

Deriving the Simplest Gauge/ String Duality

with Rajesh Gopakumar

- Part I: Open-Closed-Open Triality (arXiv: 2212.05999)
- Part II: The B-Model
- Part III: The A-Model

Today's Focus:
**Moments in the Gaussian Matrix Model
as Stringy Correlators via OCO-Triality**

Edward Mazenc (UChicago)
***Math/String Seminar*- June 27th 2023, IPMU**

The Big Picture

Holography as Open/Closed String Duality

Can we make
't Hooft's insight
precise?

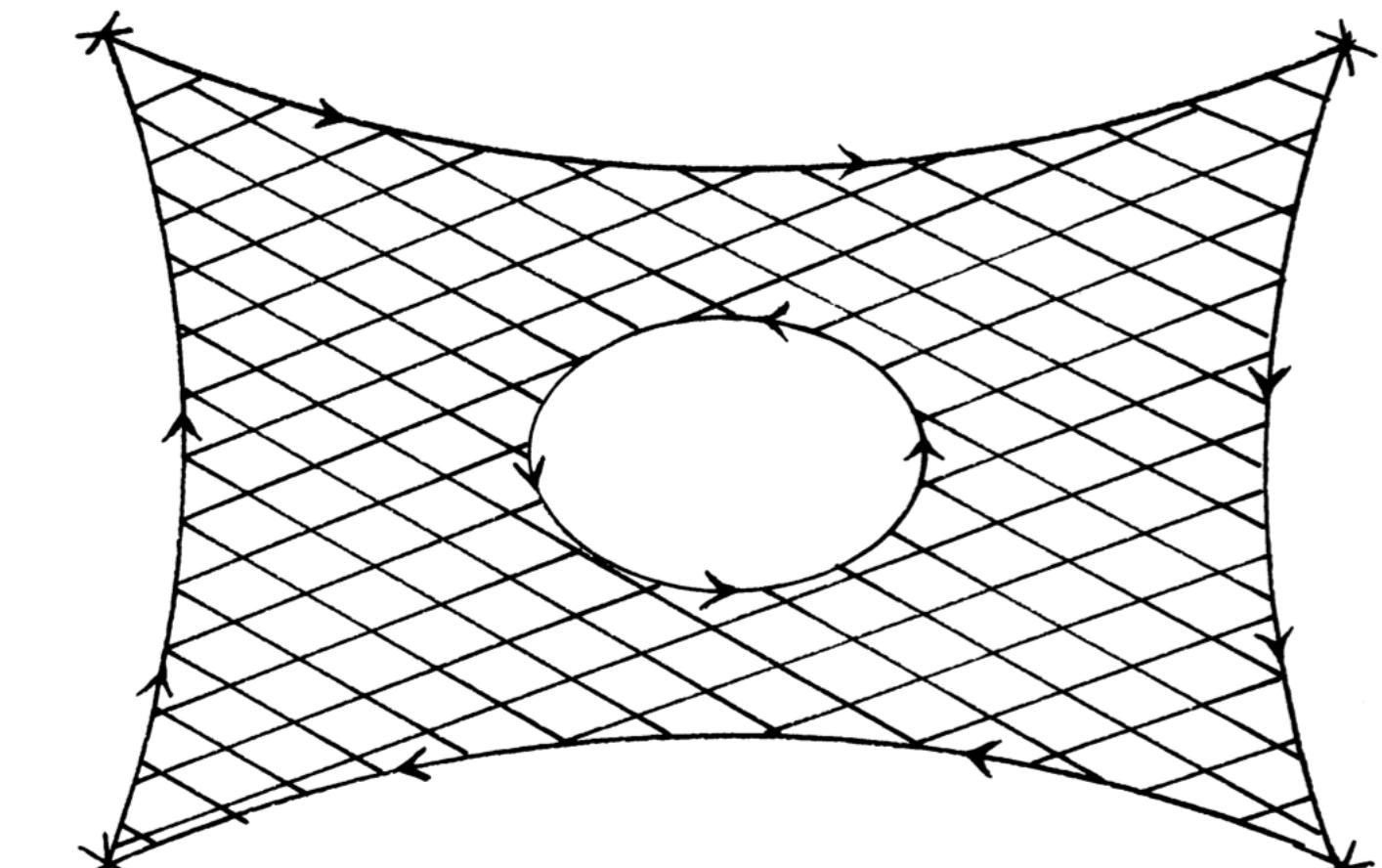
Closed String Description
Riemann surfaces with
marked points
e.g. *IIB on $AdS_5 \times S^5$*

Open String Description
Matrices
e.g. $\mathcal{N} = 4$ SYM

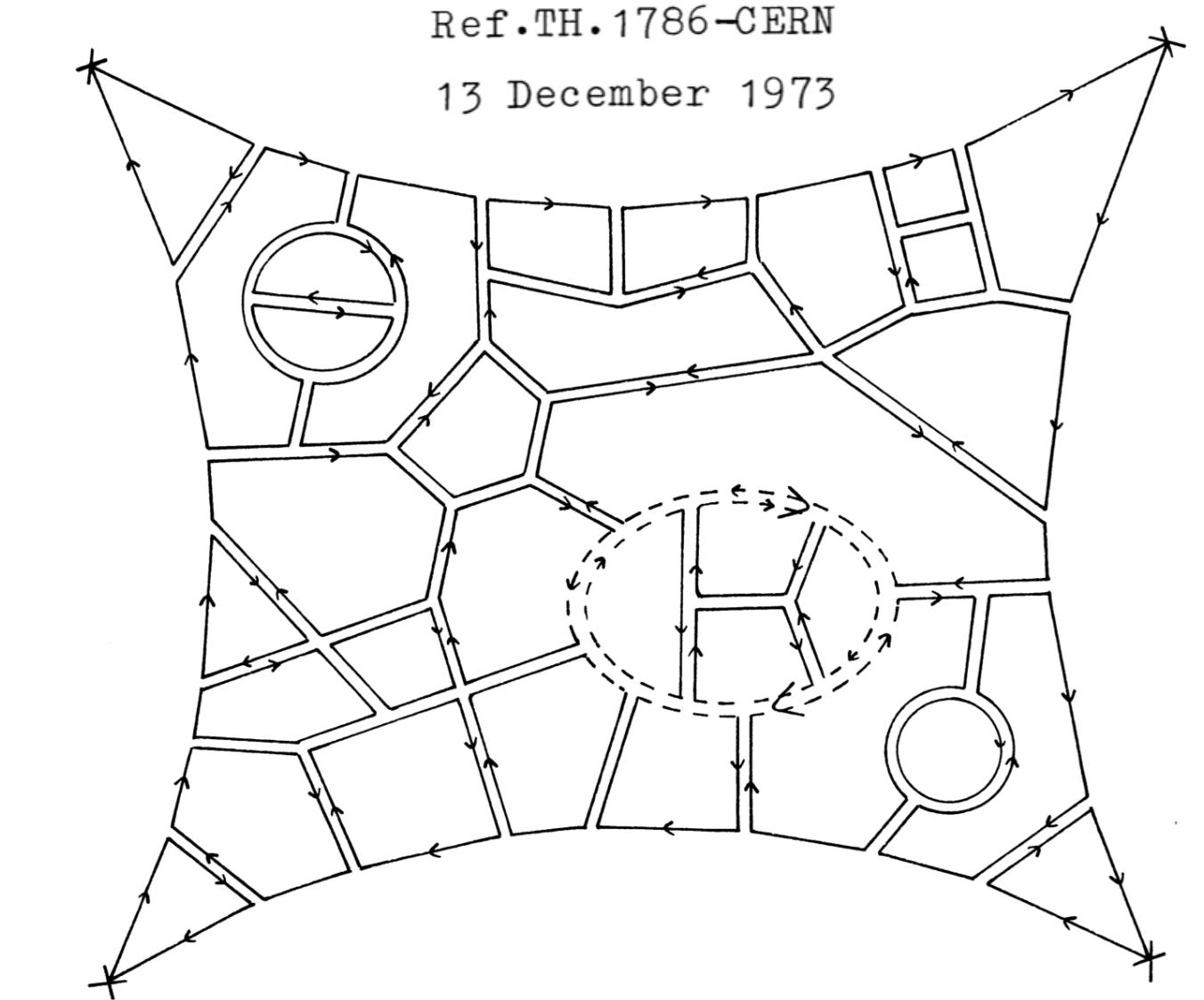
Low-energy “Effective”
Description
GR + QFT
e.g. *QG in Anti-de Sitter*

$$\int dM_{N \times N} e^{-\frac{N}{g} Tr V(M)} \prod_{i=1}^n Tr M^{k_i}$$

[BIPZ'80]



(a)



- Figure 3 -

A PLANAR DIAGRAM THEORY FOR STRONG INTERACTIONS

G. 't Hooft
CERN - Geneva

Matrix Models and Strings

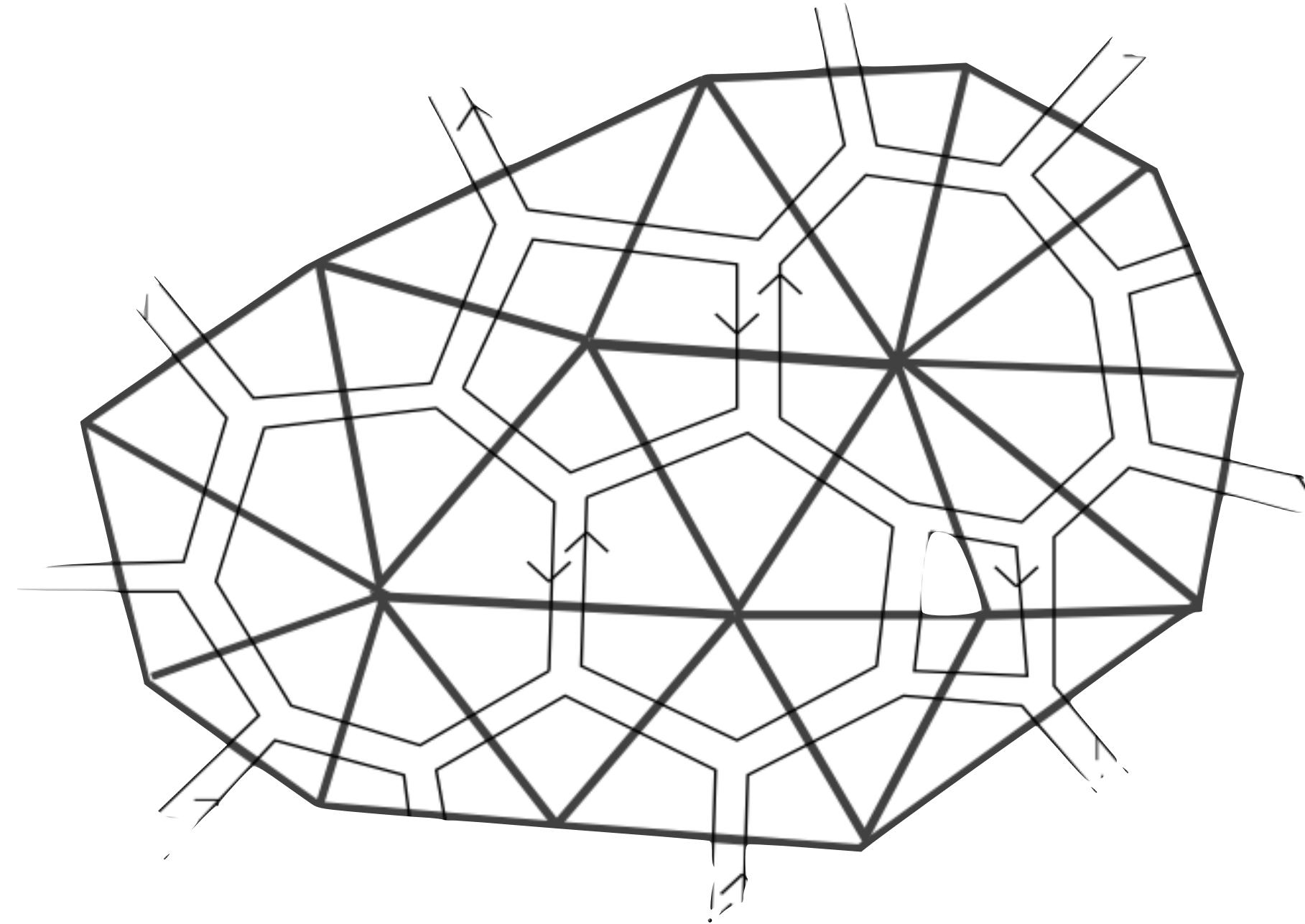
What is new here?

- **Previously required double-scaling limit:** Feynman diagrams as “latticization” of the worldsheet

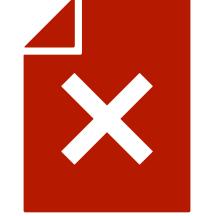
[Cf. Gross-Migdal; Douglas-Shenker; Brezin-Kazakov]

 ***Not 't Hooft limit as in AdS/CFT!***

- **Dijkgraaf-Vafa:** matrix integrals from localized holomorphic Chern-Simons Theory (i.e. open string field theory on branes in non-compact CYs)



Their proposal: closed B-model string on spectral curve of matrix model?

 ***No worldsheet theory
No operator dictionary/correlators***

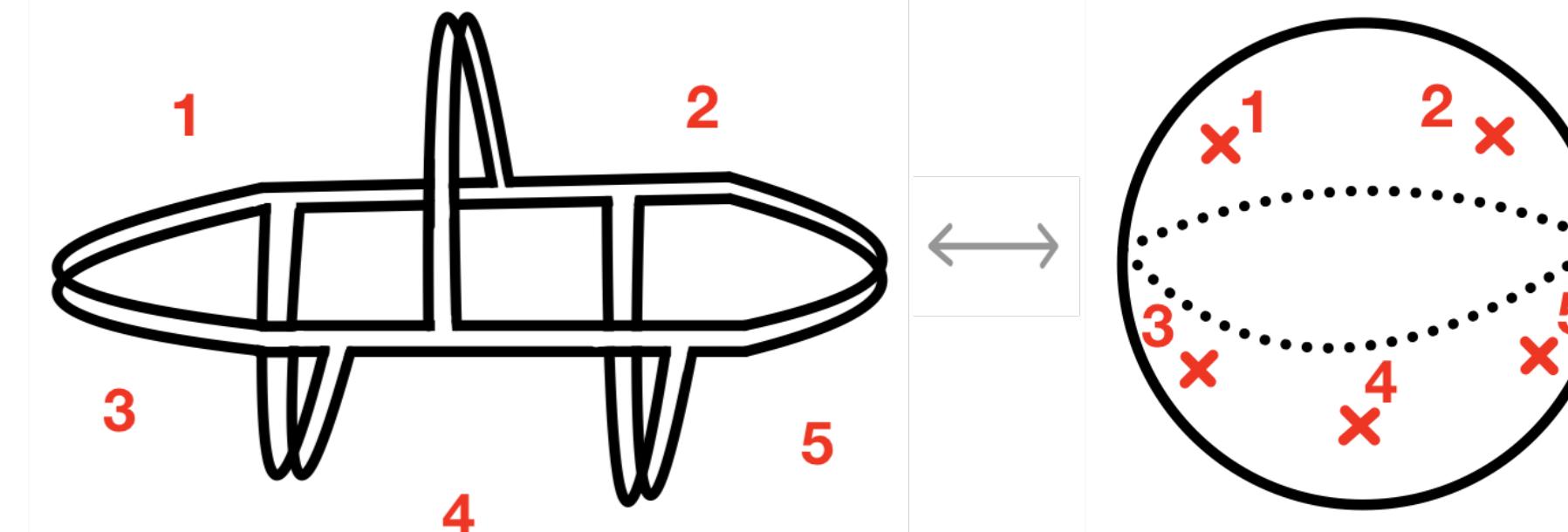
Three Main Takeaways

How do large N gauge theories reorganize themselves into closed string theories?

Use Strebel parametrization of $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$ (via lengths of edges)

[Cf. Strebel, Kontsevich]

“Each gauge theory FD as a string worldsheet”



What constitutes a derivation of open-closed duality?

We rewrite all correlators of traces exactly as integrals over $\mathcal{M}_{g,n}$

Open-Closed Operator dictionary: $\text{Tr}M^k \leftrightarrow \mathcal{O}_k$

Can we identify a world-sheet theory which gives rise to these moduli-space integrals?

2 closed string theories (related by mirror symmetry)

B-Model: Superpotential derived from matrix model spectral curve

A-Model: Twisted $(SL(2, \mathbb{R})/U(1))_1$ coset model with momentum condensate

Today's Roadmap

The Proposal

An equality of matrix and string correlators

The Verification

OCO-Triality: 2 equivalent matrix models

OCO-Triality: the broader lessons

The Derivation

A-Model: Lattice points on $\mathcal{M}_{g,n}$, Belyi Maps & the BMN-limit

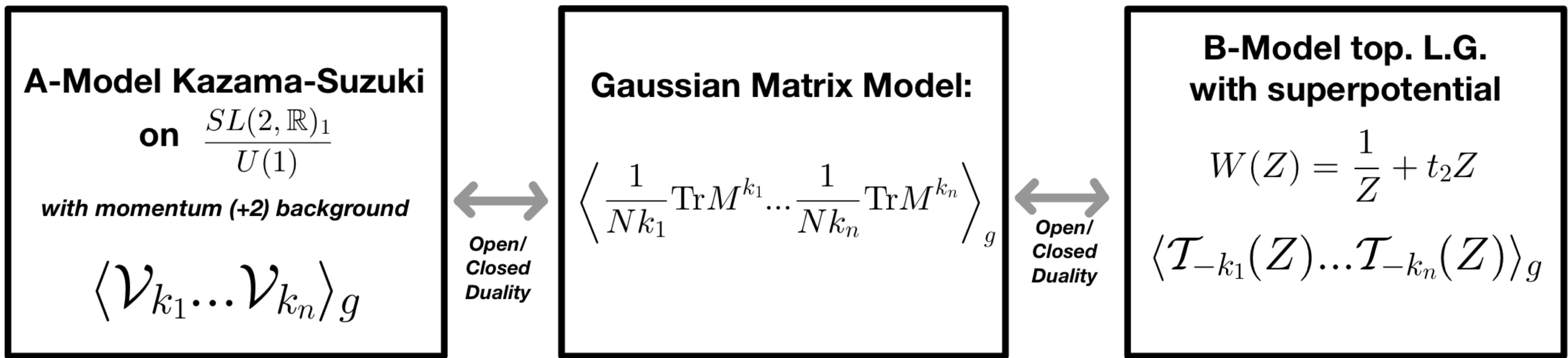
B-Model: Topological Recursion, Top. Matter + 2d gravity & the BMN-limit

The Proposal

Matrix Correlators as Stringy n -point Functions

Concrete String Dual to the Gaussian MM

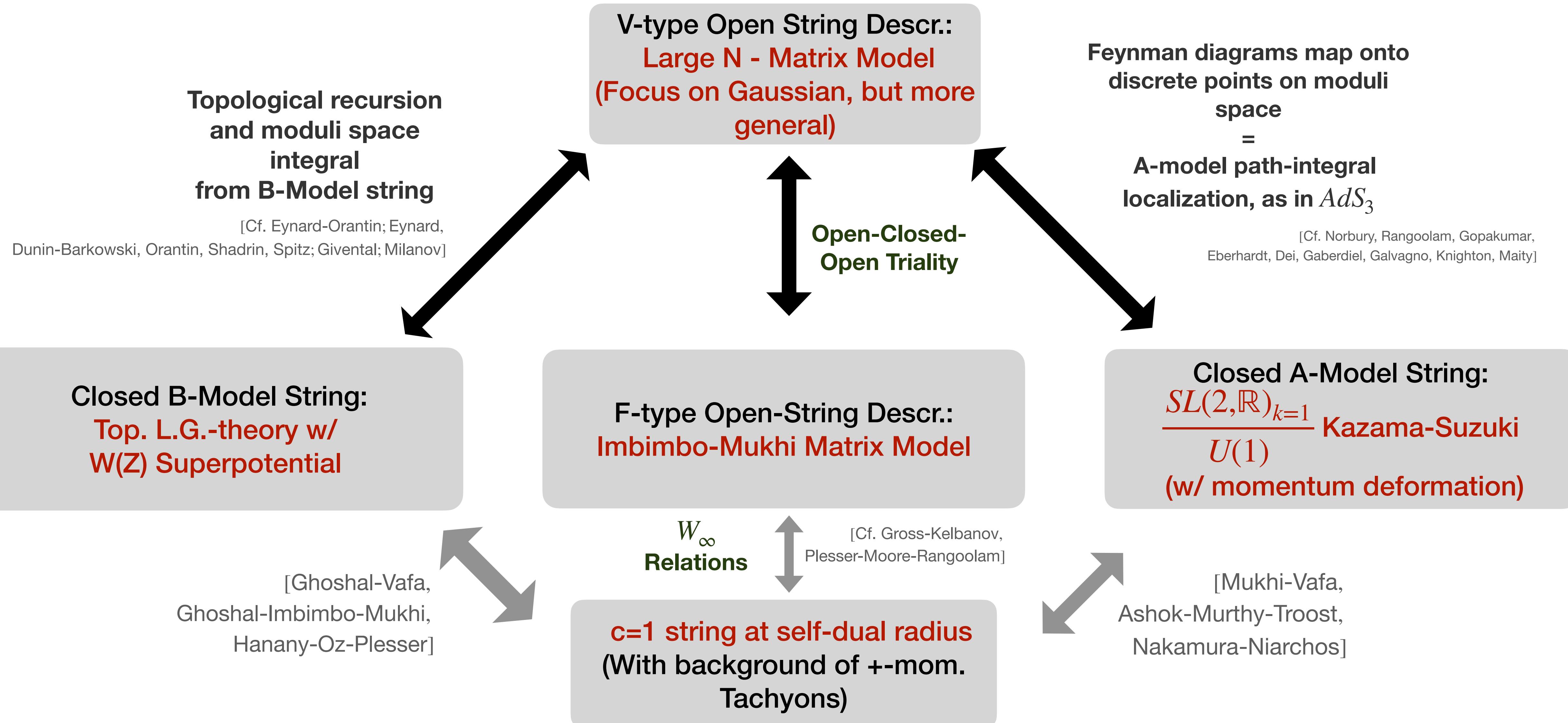
An explicit all-genus mapping of correlators



(Interacting theories correspond to deformations of background)

The Underlying Logic

A web of dualities & papers

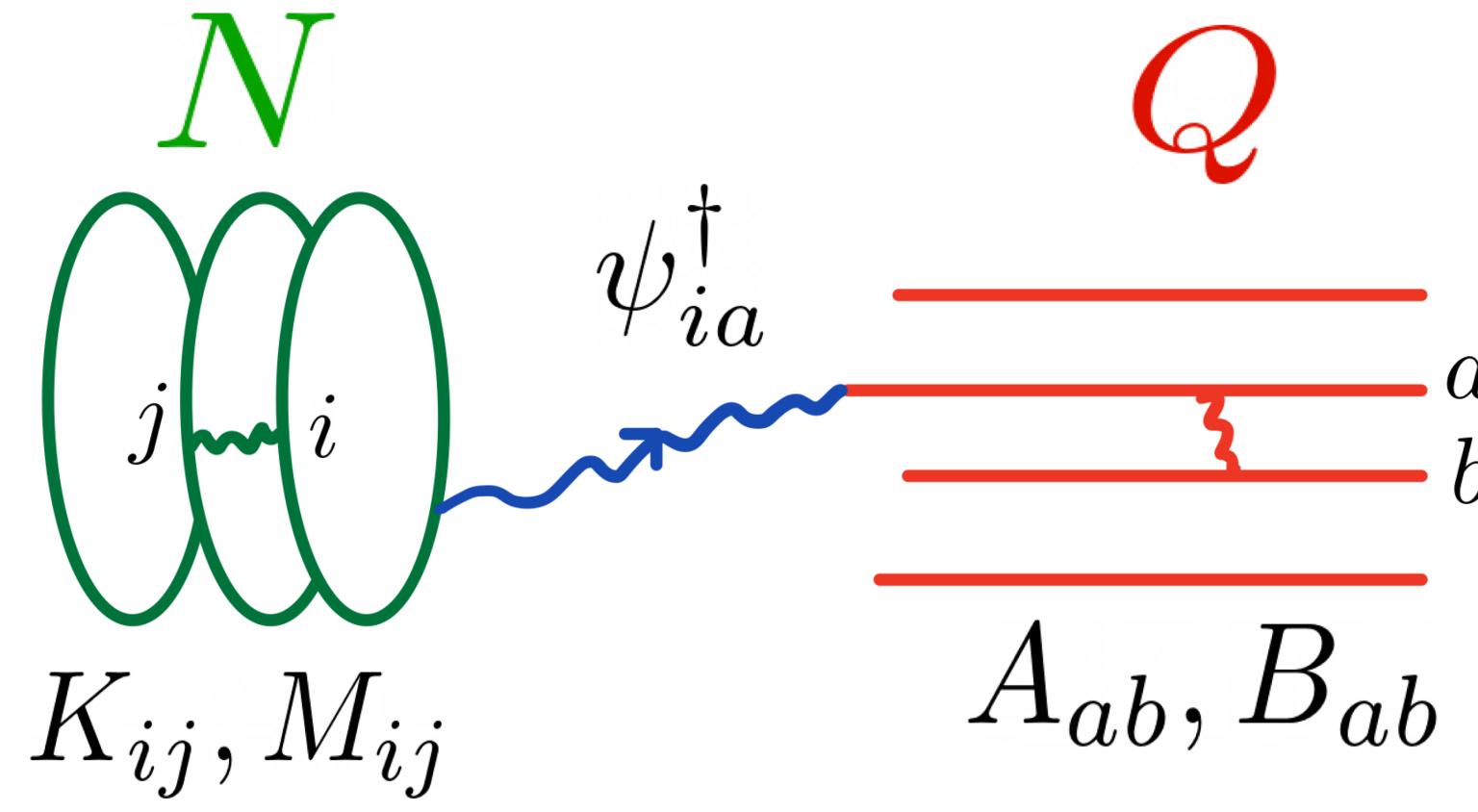


The Verification

An Equality of Matrix Integrals and the Appearance of the $c=1$ String

A New Equality of 2 Matrix Integrals

Open-Closed-Open Triality as Verification



$$\begin{aligned} Z(X, Y) &= \frac{1}{Z_N} \int dK dM_{N \times N} e^{+\frac{1}{g} \text{Tr}(V(K) - K(M - Y))} \prod_{a=1}^Q \det(x_a - M) \\ &= \frac{(-1)^{NQ}}{Z_Q} \int dA dB_{Q \times Q} e^{-\frac{1}{g} \text{Tr}(V(A) + A(B - X))} \prod_{i=1}^N \det(y_i - B). \end{aligned}$$

Key Steps

$$Z(X, Y) = \frac{1}{Z_N} \int dK d\psi d\psi^\dagger e^{\frac{1}{g} \text{Tr}_N(V(K) + KY) + \psi_{ia}^\dagger X_{ab} \psi_{ib}} \int dM e^{-\frac{1}{g} M_{ij} (K_{ji} - g\psi_{ia}^\dagger \psi_{ja})}$$

$$= \int dK d\psi d\psi^\dagger e^{\frac{1}{g} \text{Tr}_N(V(K) + KY) + \psi_{ia}^\dagger X_{ab} \psi_{ib}} \delta(K_{ji} - g\psi_{ia}^\dagger \psi_{ja})$$

$$= \int d\psi d\psi^\dagger e^{\frac{1}{g} \text{Tr}_N V[-g\psi\psi^\dagger] + \psi_{ia}^\dagger (X_{ab}\delta_{ij} - \delta_{ab}Y_{ij}) \psi_{jb}}$$

“Color-Flavor Transformation”

$$\text{Tr}_N [(\psi\psi^\dagger)^k] = (-1)^{2k-1} \text{Tr}_Q [(\psi^\dagger\psi)^k]$$

→ Reverse Steps

[Cf. Maldacena-Moore-Seiberg-Shih,
Aganagic-Dijkgraaf-Klemm-Marino-Vafa,
Goel - H. Verlinde,
Altland-Sonner]

$$A_{ba} = -g\psi_{ia}^\dagger \psi_{ib}$$

The Imbimbo-Mukhi Matrix Model

Traces as tachyon modes in c=1 at self-dual radius

[Cf. "Kontsevich-Penner-Model", Chekhov et al.,
Bonora-Xiong, Moore-Plesser-Rangoolam]

$$\frac{1}{Z_N} \int dK dM_{N \times N} e^{-\frac{1}{g} \text{Tr}(V_p(K) - K(M - P))} \prod_{a=1}^Q \det(x_a - M)$$

$$=$$

$$\frac{1}{Z_Q} \int dA dB_{Q \times Q} e^{+\frac{1}{g} \text{Tr}(V_p(A) + A(B - X))} \prod_{i=1}^N \det(\gamma_i - B)$$

$$\frac{1}{Z_N} \int dK dM_{N \times N} e^{+N \text{Tr}(V_p(K) - KM + \sum_{k=1}^{\infty} \bar{t}_k M^k)}$$

$$=$$

$$\det(X)^{-N} \int dA_{Q \times Q} e^{-N \text{Tr}(V_p(A) - AX) - (N+Q) \text{Tr} \log(A)} \times (\text{Penner Model})$$

$$Z_{IM}(t_k, \bar{t}_k) = \det(X)^{-i\mu} \int dA_{Q \times Q} e^{+i\mu \sum_{k=1}^{\infty} t_k \text{Tr}(A^k) + i\mu \text{Tr}(AX) - (i\mu + Q) \text{Tr} \log(A)}$$

$$N = i\mu \quad \begin{matrix} \text{Genus-expansion} \\ \text{= large } N \text{ expansion} \end{matrix}$$

$$\bar{t}_k = \frac{1}{k} \text{Tr}_Q (X^{-k})$$

Exact Operator Dictionary:

$$\frac{1}{Nk} \text{Tr} M^k \leftrightarrow \frac{\partial}{\partial \bar{t}_k} \leftrightarrow T_{-k}$$

Generating Function of “Tachyon”
correlators in “c=1 2d-string theory” in large
phase space

All Genus 1-pt Function

A detailed sanity check

Gaussian Matrix Model \leftrightarrow c=1 string at self-dual radius with momentum +2 tachyon-background

$$\left\langle \frac{1}{N} \text{Tr} M^{2n} \right\rangle_{Gaussian} = 2n \langle T_{-2n} \rangle_{t_2}$$

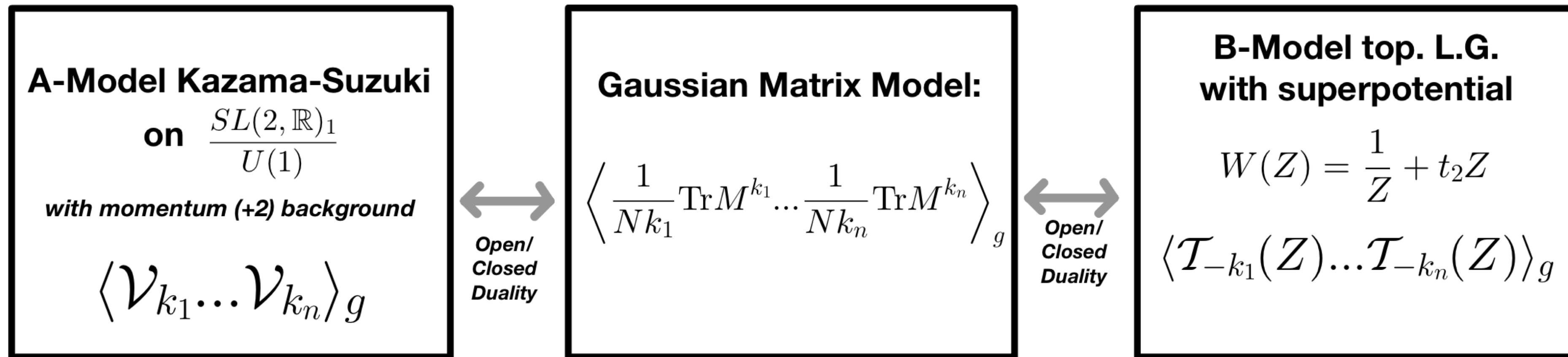


[Gopakumar-Mukhi ('95),
–unpublished]

$$= \frac{1}{N^{2n+1}} \frac{1}{2n+1} \oint dz z^{-N} e^{-\frac{N}{2} t_2 z^2} \partial_z^{2n+1} \left(z^N e^{+\frac{N}{2} t_2 z^2} \right)$$

How This Verifies Our Proposal

From $c=1$ to A- and B-model string



$$T_{-k} \leftrightarrow \mathcal{V}_k = ce^{-\frac{(k-2)}{\sqrt{2}}\phi} e^{-i\frac{k}{\sqrt{2}}X}$$

[Mukhi-Vafa,
Ashok-Murthy-Troost,
Nakamura-Niarchos]

$$\frac{1}{Nk} \text{Tr} M^k \leftrightarrow T_{-k}$$

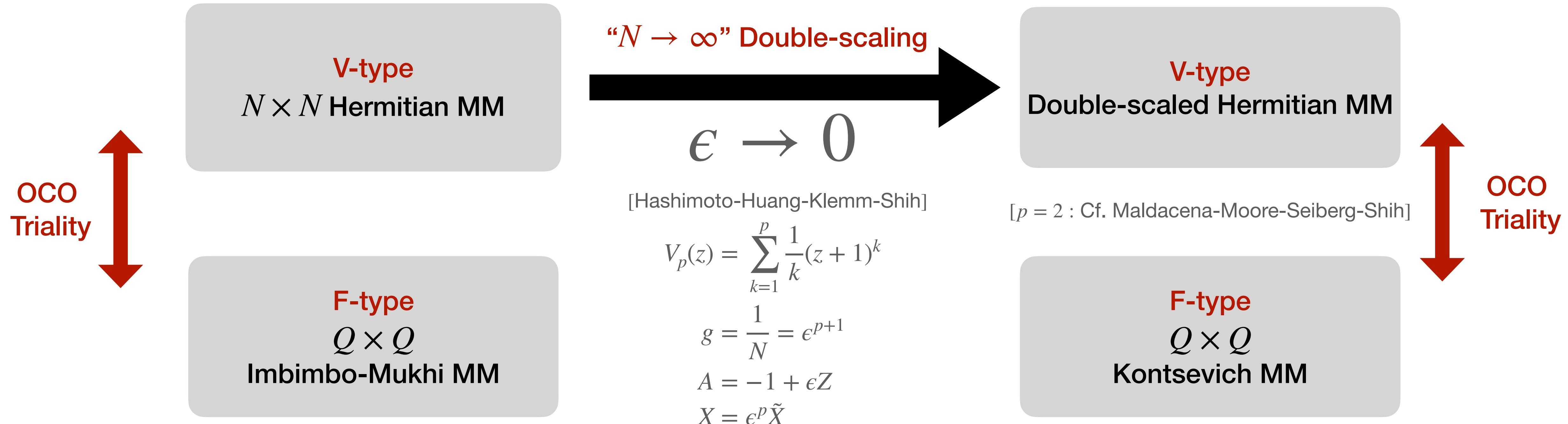
$$T_{-k} \Leftrightarrow \mathcal{T}_{-k}(Z) \equiv \left(\frac{\partial}{\partial Z} W(Z, t)^k \right)_-$$

[Ghoshal-Vafa,
Ghoshal-Imbimbo-Mukhi,
Hanany-Oz-Plesser]

From duality with $c=1$ string

Role of double-scaling?

Imbimbo-Mukhi vs. Kontsevich



Bottom Path: “N” Double-scaling Q fixed!

$Z_{IM} \propto \int dA_{Q \times Q} e^{-\frac{1}{g} \text{Tr} (V_p(A) - AX) - (N+Q) \text{Tr} \log(A)}$

$Z_{Kontsevich} \propto \int dZ_{Q \times Q} e^{\text{Tr} \left(\frac{Z^{p+1}}{p+1} + Z \tilde{X} \right)}$

[Gaiotto-Rastelli]

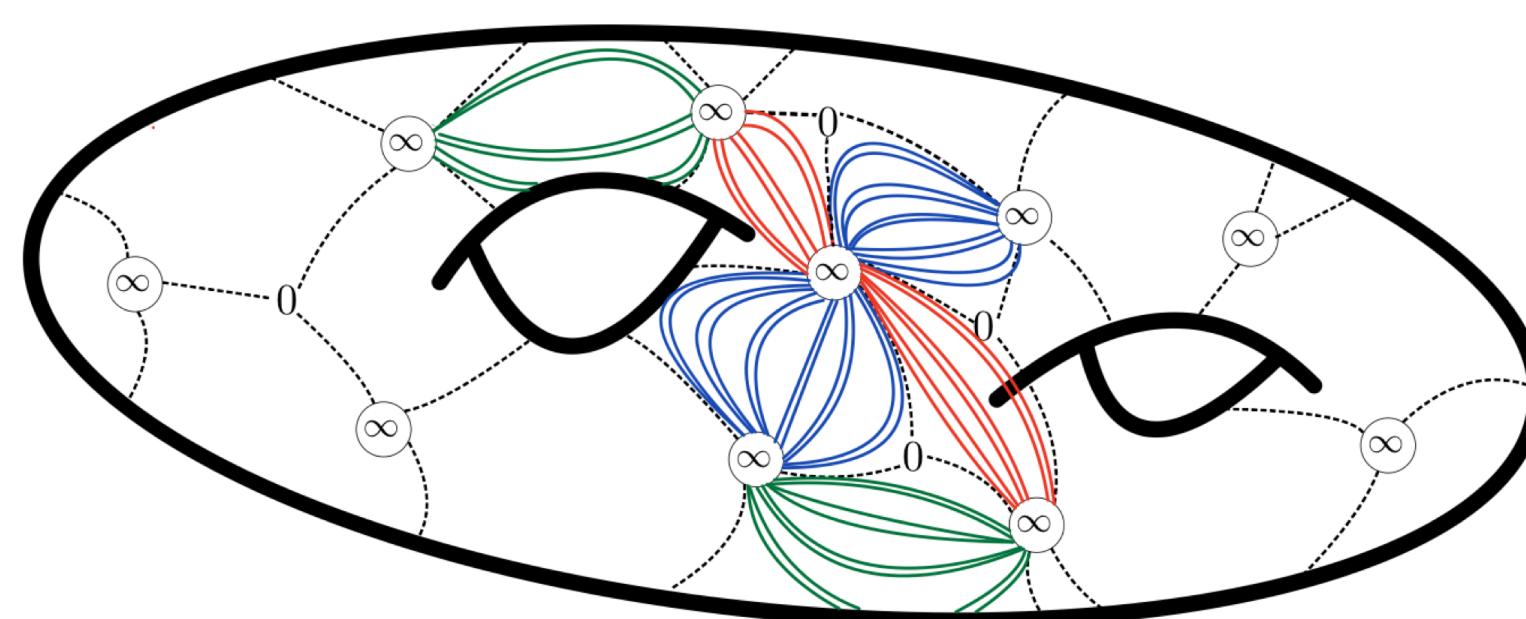
p=2: OSFT c=28 Liouville +c=-2 on Q FZZT

OCO-Triality beyond Matrix Models

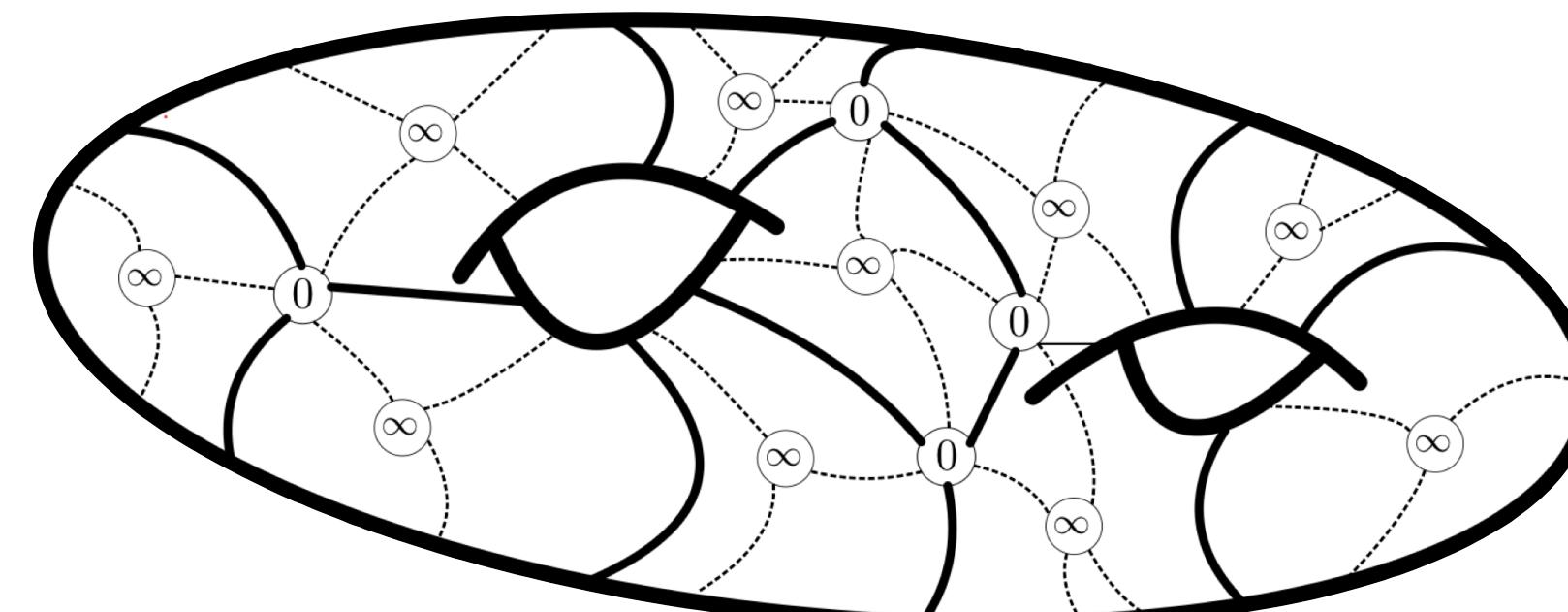
2 ways to reconstruct closed strings from open ones

OCO-Triality as Graph Duality

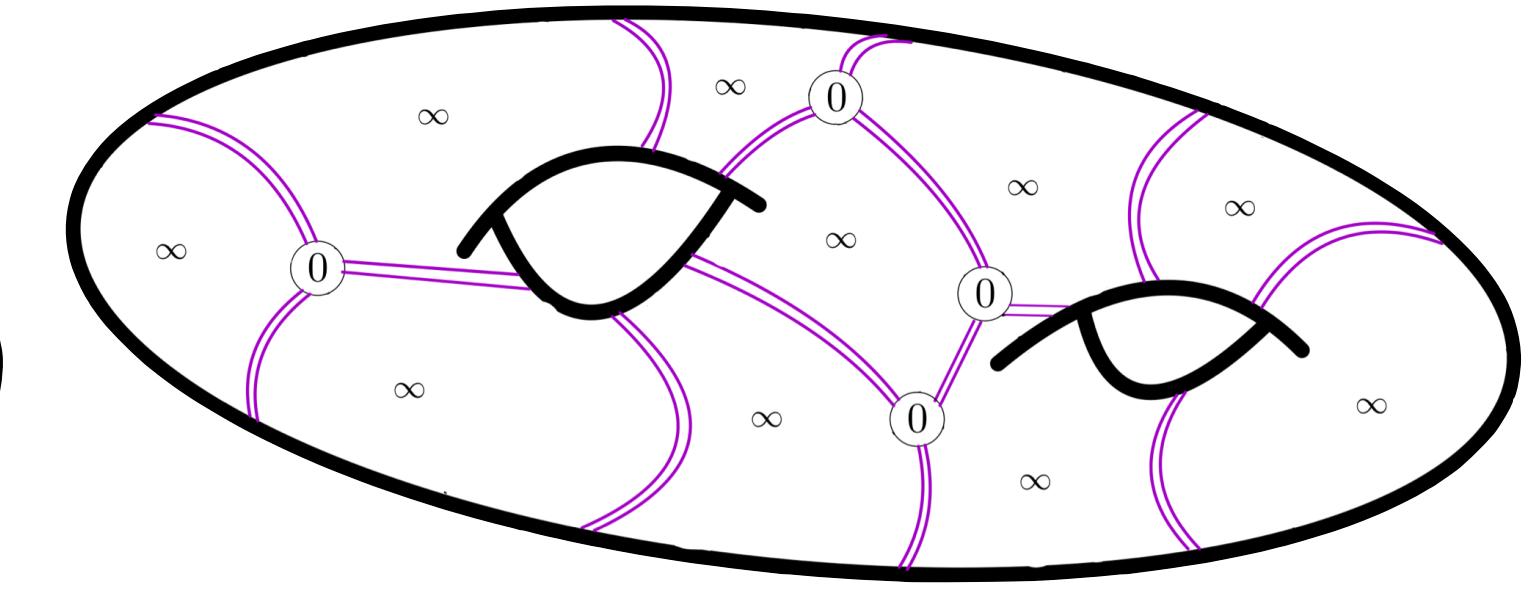
V/F-type: 2 ways to reconstruct closed strings from gauge theory FDs



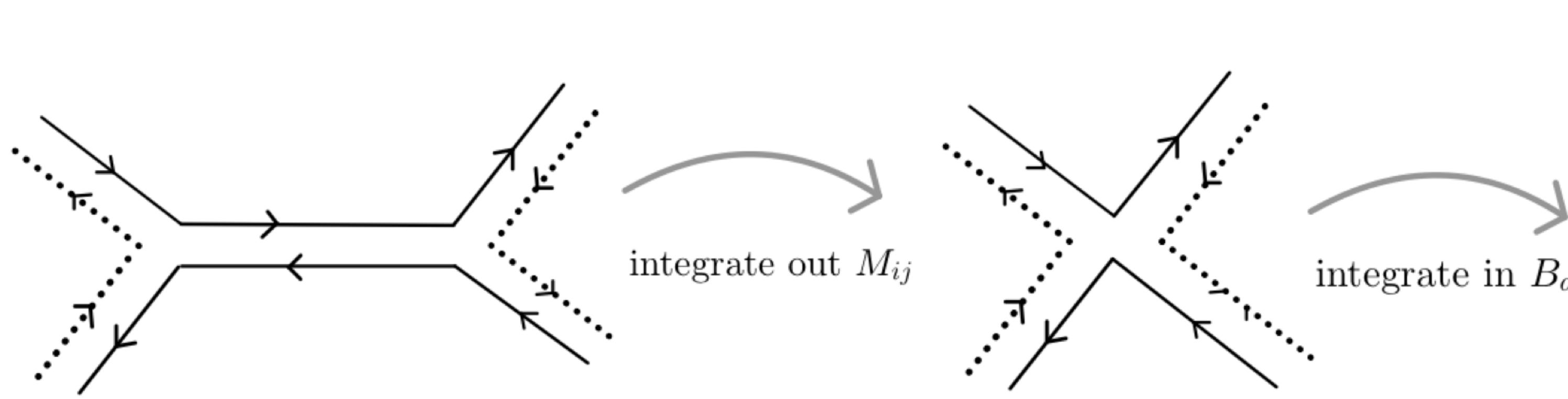
V-type Duality



Closed String WS



F-type Duality

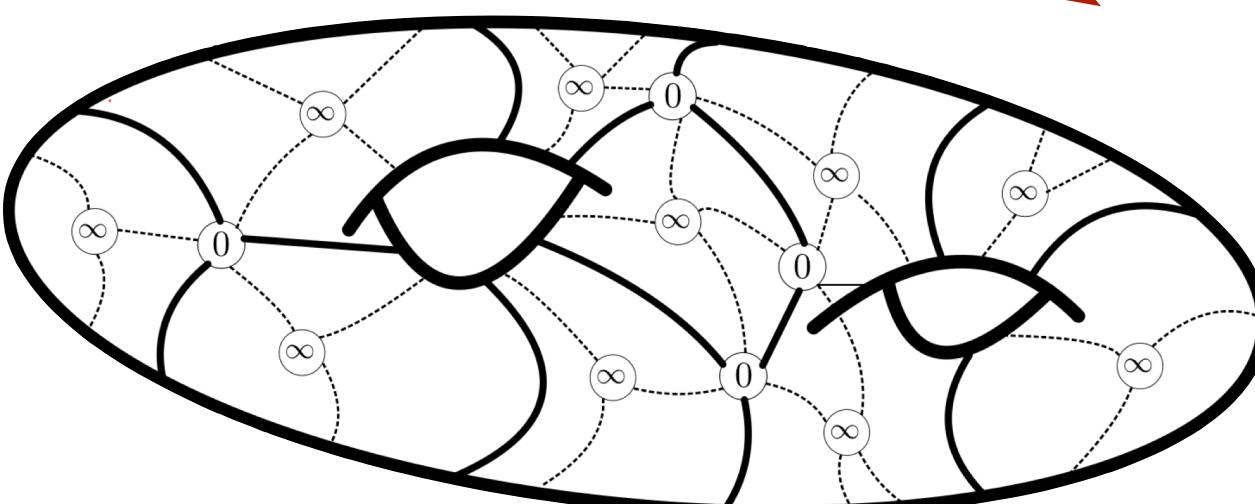


Edge →
Dual Edge

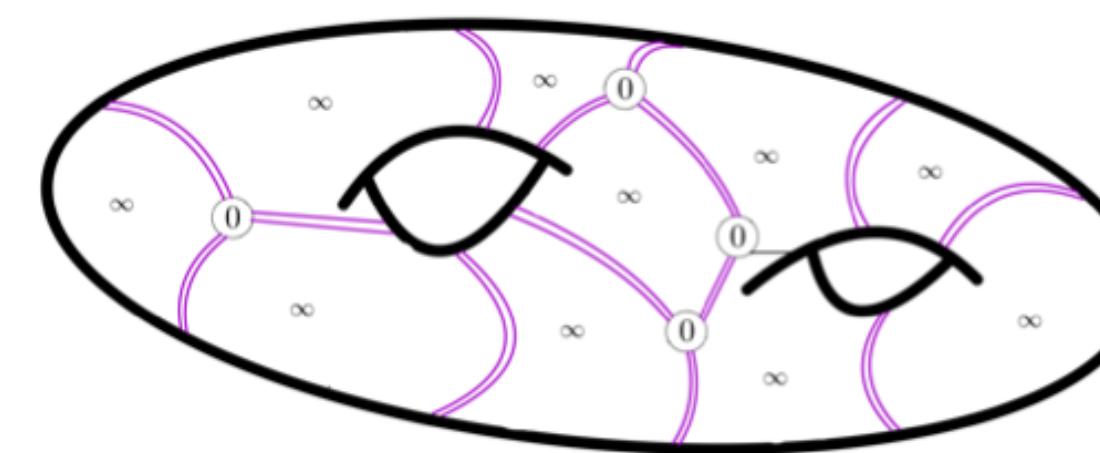
More on V- versus F-type

Reconstructing closed strings from open string strips

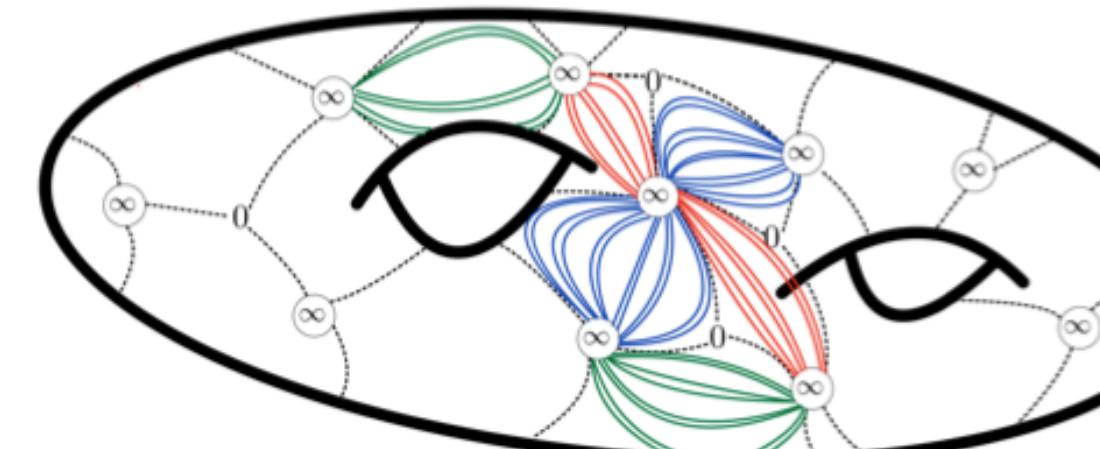
Closed String WS



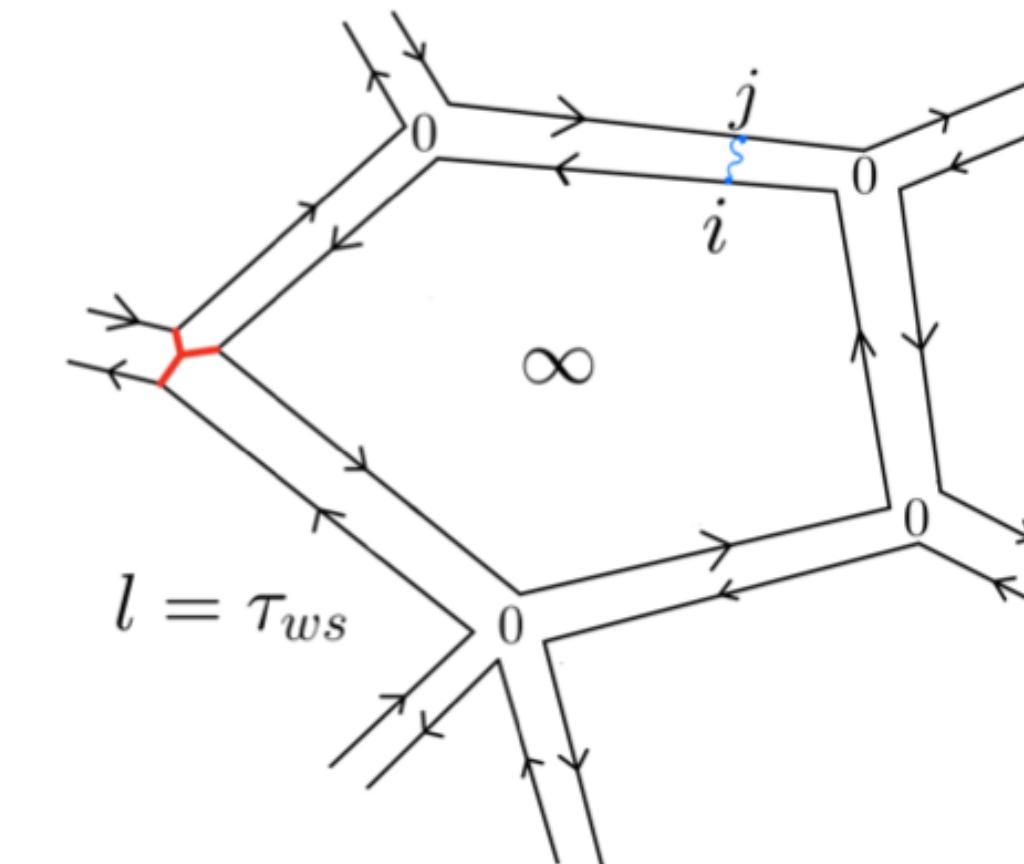
Cf. Gopakumar



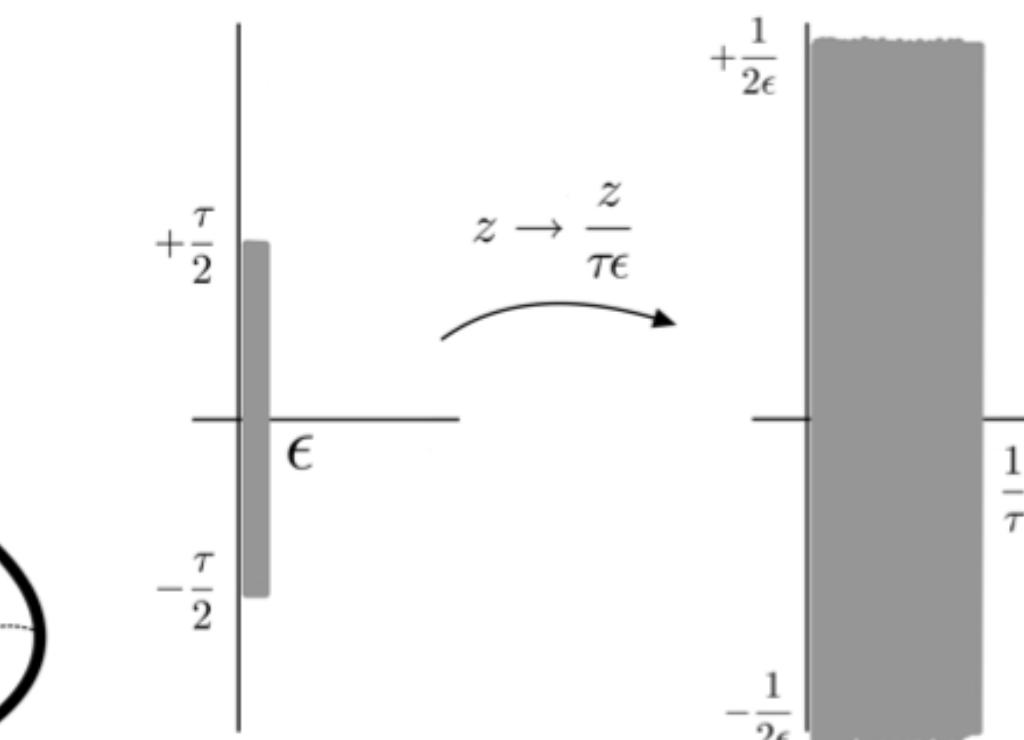
(a) F-Dual Reconstruction



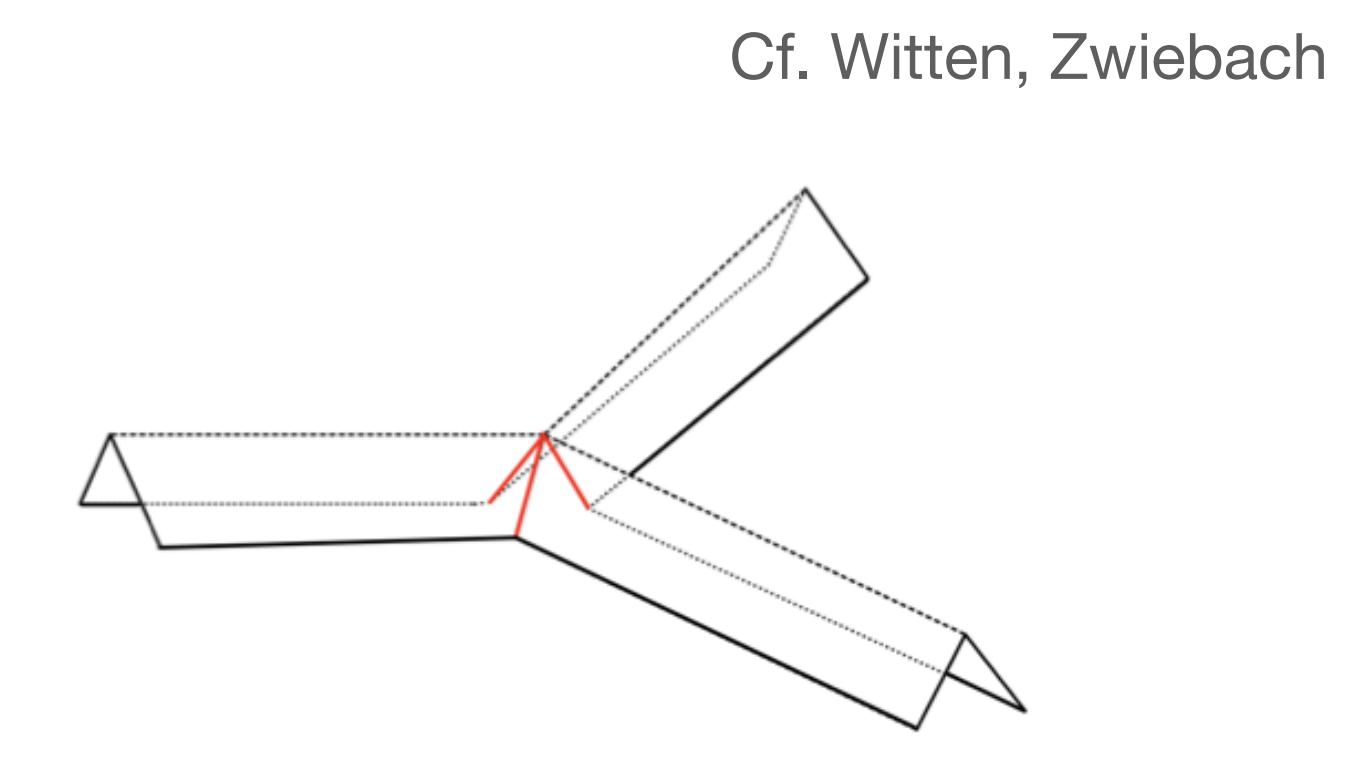
(a) V-Dual Reconstruction



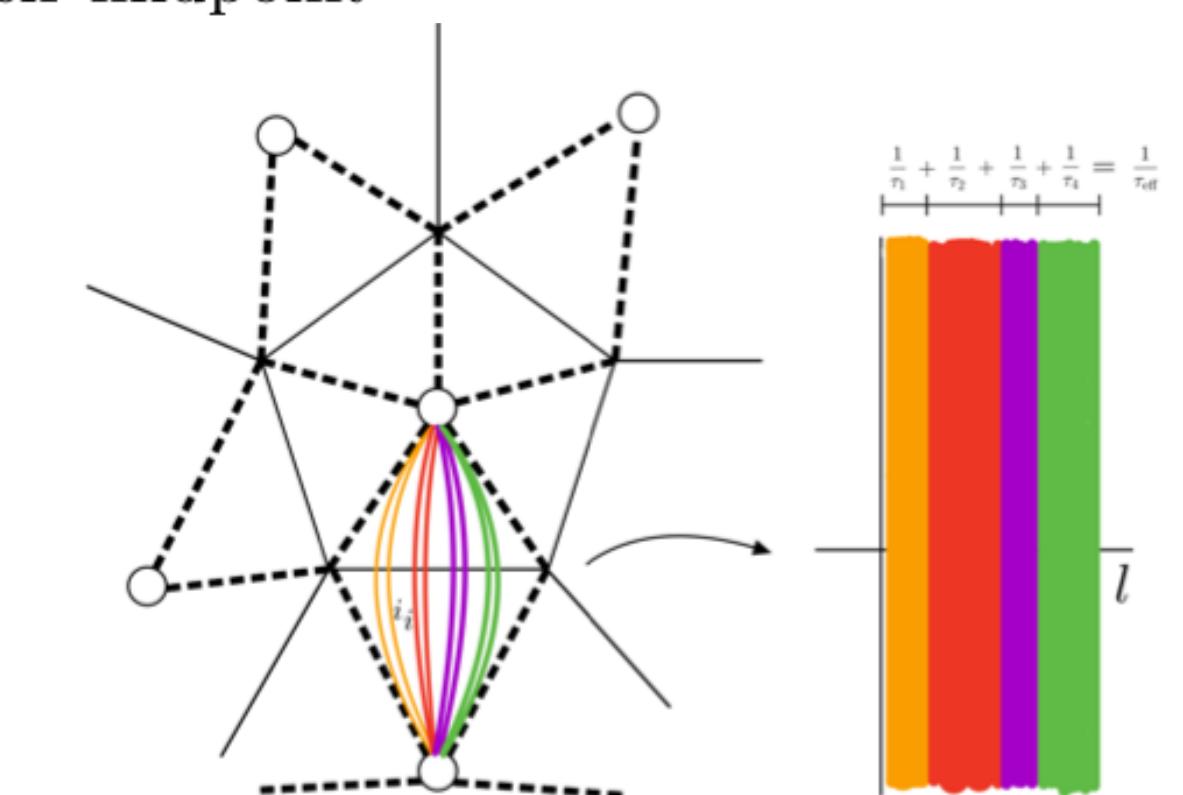
(b) Edges as open string strips



(b) From worldlines to open string strips



(c) Glueing 3 open strings along their midpoint



(c) Gluing homotopically equivalent ribbons (string bits)

Cf. Witten, Zwiebach

V- and F-type Open/Closed String duality

Extending holographic duality to a triality

Vertex
Type

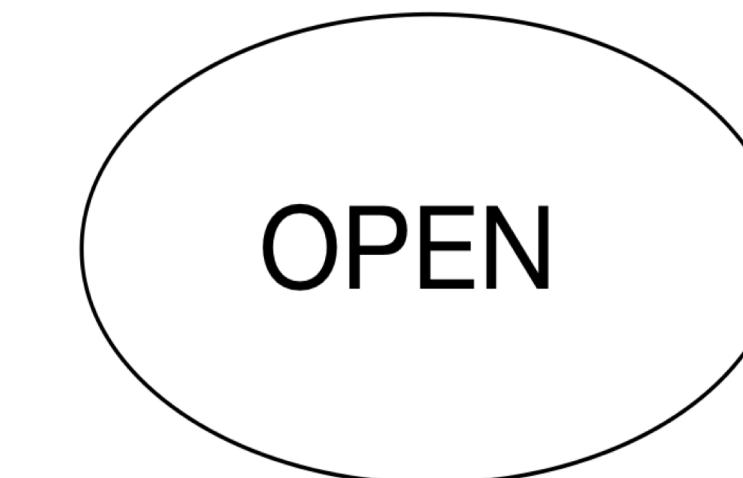
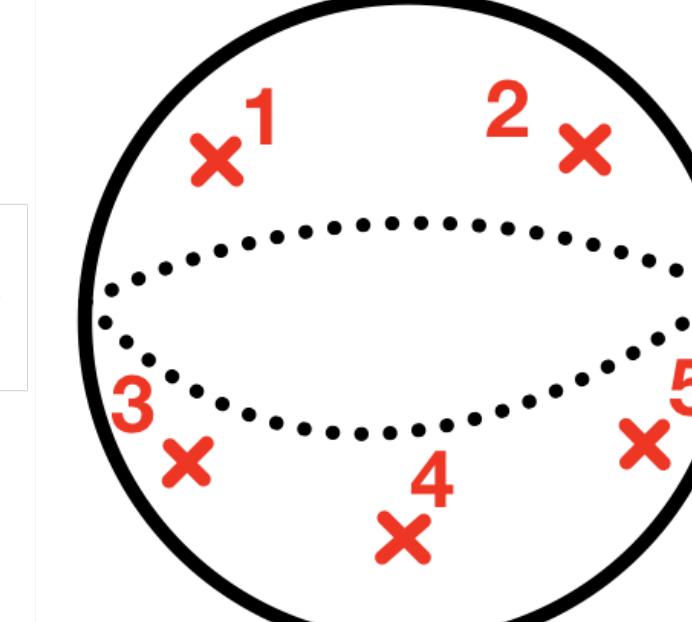
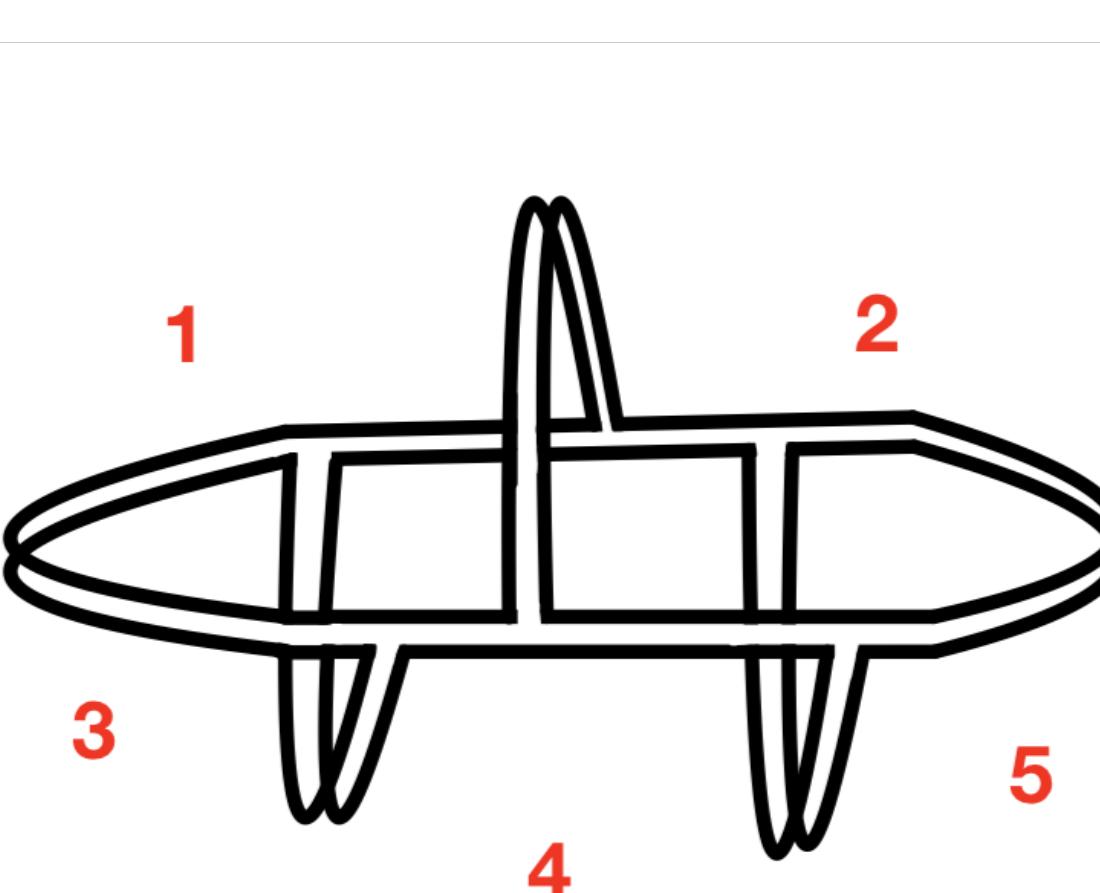
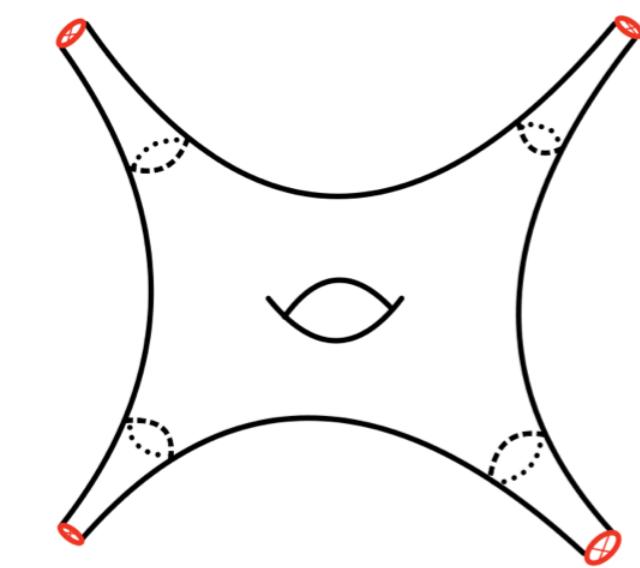
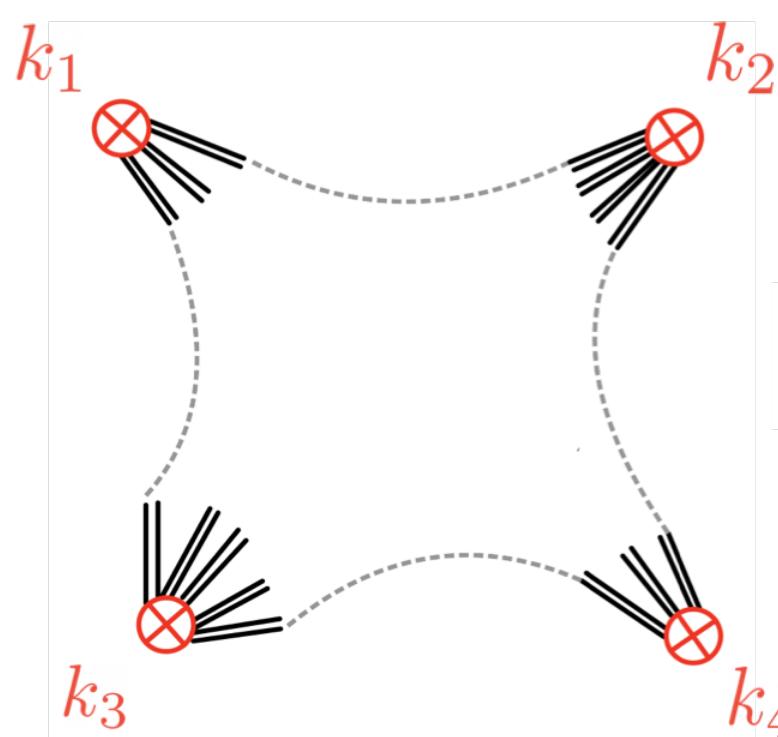
E.g.

$\mathcal{N} = 4$ SYM

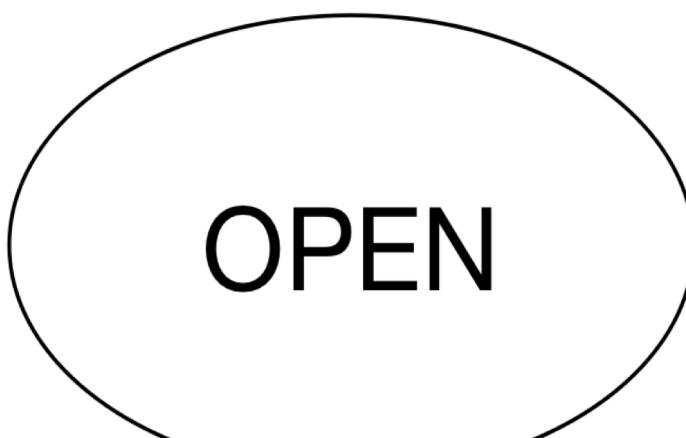
Face
Type

E.g.

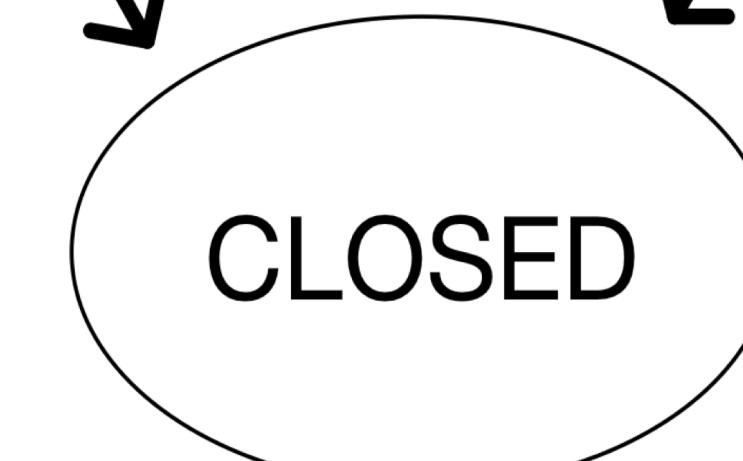
Gopakumar/
Vafa duality



Feynman
Graph Duals



"V-type"
Duality



"F-type"
Duality

The Derivation

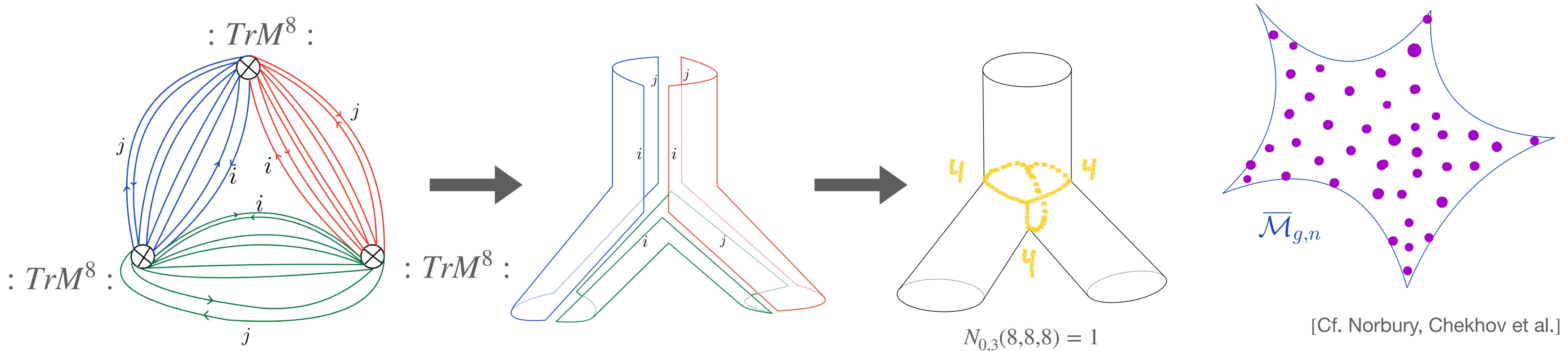
Why do these closed string theories appear?

The A-Model

How do we see the holomorphic maps from the WS to the TS from the matrix?

Putting V-Type Open/Closed Duality to the Test

→ Gaussian correlators count lattice points on moduli space $\mathcal{M}_{g,n}$



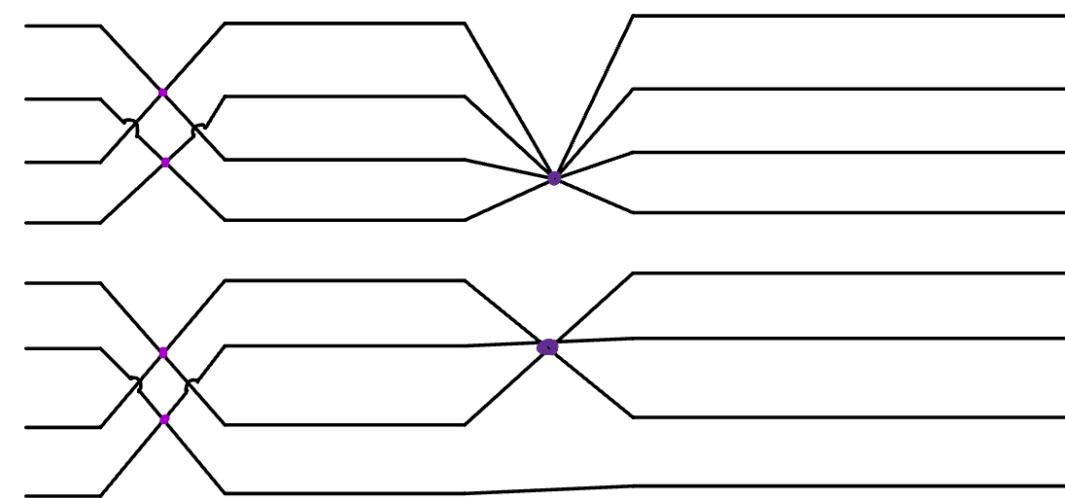
$$\left\langle \prod_{i=1}^n \frac{1}{Nk_i} : TrM^{k_i} : \right\rangle_c^g = \text{Sum over integer length Strebel graphs} \equiv N_{g,n}(k_1, \dots, k_n)$$

Explicit sanity checks: $\langle \prod_{i=1}^n \frac{1}{N2k_i} : TrM^{2k_i} : \rangle_c^{g=0} = N_{g=0,n}(2k_1, \dots, 2k_n)$ & $\langle \frac{1}{N2k_1} : TrM^{2k_1} : \rangle_c^{g=1} = N_{g=1,n}(2k_1)$

From Wick Contractions to Belyi Maps

Gaussian Correlators as Holomorphic Branched Covers

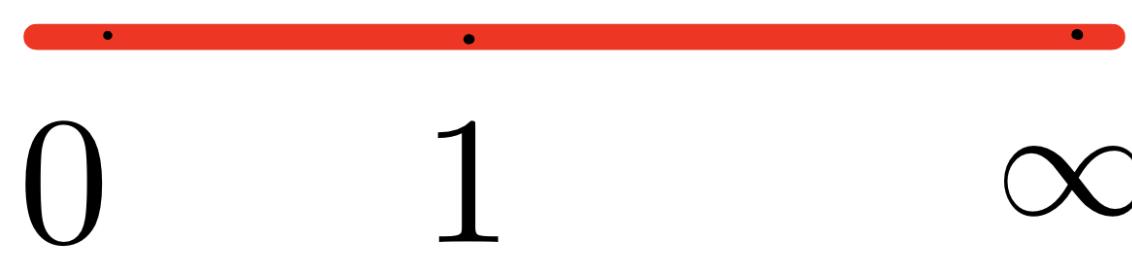
[Cf. Rangoolam, Gopakumar]



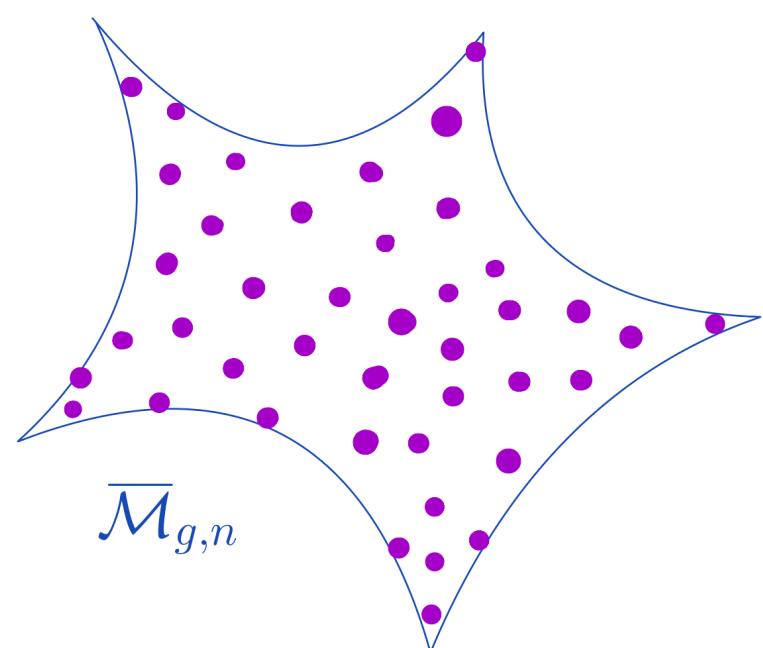
$$\left\langle \prod_{i=1}^n \text{Tr} M^{2k_i} \right\rangle = \sum_{\alpha, \gamma \in S_{2k}} \delta(\alpha \cdot \beta \cdot \gamma) N^{-k + C_\gamma}$$

$\alpha \in (2)^k$
 $\beta \in (2k_1) \dots (2k_n)$
 $\gamma = (\alpha \cdot \beta)^{-1}$

Wick Contractions
= Edges
Cyclicity of Traces
= Vertices
Closed index loop post Wick
= Faces



Belyi Maps: Holomorphic covering maps $\Sigma_{g,n} \rightarrow \mathbb{P}^1$ of degree $2k$ with exactly three branch points $(0, 1, \infty)$ and branching profile $(2k_1, \dots, 2k_n)$ at ∞ ; $(2)^k$ branching at 1



Belyi Thm: Such Maps only exist for these integer points on $\mathcal{M}_{g,n}$!

[Cf. Mulase-Penkava]

Such localization on moduli space $\mathcal{M}_{g,n}$ indeed already seen in $\frac{SL(2, \mathbb{R})_1}{U(1)}$ string!

[Cf. Eberhardt, Dei, Gaberdiel, Gopakumar, Knighton, Maity]

The « BMN-Limit »

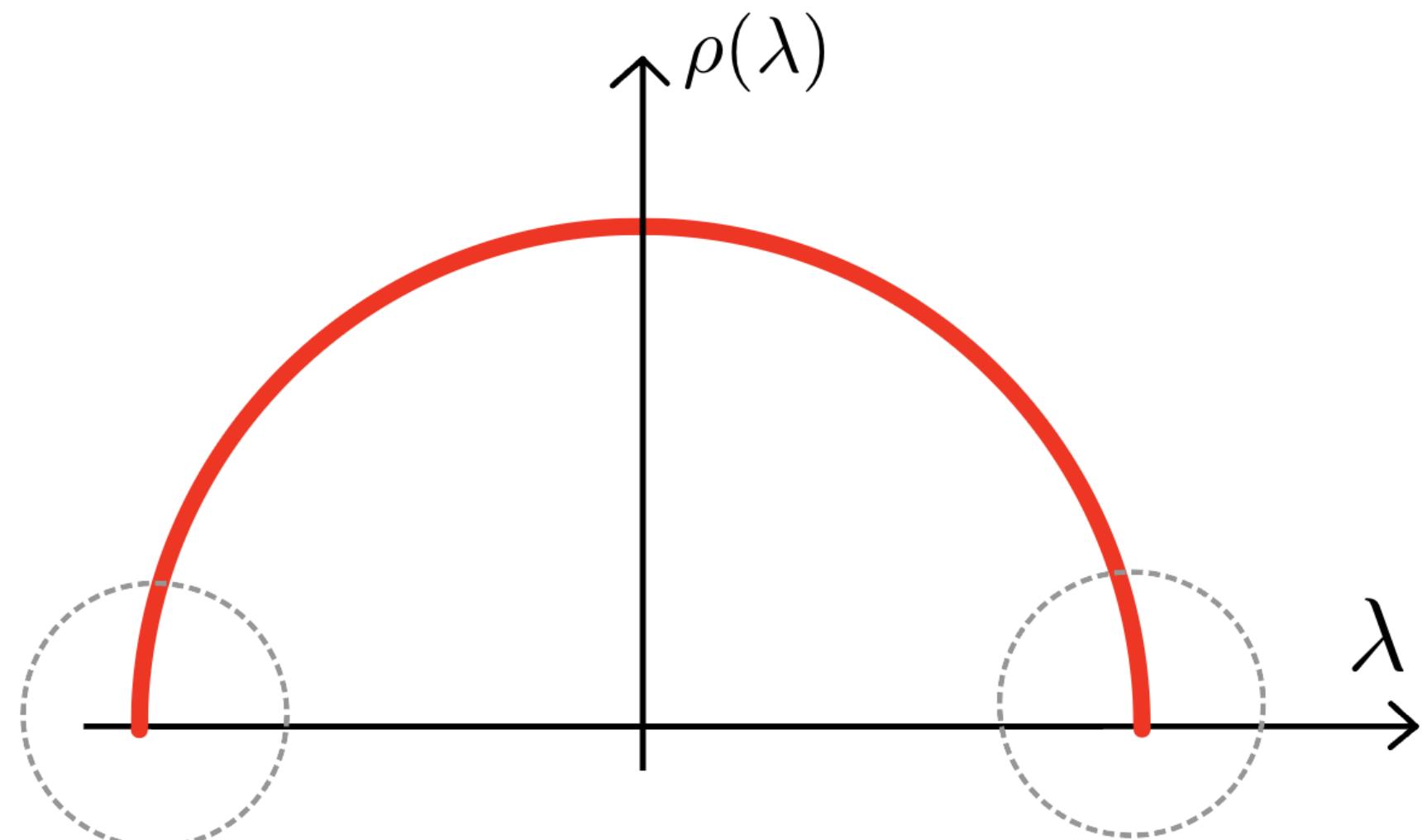
A New Perspective on Double-Scaling

$$\omega_{\text{Kontsevich}} = \sum_{i=1}^n k_i^2 \psi_i$$

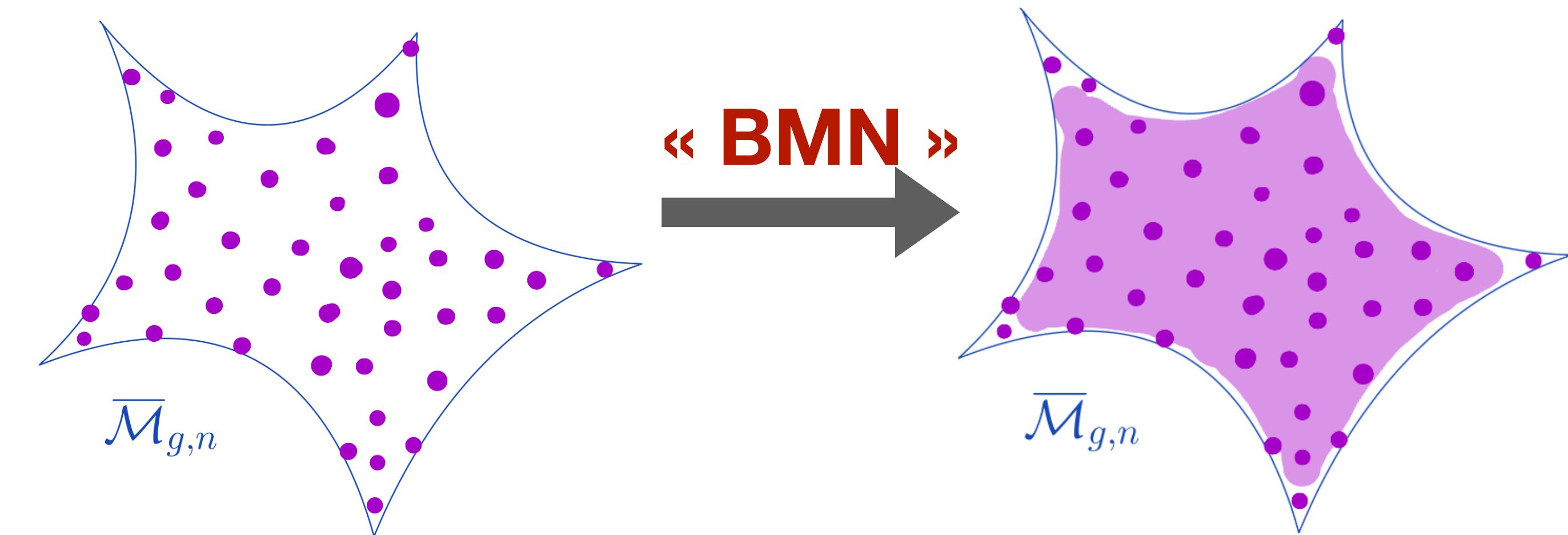
$$\lim_{k_i \rightarrow \infty} \left\langle \prod_{i=1}^n \frac{1}{Nk_i} : \text{Tr} M^{2k_i} : \right\rangle_c = \lim_{k_i \rightarrow \infty} N_{g,n}(2k_1, \dots, 2k_n) \rightarrow \text{Vol}_{\text{Kontsevich}}(2k_1, \dots, 2k_n)$$

[Cf. Ehrhart, Norbury]

Pure 2d top gravity in BMN limit



Wigner Semicircle \leftrightarrow full AdS
Edge Region (Airy) \leftrightarrow pp-Wave geometry



Cover all of moduli space!

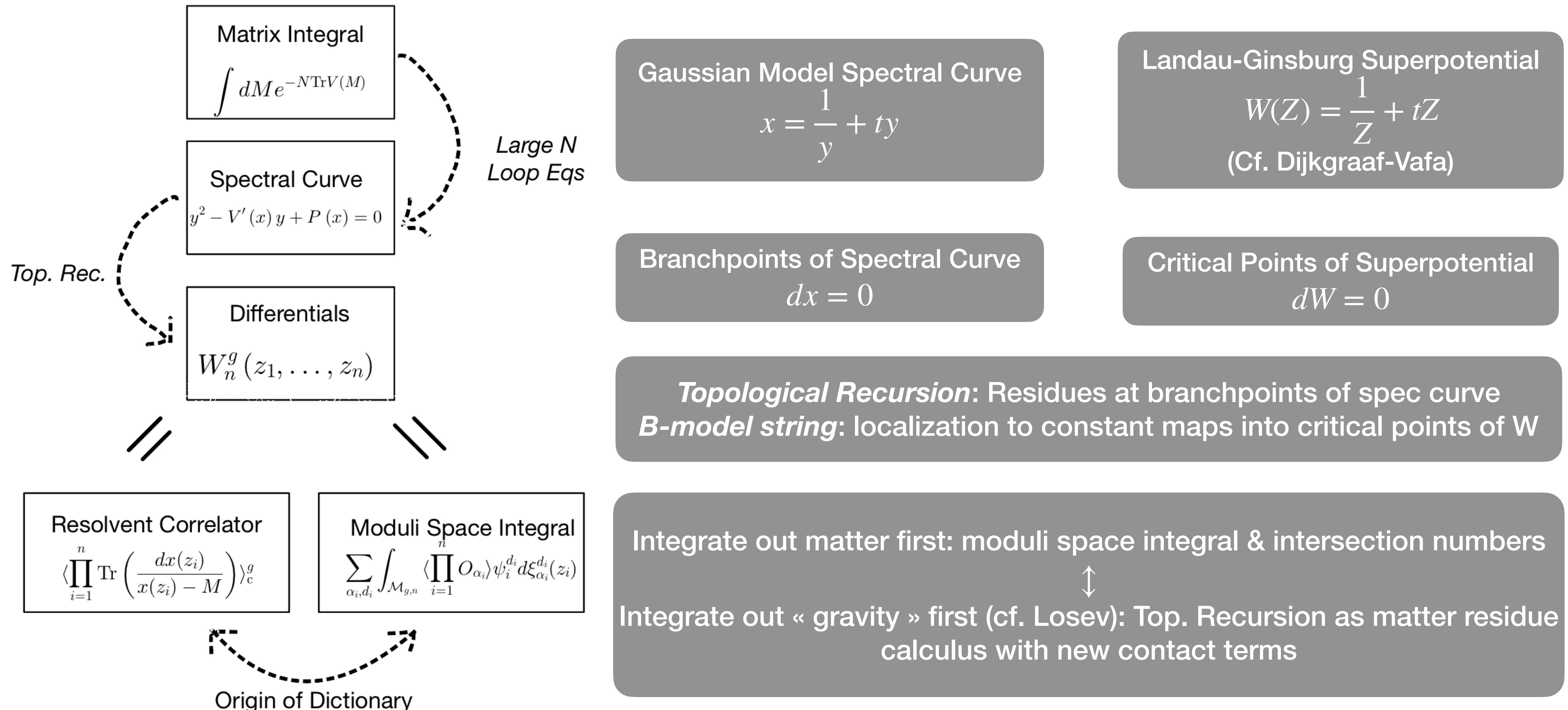
The B-Model

**How do we see the constant maps from the WS to the critical points
of the super potential?**

Finding a B-Model in Disguise

The Many Faces of Topological Recursion

[Cf. Eynard-Orantin, Eynard, DOSS, Milanov]



CohFT Correlators

Traces as Matter Primaries + Gravitational Descendants

$$\int dM_{N \times N} e^{-N \text{Tr} V(M)} \prod_{i=1}^N \text{Tr} M^{k_i}$$

Potential
determines
matter theory

large N expansion
= genus expansion

$$\sum_g N^{2-2g-n} \sum_{\alpha_i, d_i} c_{\alpha_i, d_i}^{k_i} \int_{\overline{\mathcal{M}}_{g,n}} \langle O_{\alpha_1} \dots O_{\alpha_n} \rangle \prod_{i=1}^n \psi_i^{d_i}$$

Extra psi-class
Insertions

Main tool: TR as CohFT
(Eynard 2011 + DOSS 2014 + Giachetto Thesis+...)

Matter Primaries =
Edges of Eigenvalue Distribution

Sanity Checks:
g=0 3pt & 4pt, and g=1 1pt correlators from
explicit moduli-space integrals

$$\text{Tr} M^{2k} \leftrightarrow \mathbf{O}_+ \sum_{d=0}^{k-1} \frac{(2k)!}{(k-d)!(k-1-d)!} \psi^{2d} + \mathbf{O}_- \sum_{d=0}^{k-1} \frac{(2k)!}{(k-1-d)!(k-1-d)!} \psi^{2d+1}$$

Very Explicit Universal Operator Dictionary!

The BMN Limit - Take 2

« Washing Out the Matter Theory »

Reproduce Okounkov &
Okounkov-Pandharipande!

From our operator dictionary

Maximize ψ -class insertions
 \leftrightarrow
Decouples matter theory

$$\lim_{\kappa \rightarrow \infty} \frac{\langle Tr M^{2\kappa x_1} \dots Tr M^{2\kappa x_n} \rangle^{(g)}}{2^{2\kappa|x|} \kappa^{3g-3+3n/2}} = \frac{2^g}{(\pi)^{n/2}} \sum_{d_1+\dots+d_n=d_{g,n}} \langle \prod_{\alpha=1}^n \psi_{\alpha}^{d_{\alpha}} \rangle_{\mathcal{M}_{g,n}} x_{\alpha}^{d_{\alpha}+1/2}$$

$$\lim_{k \rightarrow \infty} Tr M^{2k} \sim O_a \times 2^{2k} \sum_d k^{d+1/2} \psi^d$$

$$\langle O_{\alpha_1} \dots O_{\alpha_n} \rangle^{TFT} \times \int_{\mathcal{M}_{g,n}} \prod_{i=1}^n \sum_d c_{\mathbf{k}_i, d} \psi_i^d$$

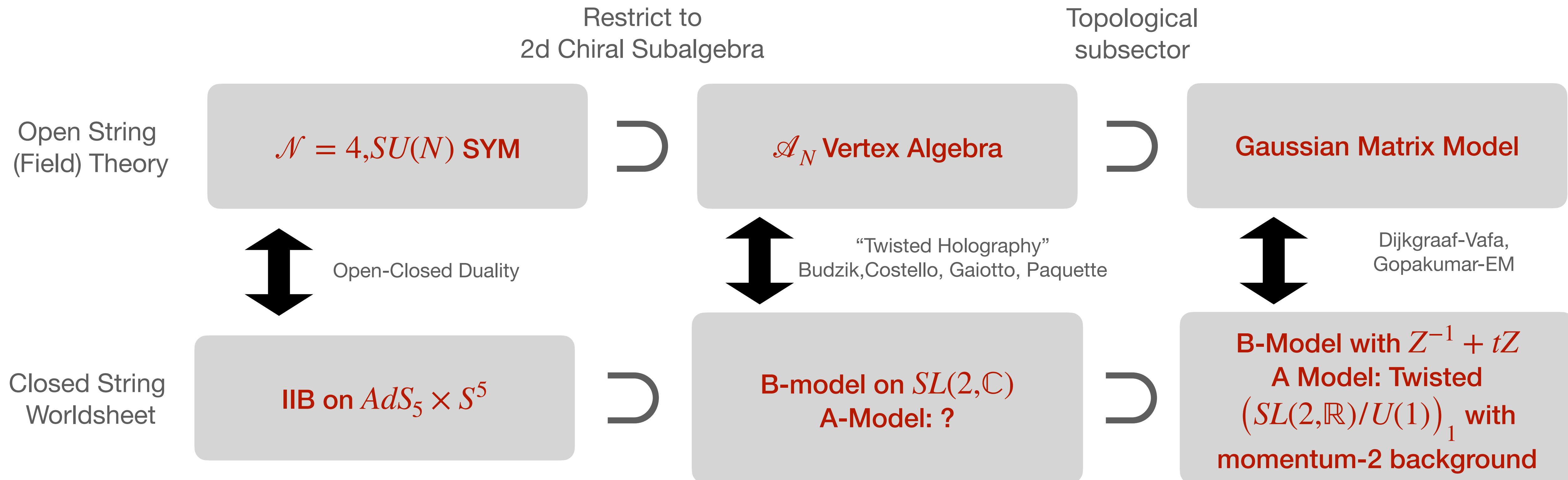
Pure 2d top
gravity in BMN
limit!

AdS/CFT

How do these lessons fit into holography?

The Even Bigger Picture

As a topological subsector of “standard” AdS/CFT



Thank You!

Happy to go into more details!

All-genus
1-pt fn

Two
B-Model $g=0$ 4pt fn
calculations

More on A-model
“cigar”

Two B-Model Perspectives on 4-pt Fn.

“Pure Matter” vs. Intersection Theory Computations of $N_{g=0,n=4}(2k_1, \dots, 2k_4)$

Matrix Model Answer: $\langle \prod_{i=1}^{n=4} \frac{1}{N2k_i} : TrM^{2k_i} : \rangle_c^{g=0} = N_{g=0,n=4}(2k_1, \dots, 2k_4) = k_1^2 + k_2^2 + k_3^2 + k_4^2 - 1$

“Pure Matter” LG w/ contact terms (cf. Losev)

$$W(Z) = \frac{1}{Z} + Z \quad \frac{1}{Nk} : TrM^k : \leftrightarrow \mathcal{O}_k \equiv \frac{1}{Z^{k+1}}$$

$$\begin{aligned} N_{g=0,n=3}(k_1, k_2, k_3) &= \langle \mathcal{O}_{k_1} \mathcal{O}_{k_2} \mathcal{O}_{k_3} \rangle \\ &= \oint \frac{1}{W'(z)} \frac{1}{z^{k_1+1}} \frac{1}{z^{k_2+1}} \frac{1}{z^{k_3+1}} \\ &= \underset{z \rightarrow 1}{\text{Res}} \frac{z^2}{(z^2 - 1)} \frac{1}{z^{k_1+1}} \frac{1}{z^{k_2+1}} \frac{1}{z^{k_3+1}} + \underset{z \rightarrow -1}{\text{Res}} \frac{z^2}{(z^2 - 1)} \frac{1}{z^{k_1+1}} \frac{1}{z^{k_2+1}} \frac{1}{z^{k_3+1}} \\ &= \left(\frac{1}{2} \right) + (-1)^{k_1+k_2+k_3} \left(\frac{1}{2} \right) \end{aligned}$$

Start with 3-pt Fn.
(Cf. Vafa)

Matter Theory as Iterated Residue Calculus

B-Model “after integrating out gravity”

“Pure Matter” LG w/ contact terms (cf. Losev)

$$C_W(\mathcal{O}_{k_i}, \mathcal{O}_{k_j}) = \frac{d}{dz} \left(\frac{\mathcal{O}_{k_i} \mathcal{O}_{k_j}}{W'(z)} \right)_- = \sum_{l=1}^{k_i+k_j} 2l \mathcal{O}_{2l}$$

$$\begin{aligned} \langle \mathcal{O}_{2k_1} \mathcal{O}_{2k_2} \mathcal{O}_{2k_3} \mathcal{O}_{2k_4} \rangle &= \frac{d}{dt} \langle \mathcal{O}_{2k_1} \mathcal{O}_{2k_2} \mathcal{O}_{2k_3} \rangle_{W+t\mathcal{O}_{2k_4}} \Big|_{t=0} + \sum_{i=1}^3 \langle C_W(\mathcal{O}_{2k_4}, \mathcal{O}_{2k_i}) \prod_{j \neq i} \mathcal{O}_{2k_j} \rangle \\ &= -(2k_4 + 1)(k_1 + k_2 + k_3 + k_4 + 1) + k_1(1 + k_1) + k_2(1 + k_2) + k_3(1 + k_3) + 2k_4(k_1 + k_2 + k_3) + 3k_4(1 + k_4) \end{aligned}$$

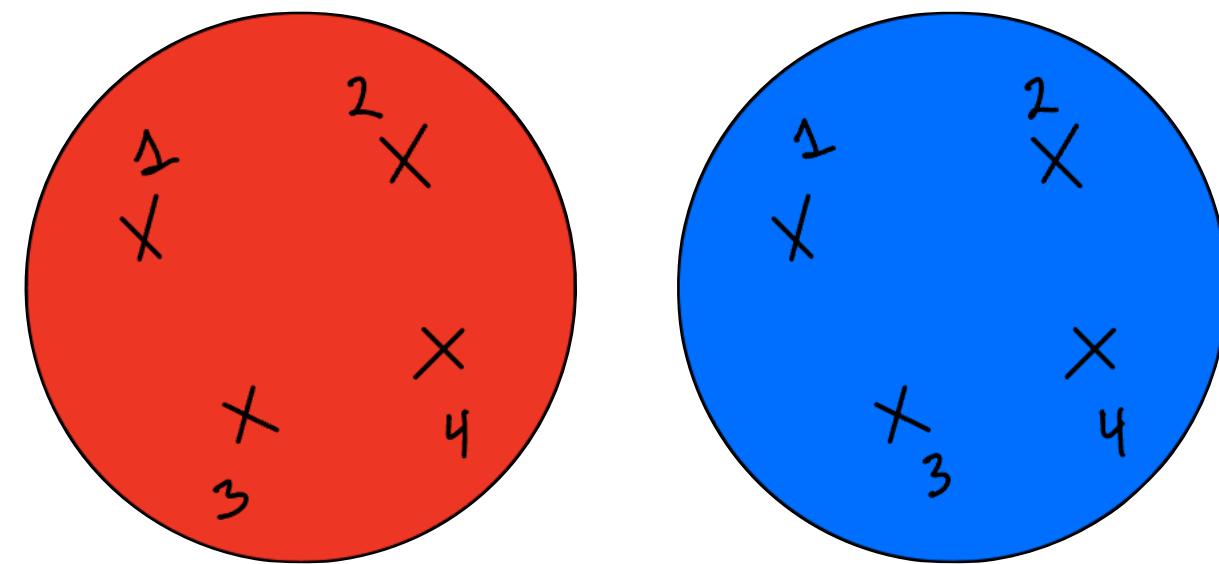
Contributions from deformed 3-pt fn

Contributions from contact terms

$$= k_1^2 + k_2^2 + k_3^2 + k_4^2 - 1 = N_{g=0, n=4}(2k_1, \dots, 2k_4)$$

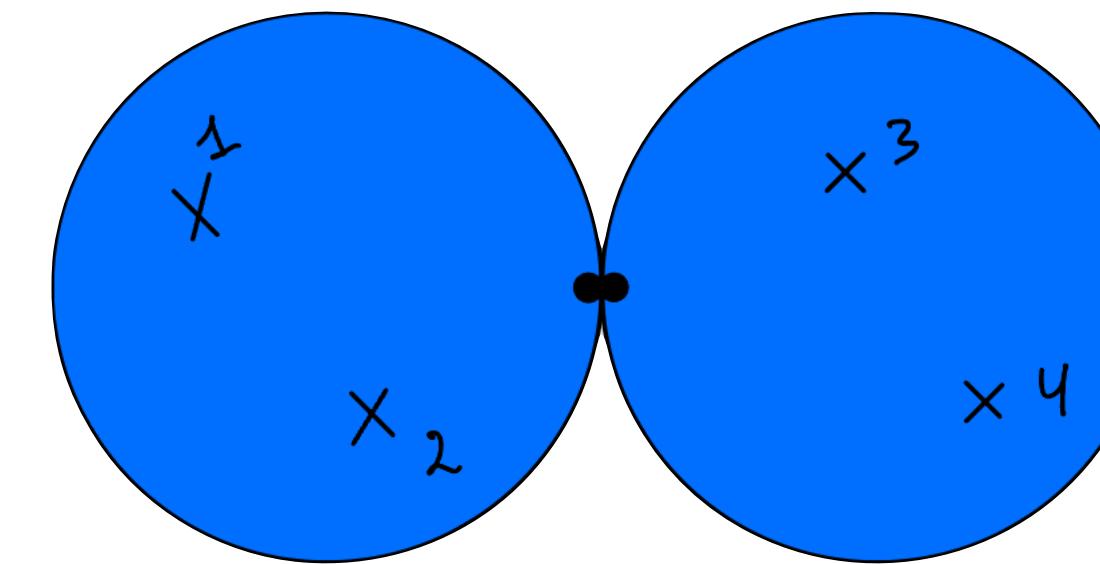
4-pt Function from Moduli Space Integral

B-Model “after integrating out matter”

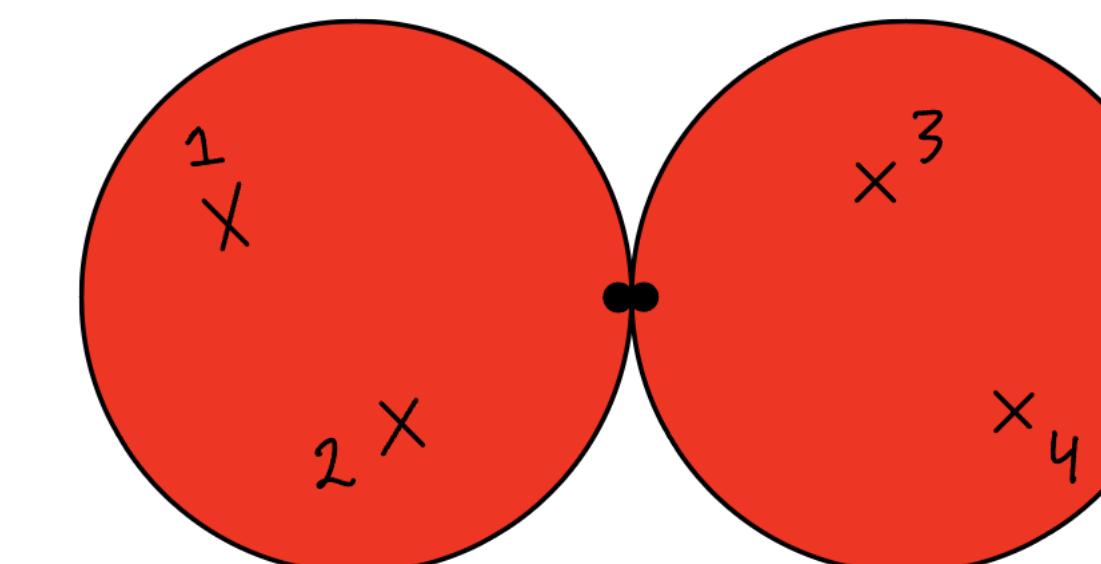


$$\frac{1}{2}(k_1^2 - \frac{1}{16})\langle\psi_1\rangle_{\mathcal{M}_{0,4}} + \text{Perm}(1,2,3,4)$$

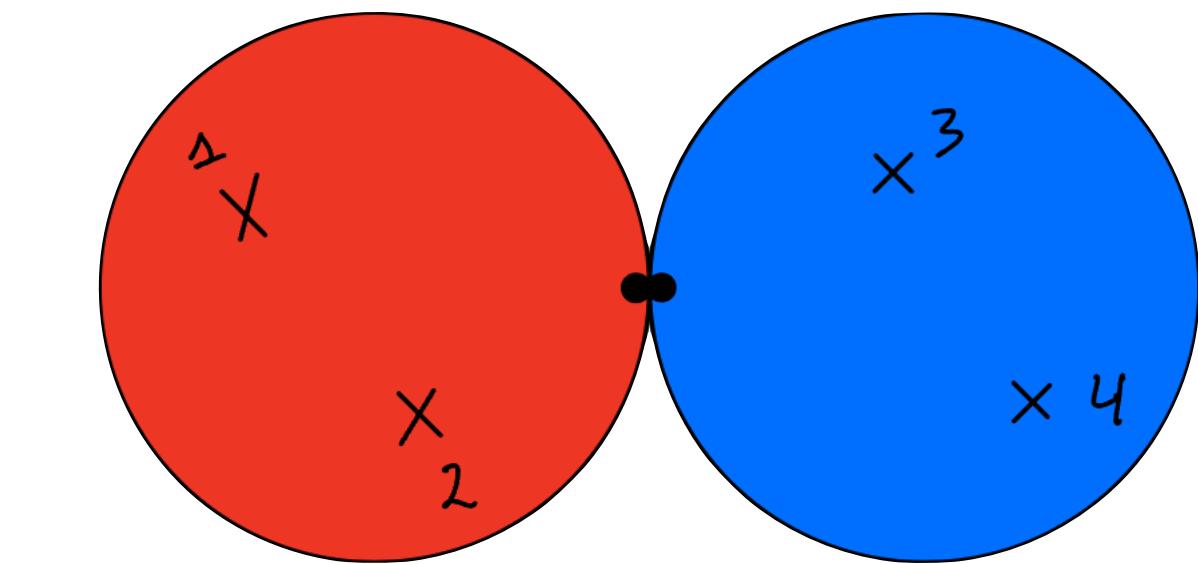
$$\frac{1}{2}(k_1^2 - \frac{1}{16})\langle\psi_1\rangle_{\mathcal{M}_{0,4}} + \text{Perm}(1,2,3,4)$$



$$-\frac{3}{64}\langle\psi_1^0\psi_2^0\psi^0\rangle_{\mathcal{M}_{0,3}}\langle\psi_3^0\psi_4^0\psi^0\rangle_{\mathcal{M}_{0,3}}$$



$$-\frac{3}{64}\langle\psi_1^0\psi_2^0\psi^0\rangle_{\mathcal{M}_{0,3}}\langle\psi_3^0\psi_4^0\psi^0\rangle_{\mathcal{M}_{0,3}}$$



$$-\frac{3}{32}\langle\psi_1^0\psi_2^0\psi^0\rangle_{\mathcal{M}_{0,3}}\langle\psi_3^0\psi_4^0\psi^0\rangle_{\mathcal{M}_{0,3}}$$

From operator insertions

$$-\frac{3}{64}\langle\kappa_1\rangle_{\mathcal{M}_{0,4}} - \frac{3}{64}\langle\kappa_1\rangle_{\mathcal{M}_{0,4}}$$

From “background” dual
to matrix potential

Coefficients fixed both by local behavior of spectral curve & Bergmann kernel near branchpoints) and our new operator dictionary

$$= k_1^2 + k_2^2 + k_3^2 + k_4^2 - 1 = N_{g=0,n=4}(2k_1, \dots, 2k_4)$$

Traces from Hodge-GUE

All genus-0 and genus-1 correlators

$$\langle \text{Tr}M^{2k_1} \dots \text{Tr}M^{2k_a} \rangle_c^g = \sum_{h=0}^{\lceil g/2 \rceil} \frac{2^g}{2^{3h}(2h)!} \sum_{l=0}^h \frac{1}{l!} \int_{\bar{\mathcal{M}}_{g-h,a+l+2h}}$$

$$\Lambda(x) \equiv \sum_{j=0}^g c_j(\mathbb{E}) x^j$$

$$\Lambda(-1)\Lambda(-1)\Lambda(1/2) \prod_{i=1}^a \frac{1}{1 - k_i \psi_i} \frac{(2k_i)!}{(k_i)!(k_i-1)!} \prod_{j=a+1}^{a+l} \frac{-\psi_j^2}{1 - \psi_j}$$

Cf. Borot & Garcia-Failde

Reproduce
 $g = 0$ n-pt
 Matrix Model answer

$$\langle \prod_{i=1}^n \text{Tr}M^{2k_i} \rangle_c^{g=0} = \frac{(k_{tot}-1)!}{(k_{tot}-n+2)!} \prod_{i=1}^n \frac{(2k_i)!}{(k_i)!(k_i-1)!}$$

Hodge bundle trivial,
 ψ -class intersection numbers
 purely combinatorial

$g = 1$ n-pt
 (Also beyond MM answers)

Cf. Morozov-Shakirov

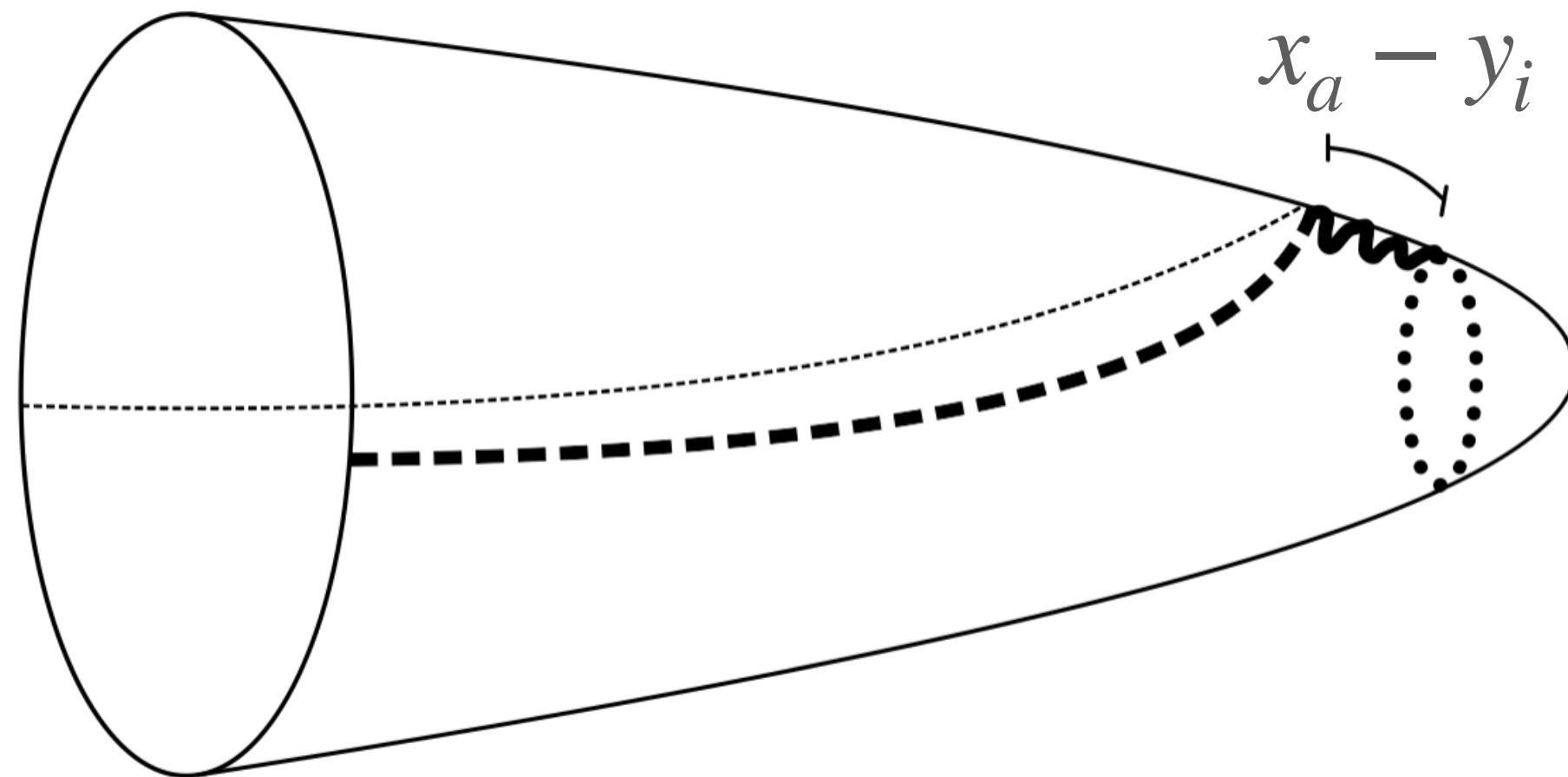
$$\langle \text{Tr}M^{2k_1} \text{Tr}M^{2k_2} \rangle_{conn}^{g=1} = \left(\frac{k_1^2 + k_1 k_2 + k_2^2 - 2(k_1 + k_2) + 1}{12} \right) \prod_{i=1}^2 \frac{(2k_i)!}{(k_i)!(k_i-1)!}$$

Requires the λ_g -formula (Faber-Pandharipande)

$$\int_{\bar{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \dots \psi_n^{\alpha_n} \lambda_g = \frac{(2g+n-3)!}{\alpha_1! \dots \alpha_n!} \times \frac{2^{2g-1} - 1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!}$$

The A-Model: the Cigar

From $c=1$ at self-dual radius to the topological coset



$$\int dK dM_{N \times N} e^{-\frac{1}{g} \text{Tr}(V_p(K) - K(M - Y))} \prod_{a=1}^Q \det(x_a - M)$$

- Can map tachyon vertex operators in $c=1$ at self-dual radius to operators in coset model, giving operator dictionary for traces :

$\text{Tr}M^k \leftrightarrow D_{j=1/2}^k$, where the momentum background makes us

consider vertex operators in the spectrally flowed $j = 1/2$
sector

- Compact $\sim ZZ$ // Non-compact $\sim FZZT$

All Genus 1-pt Function

A detailed sanity check

$$\left\langle \frac{1}{N} \text{Tr} M^{2n} \right\rangle_{Gaussian} = 2n \langle T_{-2n} \rangle_{t_2}$$

Gaussian Matrix Model \leftrightarrow c=1 string at self-dual radius with momentum +2 tachyon-background

c=1 string calculation:

$$2n \langle T_{-2n} \rangle_{t_2} = \frac{1}{2n+1} \left(\frac{1}{i\mu} \right)^{2n+1} \oint dz W^{2n+1}(z)$$

$$2n \langle T_{-2n} \rangle_{t_2} = \frac{1}{N^{2n+1}} \frac{1}{2n+1} \oint dz z^{-N} e^{-\frac{N}{2} t_2 z^2} \partial_z^{2n+1} \left(z^N e^{+\frac{N}{2} t_2 z^2} \right)$$

W_∞ currents $W^{2n+1}(z) \equiv \bar{\psi}(z) \partial_z^{2n+1} \psi(z)$

Bosonization $\psi(z) = e^{i\mu\phi(z)}$

Background $\phi(z) = \log z + \sum_{k=1}^{\infty} \frac{1}{k} t_k z^k$

All Genus 1-pt Function

A detailed sanity check

$$\left\langle \frac{1}{N} \text{Tr} M^{2n} \right\rangle_{Gaussian} = 2n \langle T_{-2n} \rangle_{t_2}$$

Gaussian Matrix calculation:

$$\left\langle \frac{1}{N} \text{Tr} M^{2n} \right\rangle = \frac{1}{Z} \int dM e^{-\frac{N}{2t_2} \text{Tr} M^2} \text{Tr} M^{2n} = \left(\frac{2t_2}{N} \right)^n \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{h_k} \int dx e^{-x^2} F_k(x) F_k(x)$$

$$= \left(\frac{2t_2}{N} \right)^n \frac{1}{N} \frac{1}{h_{N-1}} \frac{2}{2n+1} \int dx e^{-x^2} x^{2n+1} F_N(x) F_{N-1}(x)$$

Christoffel-Darboux

$$\sum_{k=0}^{N-1} \frac{1}{h_k} F_k(x) F_k(x) = \frac{1}{h_{N-1}} (F'_N(x) F_{N-1}(x) - F_N(x) F'_{N-1}(x))$$

EOM

$$(2x - \partial_x) F'_k(x) = 2k F_k(x)$$

$$= \frac{1}{N^{2n+1}} \frac{1}{2n+1} \oint dz z^{-N} e^{-\frac{N}{2} t_2 z^2} \partial_z^{2n+1} \left(z^N e^{+\frac{N}{2} t_2 z^2} \right) = 2n \langle T_{-2n} \rangle_{t_2}$$

Integral Representation

$$F_N(x) = 2^{-N} (-1)^N e^{x^2} \frac{1}{2\sqrt{\pi}} \int (is)^N e^{isx - s^2/4}$$

$$F_{N-1} = 2^{-(N-1)} (N-1)! \oint \frac{e^{2ux - u^2}}{u^N}$$

Orthogonal Polynomials

$$\int d\lambda e^{-\lambda^2} F_m(\lambda) F_n(\lambda) = h_m \delta_{mn}$$
