tau} = qF^a_{\ b}u^b \qquad e/m_e \approx 5.2728\times10^{17}\ \mathrm{g}^{-1/2}\mathrm{cm}^{3/2}\mathrm{s}^{-1}$$

Effect of magnetic field on charged particles

Cyclotron frequency: $$\Omega_c = \frac{|qB|c}{E}$$

Keplerian frequency: $$\Omega_K = \frac{r_g^{1/2} c}{r^{3/2}\sqrt{2}},$$

$$r_g = \frac{2MG}{c^2}$$

$$b = \Omega_c / \Omega_K = \frac{qBMG}{mc^4}$$

<u>For large b magnetic field essentially modifies motion of a charged particle</u>

Characteristic scales of the magnetic field

Stellar mass black holes: $M_1 \sim 10 M_{\otimes}, \quad B_1 \sim 10^8 G$

Supermassive black holes: $M_2 \sim 10^9 M_{\otimes}, \quad B_2 \sim 10^4 G$

M. Yu. Piotrovich, et. al, arXiv:1002.4948 (2010)

$$b_1^p = \frac{eB_1 G M_1}{m_p c^4} \approx 4.7180 \times 10^7$$

$$b_2^p = \frac{B_2 M_2}{B_1 M_1} b_1^p = 10^4 b_1^p$$

Related work

Analysis of circular orbits of a charged particle in the equatorial plane:

D. V. Gal’tsov and V. I. Petukhov, Sov. Phys. JEPT **47**(3), 419 (1978)

- N. Aliev and D. V. Gal’tsov, Sov. Phys. Usp. **32**(1), 75 (1989)

Analysis of marginally stable orbits around rotating black hole:

A. N. Aliev and N. Özdemir, MNRAS. **336**, 241 (2002)

Motion of charged particle around in a dipolar magnetic field:

A. R. Prasana and R. K. Varma, Pramana, **8**, 229 (1977)

G. Preti, Class. Quantum Grav. **21**, 3433 (2004)

P. Bakala, E. Šrámková, Z. Stuchlík and G. Török,
Class. Quantum Grav. **27**, 045001 (2010)

The problem set up

- We consider a Schwarzschild black hole immersed into static and axisymmetric magnetic field which approaches a constant value far away of the black hole.

- The magnetic field is regular between the black hole horizon and accretion disk.

- We study motion of charged particles in the presence of the magnetic and the black hole gravitational fields, neglecting their mutual interaction.

- We restrict ourselves to the equatorial plane of the black hole, which is orthogonal to the direction of the magnetic field.

Magnetized black hole

The Schwarzschild space-time:

$$ds^2 = -\left(1 - \frac{r_g}{r}\right)dt^2 + \left(1 - \frac{r_g}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Killing vectors:

$$\xi_{(t)} = \frac{\partial}{\partial t}, \quad \xi_{(\phi)} = \frac{\partial}{\partial \phi}, \quad \xi^{a;b}{}_{;b} = 0, \quad (R_{ab} = 0)$$

The electromagnetic 4-potential: $A^a = \frac{1}{2} B \xi^a_{(\phi)}; \quad A^a{}_{;a} = 0$

Static, axisymmetric, uniform at infinity magnetic field:

$$B^a \partial_a = B\left(1 - \frac{r_g}{r}\right)^{1/2}\left[\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right]$$

Dynamical equations

Motion of a charged particle:

$$m\frac{du^a}{d\tau} = qF^a{}_b u^b, \qquad u^a u_a = -1$$

Conserved quantities:

$$E \equiv -\xi^a_{(t)} P_a = m\frac{dt}{d\tau}\left(1 - \frac{r_g}{r}\right), \qquad \tilde{E} \equiv \frac{E}{m}, \quad \tilde{L} \equiv \frac{L}{m}, \quad \tilde{B} \equiv \frac{qB}{2m},$$

$$L \equiv \xi^a_{(\phi)} P_a = m\frac{d\phi}{d\tau} r^2 \sin^2\theta + \frac{qB}{2} r^2 \sin^2\theta$$

$$B \rightarrow -B, \quad L \rightarrow -L, \quad \phi \rightarrow -\phi$$

Dynamical equations in the equatorial plane: $\theta = \pi / 2$

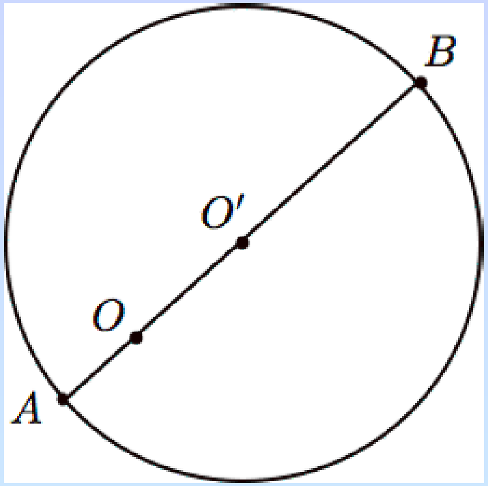
$$\frac{d^2 r}{d\tau^2} = r\left(\frac{\tilde{L}}{r^2} - \tilde{B}\right)\left[\frac{\tilde{L}}{r^2}\left(1 - \frac{3r_g}{2r}\right) + \tilde{B}\left(1 - \frac{r_g}{2r}\right)\right] - \frac{r_g}{2r^2}$$

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \left(1 - \frac{r_g}{r}\right)\left[1 + r^2\left(\frac{\tilde{L}}{r^2} - \tilde{B}\right)^2\right],$$

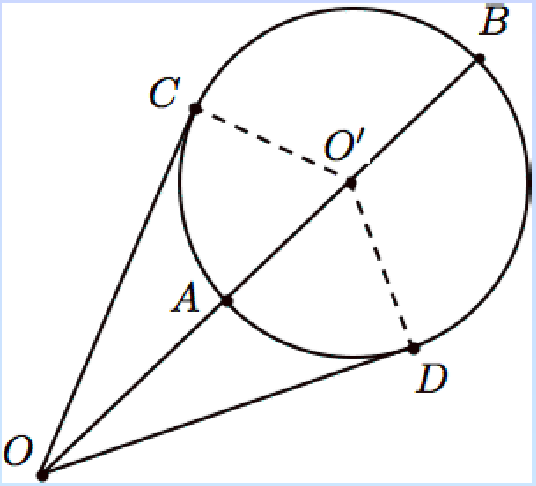
$$\frac{d\phi}{d\tau} = \frac{\tilde{L}}{r^2} - \tilde{B}, \qquad \frac{dt}{d\tau} = \tilde{E}\left(1 - \frac{r_g}{r}\right)^{-1}$$

Flat space-time limit

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - 1 - r^2\left(\frac{\tilde{L}}{r^2} - \tilde{B}\right)^2, \quad \frac{d\phi}{d\tau} = \frac{\tilde{L}}{r^2} - \tilde{B}, \quad \frac{dt}{d\tau} = \tilde{E}$$



$\tilde{L} < 0$



$\tilde{L} > 0$

$$r_c = \sqrt{|\tilde{L}|/\tilde{B}}, \quad \Omega_c = \frac{2\tilde{B}}{\tilde{E}}, \quad \tilde{E} = \sqrt{1 + 4\tilde{B}|\tilde{L}|}.$$

Weak Gravitational Field

Far away from the black hole:

r_g / r small parameter

Dynamical equations:

$$\frac{d^2 r}{d\tau^2} = \frac{\tilde{L}^2}{r^3} - \tilde{B}^2 r - g, \qquad \dot{\phi} = \frac{\tilde{L}}{r^2} - \tilde{B}, \qquad \dot{t} \approx \tilde{E}$$

Newtonian gravitational acceleration: $g = \dfrac{r_g}{2r^2}$

Approximate solution:

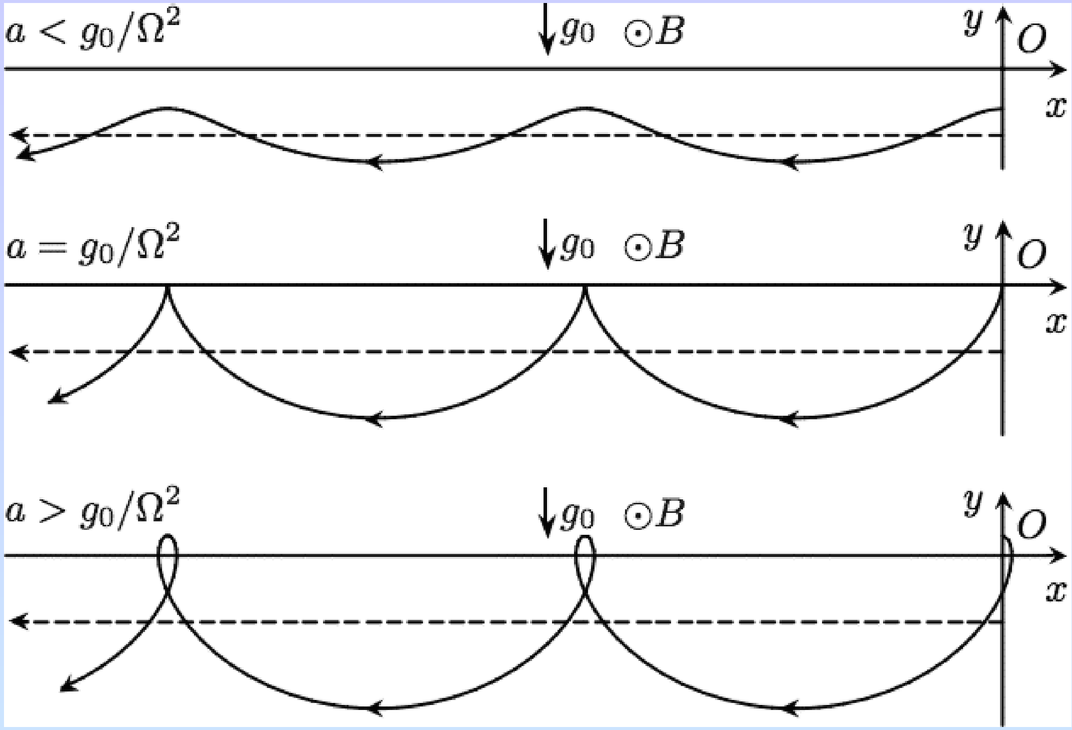
$$r \approx r_0 + y, \quad \phi \approx \phi_0 - x / r_0, \quad x \approx r_0, \quad y \approx r_0$$

$$g \approx g_0 = r_g / 2r_0^2, \quad \tilde{L} = \tilde{B} r_0^2, \quad \Omega = 2\tilde{B} = \Omega_c \tilde{E}$$

$$d^2 y / d\tau^2 = -\Omega^2 y - g_0, \quad dx / d\tau = \Omega y$$

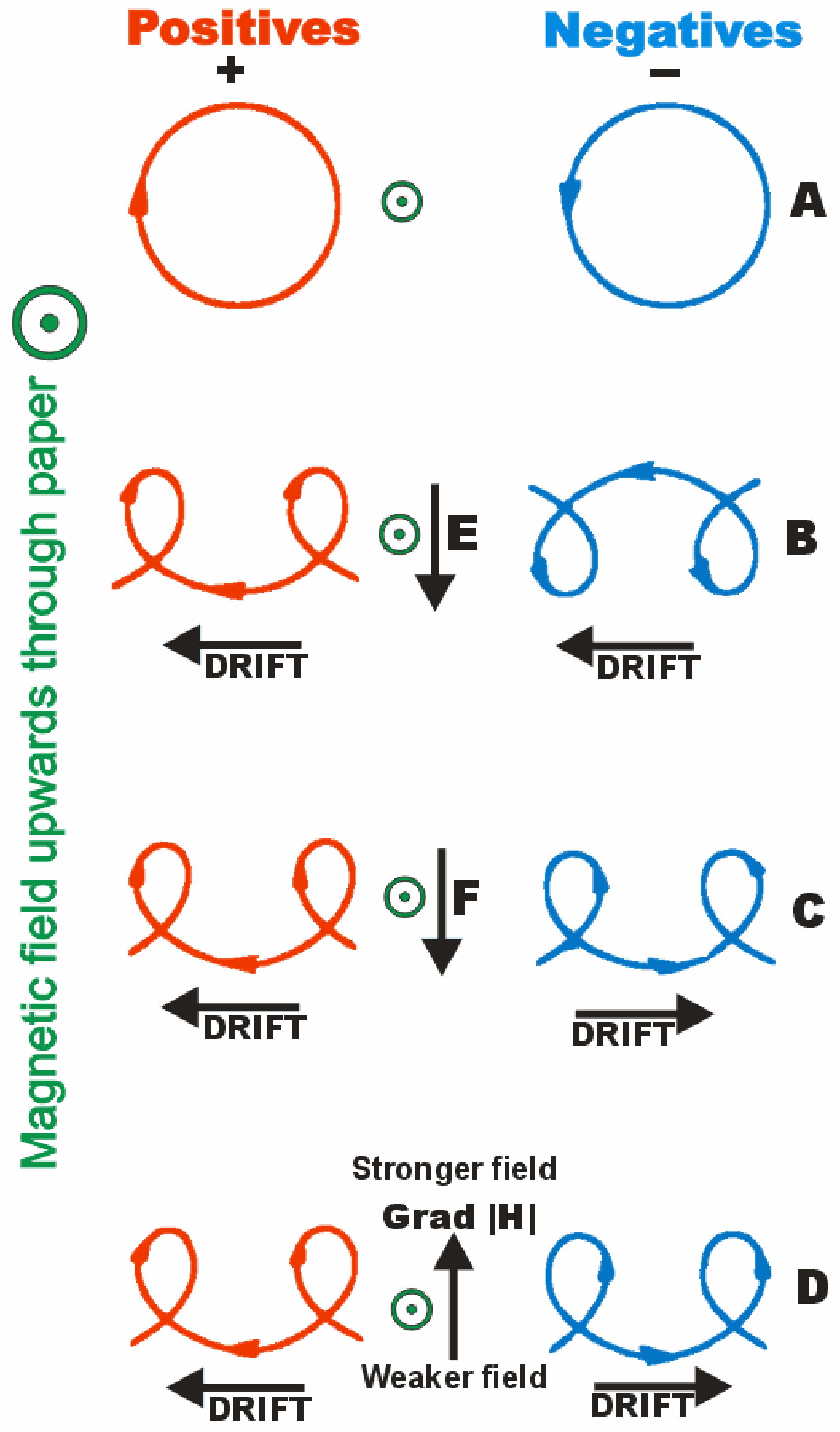
$$y(\tau) = a\cos(\Omega\tau) - \frac{g_0}{\Omega^2}, \quad x(\tau) = a\sin(\Omega\tau) - \frac{g_0}{\Omega}\tau$$

Particle trajectory in the mutually orthogonal uniform magnetic and gravitational fields



Gravitational drift velocity in the rest frame: $V = \dfrac{g_0}{\Omega_c \tilde{E}^2}$

Frame moving with the velocity V: $B \rightarrow E_y = \gamma V B$


Positives
+
Negatives
−
A
E
B
DRIFT
DRIFT
F
C
DRIFT
DRIFT
Stronger field
Grad |H|
D
Weaker field
DRIFT
DRIFT
Magnetic field upwards through paper

Motion of a charged particle

Dimensionless quantities:

$$T = t/r_g\,, \quad \rho = r/r_g\,, \quad \sigma = \tau/r_g, \quad \ell = \tilde{L}/r_g, \quad b = \tilde{B}r_g$$

Dynamical equations:

$$\left(\frac{d\rho}{d\sigma}\right)^2 = \tilde{E}^2 - U, \qquad \frac{d\phi}{d\sigma} = \frac{\ell}{\rho^2} - b, \quad \frac{dT}{d\sigma} = \frac{\tilde{E}\rho}{\rho - 1}$$

Attractive Lorentz force: $\ell = -|\ell|$

Repulsive Lorentz force: $\ell = +|\ell|$

Effective potential

$$U = \left(1 - \frac{1}{\rho}\right)\left[1 + \frac{(\ell - b\rho^2)^2}{\rho^2}\right]$$

At horizon: $U|_{\rho\to 1} \to 0$; At infinity: $U|_{\rho\to +\infty} \to b^2\rho^2$

max = # min $\Rightarrow$ even number of extremal points in $\rho \in (1, \infty)$

Extremum: $U_{,\rho} = 0 \leftrightarrow P(\rho) = Q(\rho)$

$$P(\rho) = b^2\rho^4(2\rho - 1) + \rho^2$$

$$Q(\rho) = 2\ell b\rho^2 + 2\ell^2\rho - 3\ell^2$$

Proof:

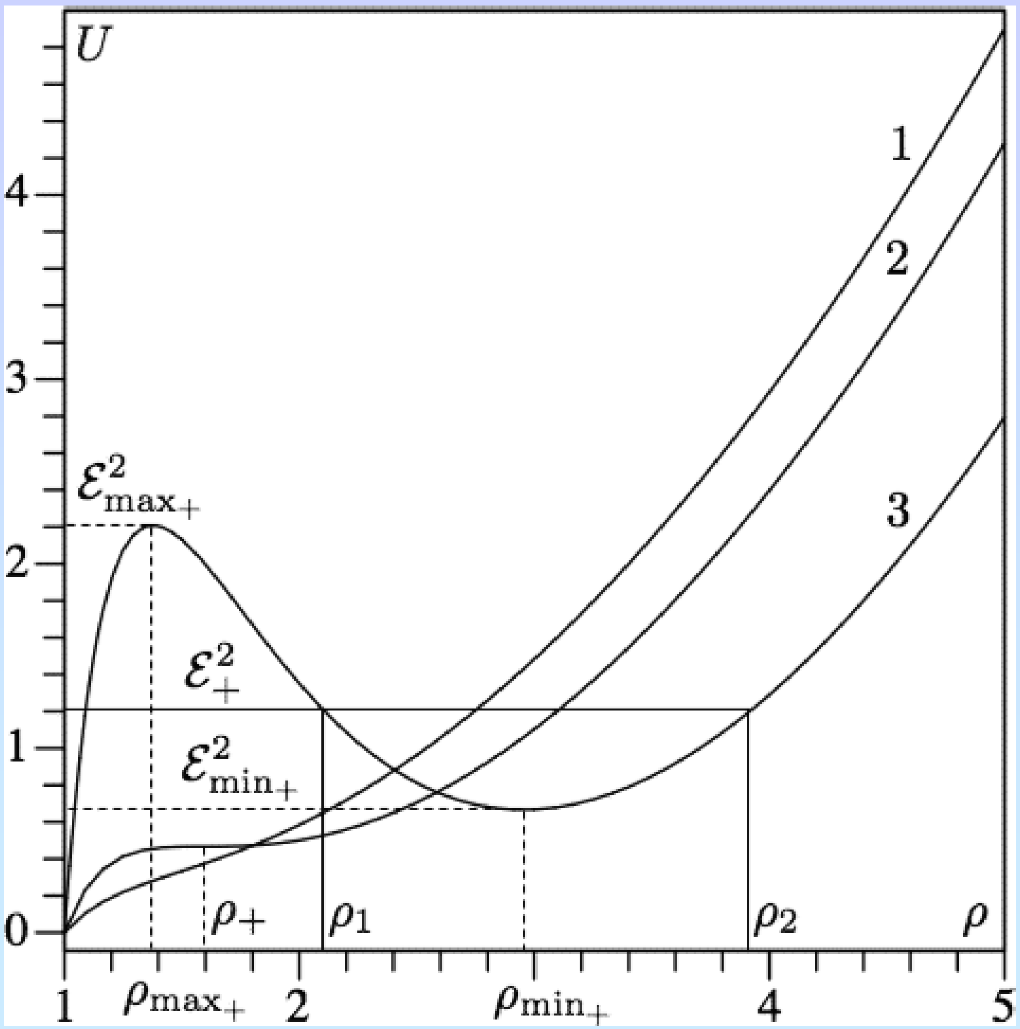
$$P(1) = 1 + b^2; \ \ Q(1) = 2\ell b - \ell^2; \quad P(1) > Q(1);$$

$$f = P' - Q'; \quad f'' = 24 b^2 \rho (5\rho - 1) > 0;$$

Suppose there exist 4 points of intersection of P and Q. At these points $f = (-, +, -, +)$.

But since $f'' > 0$ this is imposible

Effective potential has either 1 maximum and 1 minimum, or no extrema. Critical case: when maximum and minimum coincide. At this point: $U_{,\rho} = U_{,\rho\rho} = 0 \leftrightarrow P_{,\rho} = Q_{,\rho}$



(a) Stable sircular orbit

(b) Unstable circular orbit

(c) Innermost stable circular orbit

$$\rho_* \equiv \sqrt{\ell / b}, \qquad U_{,\rho}(\rho_*) = b/\ell$$

$$(\ell > 0)$$

$$\rho_* > \rho_{\min} \geq \rho_{\max}$$

Innermost stable circular orbits

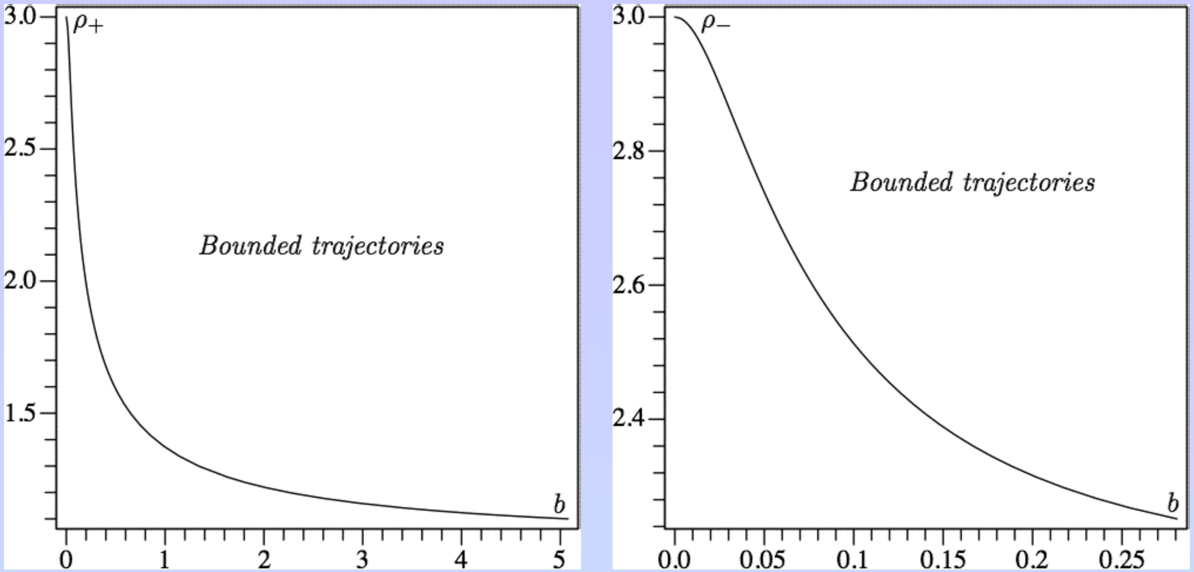
$$U_{,\rho} = 0, \qquad U_{,\rho\rho} = 0$$

$$b = \frac{\sqrt{2}(3-\rho_{\pm})^{1/2}}{2\rho_{\pm}\left(4\rho_{\pm}^{\ 2} - 9\rho_{\pm} + 3 \pm \sqrt{(3\rho_{\pm}-1)(3-\rho_{\pm})}\right)^{1/2}},$$

$$\ell_{\pm} = \pm \frac{\rho_{\pm}(3\rho_{\pm}-1)^{1/2}}{\sqrt{2}\left(4\rho_{\pm}^{\ 2} - 9\rho_{\pm} + 3 \pm \sqrt{(3\rho_{\pm}-1)(3-\rho_{\pm})}\right)^{1/2}},$$

$$\rho_{+}|_{b>>1} = 1 + \frac{1}{b\sqrt{3}} + O(b^{-2}), \quad \rho_{-}|_{b>>1} = \frac{5+\sqrt{13}}{4} + \frac{41-11\sqrt{13}}{36\sqrt{13}b^{2}} + O(b^{-4})$$

$$\ell_{+}|_{b>>1} = b + \sqrt{3} + O(b^{-1}), \quad \ell_{-}|_{b>>1} = -\frac{47+13\sqrt{13}}{8}b + O(b^{-1})$$

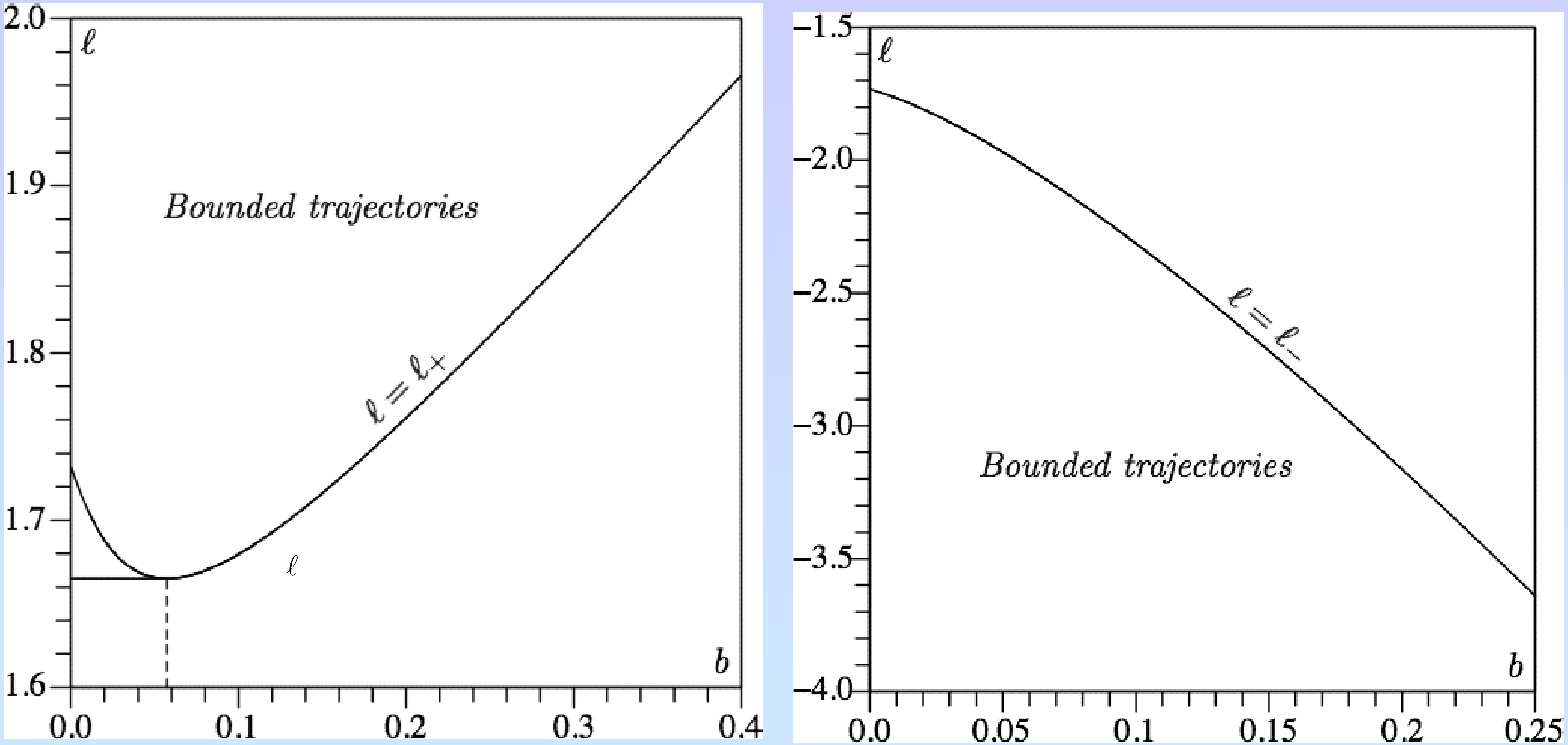


(i) In the absence of the magnetic field $\rho_+ = \rho_- = 3$. This gives the radius of the innermost stable circular orbit $r_{ISCO} = 3r_g = 6M$ (Schwarzschild value);

(ii) In both cases ($\pm$) the radius of the ISCO decreases when the magnetic field increases;

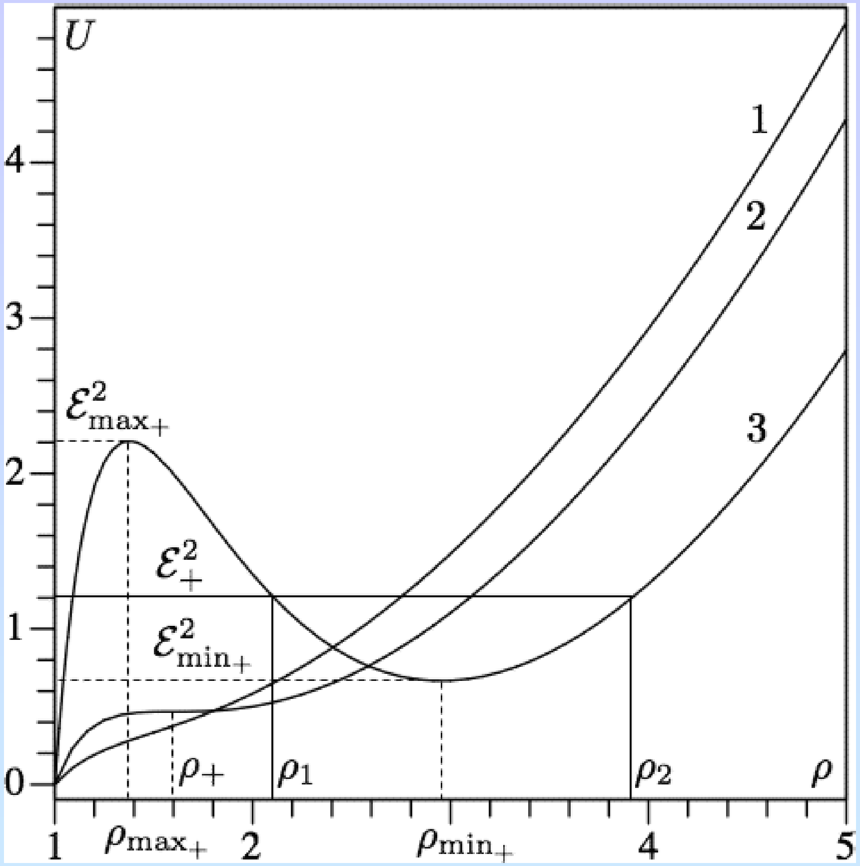
(iii) For `strong' magnetic field $r_{ISCO} \to r_g$ (for +) and

$$r_{ISCO} \to \frac{5+\sqrt{13}}{4} r_g \quad \text{(for -)}$$



2.0
1.9
1.8
1.7
1.6
ℓ
Bounded trajectories
ℓ = ℓ+
ℓ
b
0.0
0.1
0.2
0.3
0.4
−1.5
−2.0
−2.5
−3.0
−3.5
−4.0
ℓ
ℓ = ℓ−
Bounded trajectories
b
0.0
0.05
0.1
0.15
0.2
0.25

Types of trajectories



(1)Trapped orbits

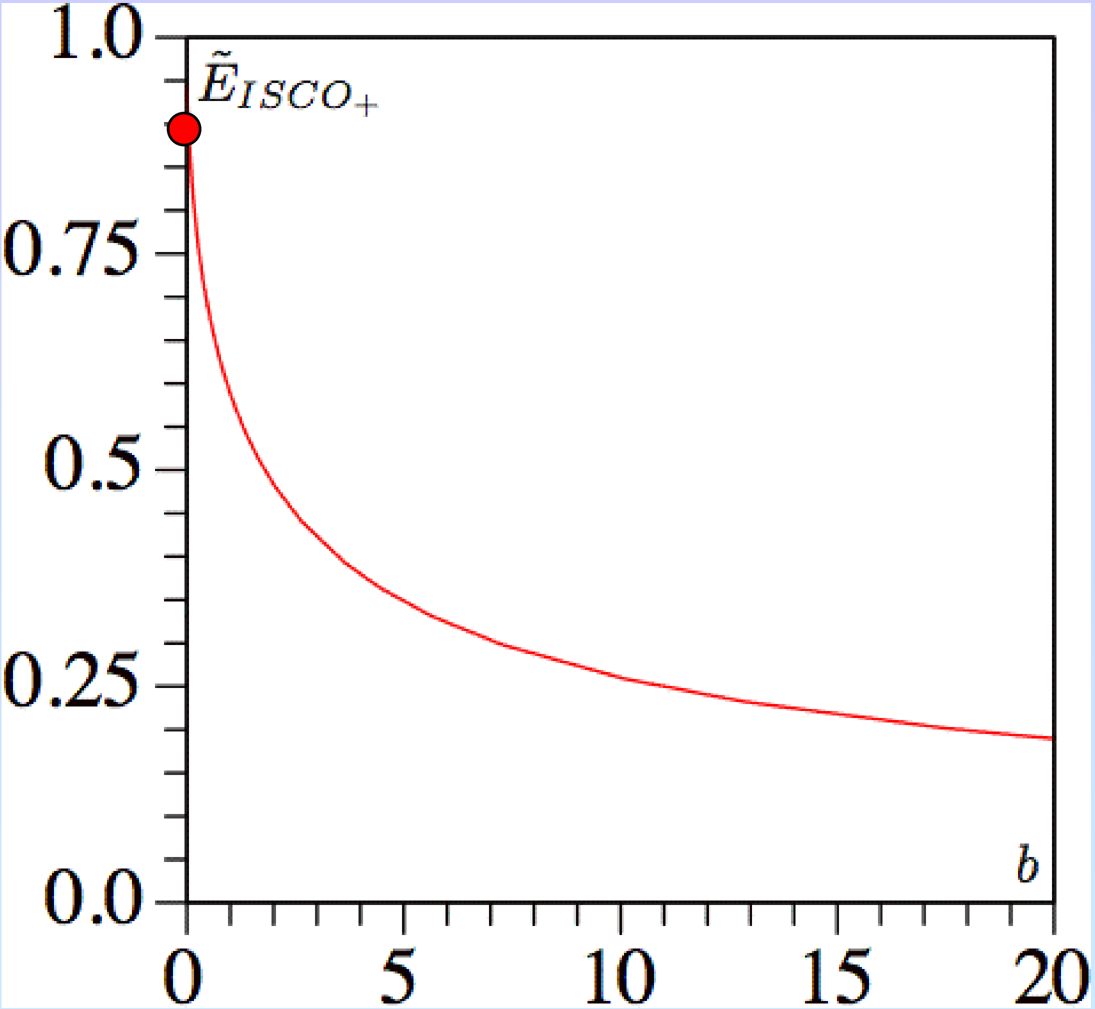
(2) Bounded orbits

$$\rho_{max} \le \rho_1 \le \rho_{min} \le \rho_2,$$

$$E_{min} \le E \le E_{max}$$

Radial motion is periodic

$\sqrt{8/9}$



`Angular' equation

Angular velocity: $$\frac{d\phi}{d\sigma} = \frac{\ell}{\rho^2} - b$$

Attractive Lorentz force: $\ell < 0 \quad \Rightarrow \quad \dot{\phi} < 0:$

Clockwise motion modulated by radial oscillations

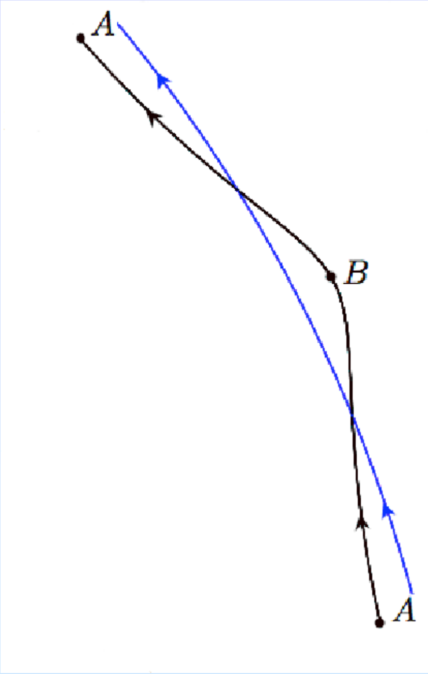
Repusive Lorentz force: $\ell > 0:$ 2 types of orbits:

(i) `smooth' and (ii) `curly'

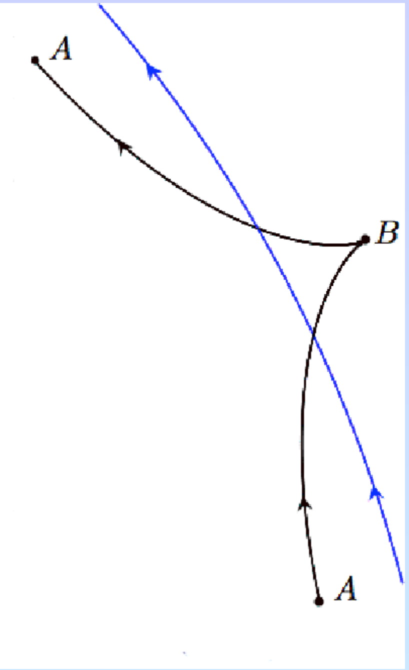
Separatrice $\tilde{E}_+(\ell) = \tilde{E}_*(\ell)$

Critical energy: $E_* = \sqrt{U(\rho_*)} = \left(1 - \frac{1}{\rho_*}\right)^{1/2}$

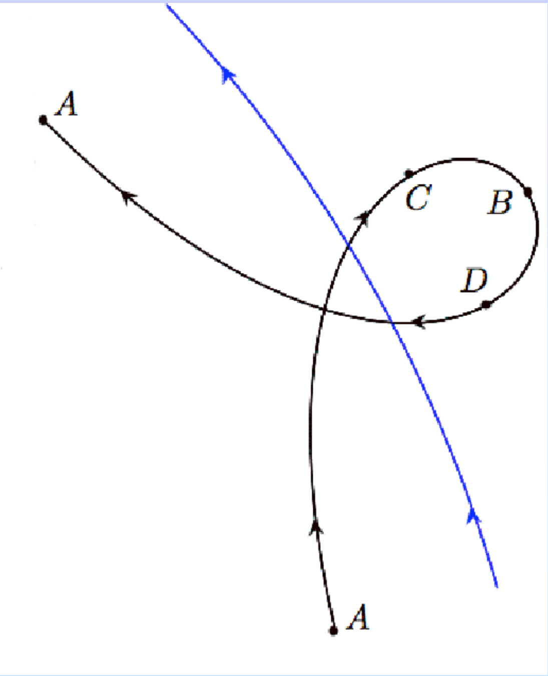
Types of orbits $\qquad \ell > 0$



$\rho_2 < \rho_*, \quad E_+ \in [E_{\min}, E_*)$

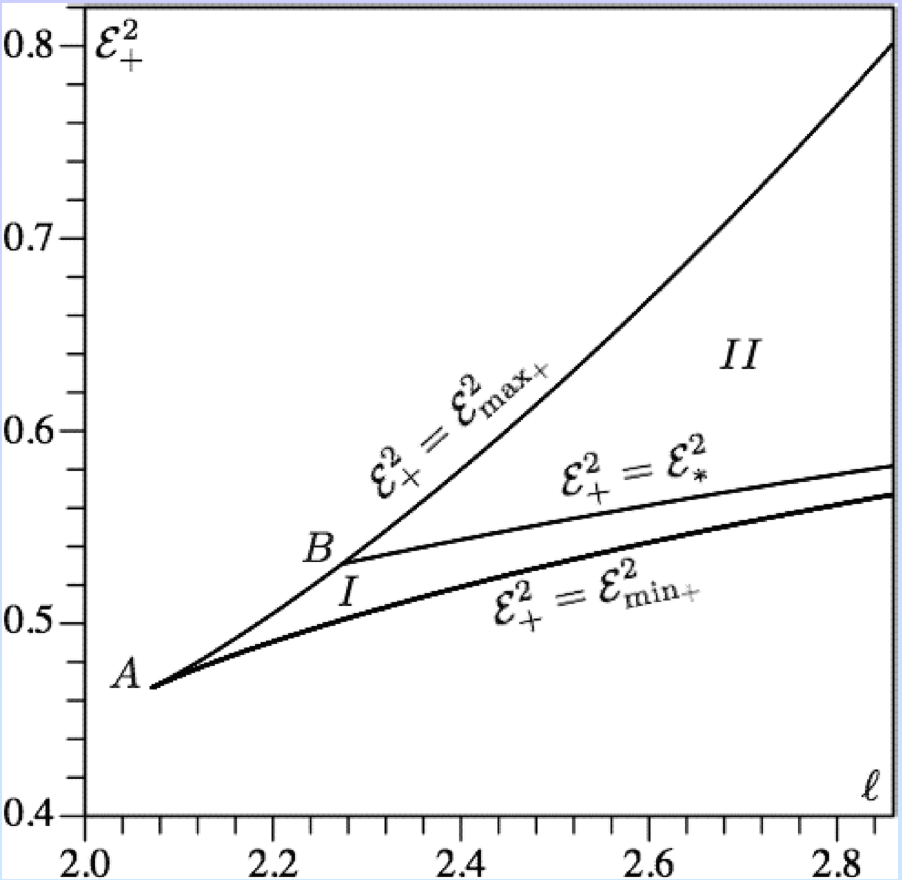


$\rho_2 = \rho_*, \quad E_+ = E_*$



$\rho_2 > \rho_*, \quad E_+ > E_*$

Energy– angular momentum plane



$$\tilde{E}_{\max}(\ell) \geq \tilde{E}_{\min}(\ell), \quad \ell \geq \ell_+ > 0$$

$$\tilde{E}_{\max}(\ell_+) = \tilde{E}_{\min}(\ell_+)$$

$$\frac{d(\tilde{E}^2)}{d\ell} = U_{,\ell}$$

$$= \frac{2b}{\rho^2}\left(1 - \frac{1}{\rho}\right)(\rho_*^2 - \rho^2),$$

$$\frac{d(\tilde{E}_{\max,\min})}{d\ell} > 0, \quad \tilde{E}_*(\ell) > \tilde{E}_{\min}(\ell)$$

(i) I+II -- bounded motion; (ii) I \`smooth' orbits; (iii) II \`curly' orbits

Separatrice $\tilde{E}_{+}(\ell) = \tilde{E}_{*}(\ell)$

Critical energy:

$$E_*^2 = m^2\left(1 - \sqrt{\frac{qBr_g^2}{2L}}\right) > 0$$

Critical magnetic field:

$$B_* = \frac{2L}{qr_g^2}\left(1 - \frac{E^2}{m^2}\right)^2$$

Approximate solution for bounded orbits

Expansion of the effective potential near its minimum:

$$U = \tilde{E}^2_{\min} + \omega_o^2 (\rho - \rho_{\min})^2 + \ldots, \quad \tilde{E}^2_{\min} = U(\rho_{\min})$$

Frequency of small oscillations about a circular orbit in the radial direction:

$$\omega_o^2 = \frac{1}{2} U_{,\rho\rho}(\rho_{\min}) = \frac{\tilde{E}^2_{\min}(\rho_{\min} - 3)}{2\rho^2_{\min}(\rho_{\min} - 1)^2} + \frac{4b^2}{\rho_{\min}}(\rho_{\min} - 1) > 0$$

Dimensional form:

$$\Omega_o^2 = \Omega_K^2 \left(1 - \frac{3r_g}{r_{\min}}\right) + \Omega_c^2 \left(1 - \frac{r_g}{r_{\min}}\right)^3$$

Stable circular orbits: $\Omega_o^2 > 0$

Marginally stable circular orbits: $\Omega_o = 0$

`Vertical' oscillations: $\Omega_{\|}^2 = \Omega_K^2$

D. V. Gal'tsov and V. I. Petukhov, Sov. Phys. JEPT **47**(3), 419 (1978)

- N. Aliev and D. V. Gal'tsov, Sov. Phys. Usp. **32**(1), 75 (1989)

Linearized dynamical equations

$$\left(\frac{d\rho}{d\sigma}\right)^2 = \tilde{E}^2 - \tilde{E}^2_{\min} - \omega_o^2(\rho - \rho_{\min})^2,$$

$$\frac{d\phi}{d\sigma} = \beta_0 + \beta_1(\rho - \rho_{\min}),$$

$$\beta_0 = \frac{\ell}{\rho^2_{\min}} - b, \quad \beta_1 = -\frac{2\ell}{\rho^3_{\min}}.$$

Validity of the approximation:

$$|\rho - \rho_{\min}| << \frac{\omega_o^2}{|U_{,\rho\rho\rho}(\rho_{\min})|}, \quad |\rho - \rho_{\min}| << \rho_{\min}$$

Approximate solution

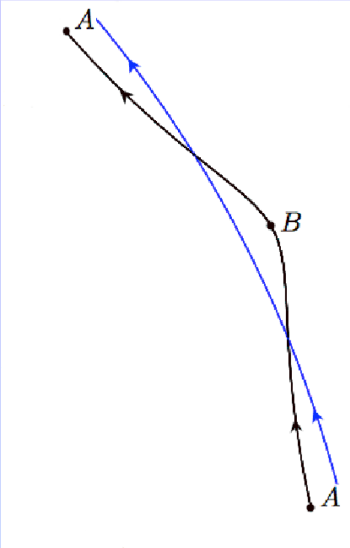
$$\rho(\sigma) = \rho_{\min} + A\cos(\omega_O \sigma),$$

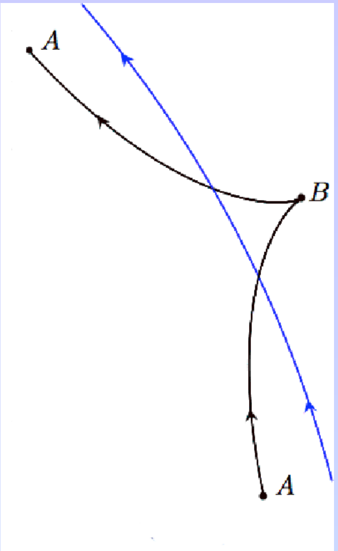
$$\phi(\sigma) = \beta_0 \sigma + \frac{\beta_1 A}{\omega_O} \sin(\omega_O \sigma)$$

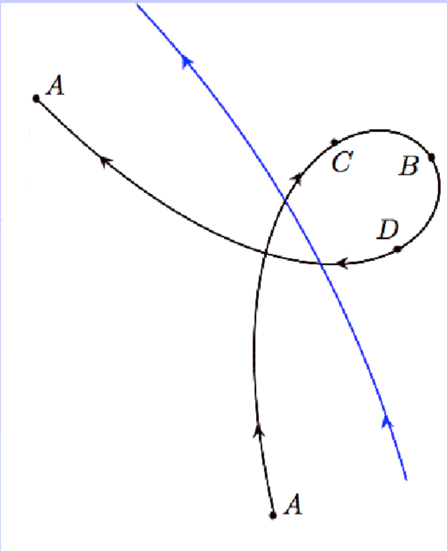
$$A = \frac{\sqrt{\tilde{E}^2 - \tilde{E}_{\min}^2}}{\omega_O}$$

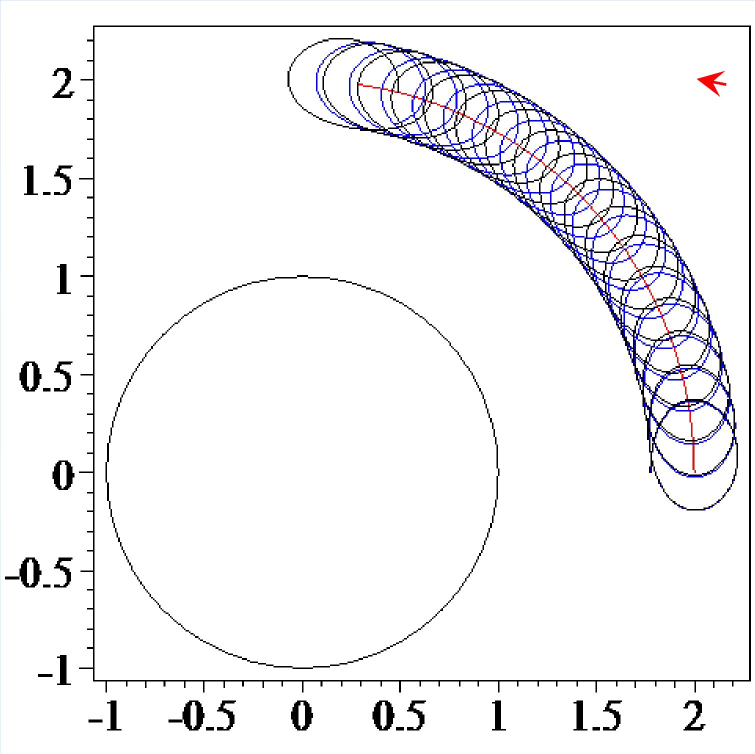
Critical amplitude

$$A = -\frac{\beta_0}{\beta_1} = \frac{\rho_{\min}}{2\rho_*^2}(\rho_*^2 - \rho_{\min}^2) \approx \rho_* - \rho_{\min} \approx \rho_{\min} \leftrightarrow \tilde{E} \approx \tilde{E}_*$$









$b \approx 8.7 \times 10^{14}, \quad \rho_* = 2.0,$

$A \approx 0.2, \quad E_{\min+} \approx 0.7,$

$\omega_o \approx 0.7, \quad N \approx 74$

Motion of the guiding center

Trajectory of the guiding center:

$$\rho = \rho_{\min}\Big|_{b>>1} \approx \rho_* - \frac{1}{8b^2\rho_*(\rho_* - 1)}$$

Drift velocity of the guiding center (anti-clockwise)

$$v = \rho_{\min}\beta_0 = \frac{\ell}{\rho_{\min}} - b\rho_{\min}$$

$$V = \frac{v}{\dot{t}} = \frac{b(\rho_{\min} - 1)}{\tilde{E}\rho_{\min}^2}(\rho_*^2 - \rho_{\min}^2)$$

Approximation of a weak gravitational field:

$$\rho_{\min} >> 1, \quad E \approx mc^2$$

$$V\Big|_{b>>1} \approx \frac{1}{4b\rho_{\min}^2} = \frac{mr_g}{2qBr_O^2} = \frac{g_0}{\Omega_c},$$

$$g_0 = \frac{r_g}{2r_o^2}, \quad \Omega_c = \frac{|qB|c}{E}$$

Approximation of a strong gravitational field:

$$V\Big|_{b>>1} \propto r_g \frac{\Omega_K^2}{\Omega_c}, \quad \Omega_K^2 = \frac{r_g^{1/2}c}{r^{3/2}\sqrt{2}}$$

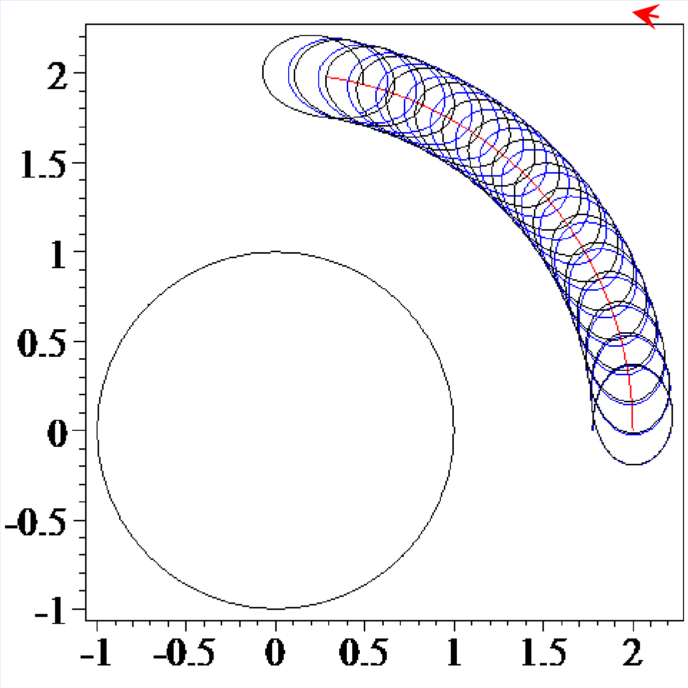
Ratio of the frequency of radial oscillations to the orbital frequency:

$$N \equiv \frac{\omega_O}{\beta_0} = \left. \frac{\sqrt{\rho_{\min}^4 (3\rho_{\min} - 1) + \rho_*^4 (\rho_{\min} - 1)}}{\rho_{\min}^{1/2} (\rho_*^2 - \rho_{\min}^2)} \right|_{b \gg 1} \approx 8 b^2 \rho_*^{3/2} (\rho_* - 1)^{3/2}$$

The ratio is equal to the number of curls per one revolution, if the amplitude is greater than the critical one:

$$A > -\frac{\beta_0}{\beta_1} = \frac{\rho_{\min}}{2\rho_*^2} (\rho_*^2 - \rho_{\min}^2)$$

Electron orbit: approximate solution (blue curve), numerical solution (black curve), motion of the guiding center (red curve).



Summary

- Effective potential corresponding to the motion of a charged particle in the equatorial plane of a weakly magnetized black hole has either two or none extrema.

- In the critical case, when the extrema coincide, the extremal point determines position of the innermost stable circular orbit (ISCO).

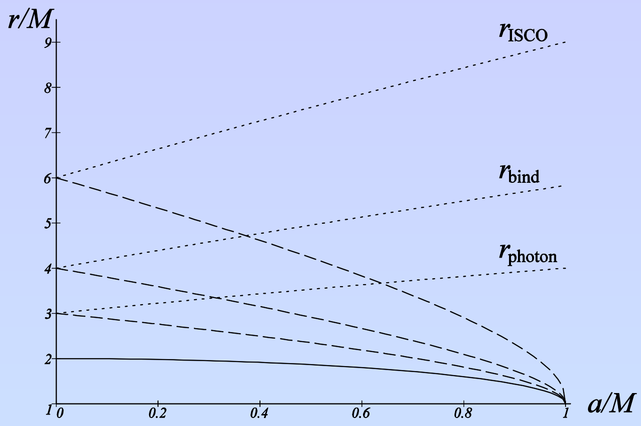
- We obtained expressions for the radii and the angular momenta of the ISCO in the approximation of a strong magnetic field.

★ We analyzed bounded trajectories of a charged particle and constructed an approximate solution to the dynamical equations. Using the approximate solution we found trajectory of the guiding center and its gravitational drift velocity, as well as, the number of curls per one revolution. The maximal value of the number grows with the increase of the magnetic field.

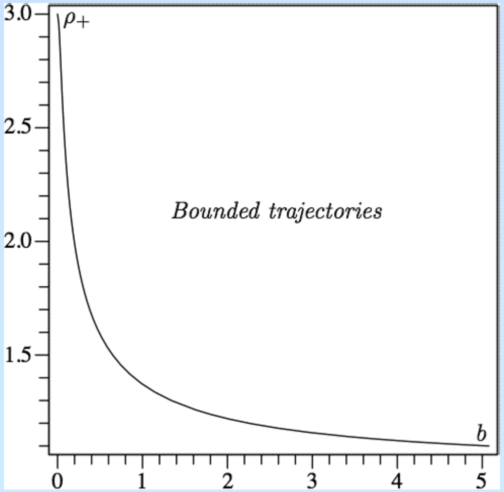
★ Result of our analysis shows that if the Lorentz force acting on a charged particle is repulsive, its bounded trajectory can be of two different types: with curls and without.The critical trajectory, which has cusps, separates these cases. We calculated the corresponding critical value of the magnetic field.

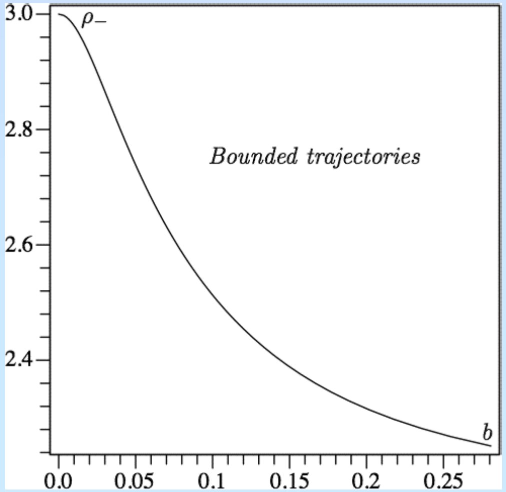
Lessons

(i) Radius of ISCO



Kerr



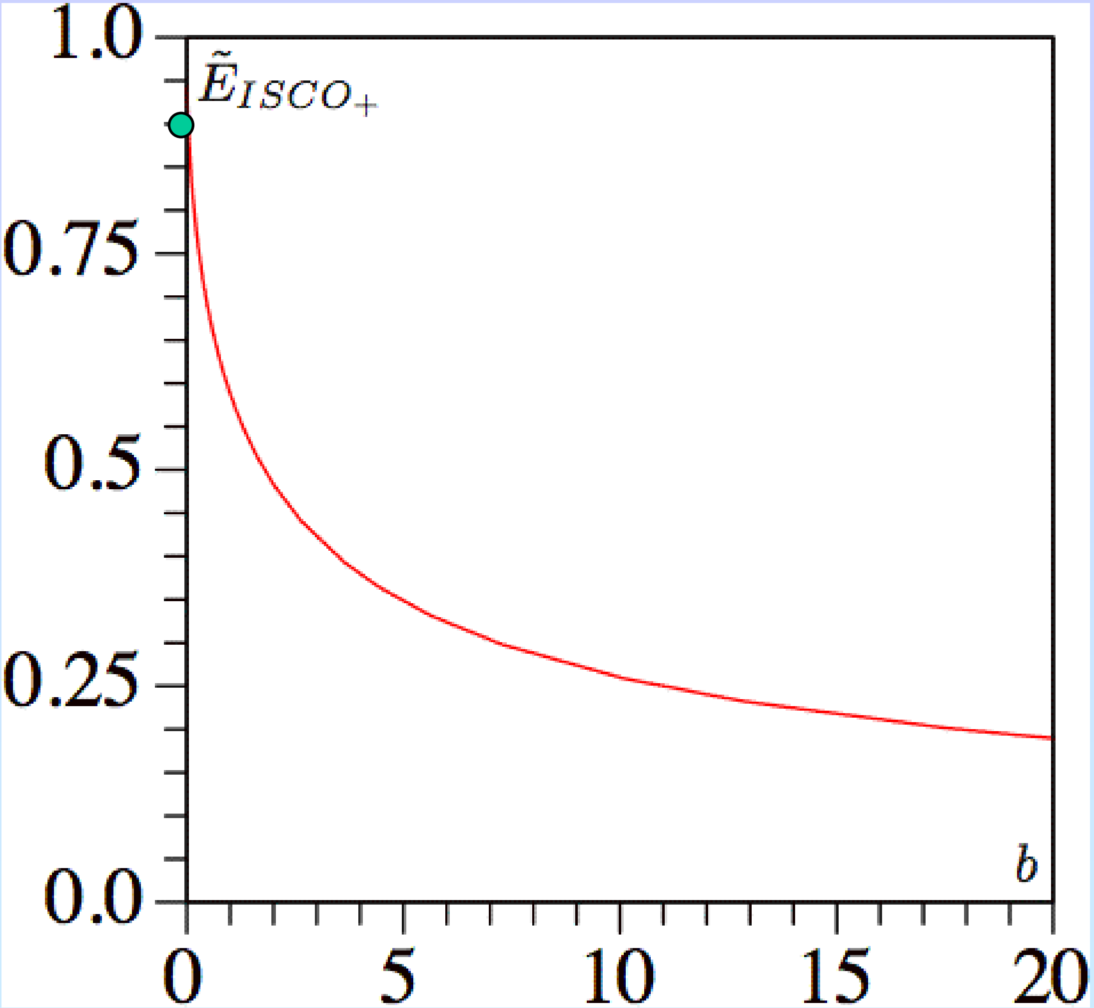


Magnetized BH

(ii) Energy release

	$a=0$	$a=M$	
		$\mathcal{L}>0$	$\mathcal{L}<0$
$\mathcal{E}$	$\sqrt{8/9}$	$\sqrt{1/3}$	$\sqrt{25/27}$
$1-\mathcal{E}$	0.0572	0.4236	0.0377
$\mid\mathcal{L}\mid/M$	$2\sqrt{3}$	$2/\sqrt{3}$	$22/3\sqrt{3}$

$$\tilde{E}_{ISCO+} \sim \frac{2}{3^{3/4} b^{1/2}}$$



(iii) Periods