# Index for non-relativistic superconfomal field theories -- from LLM to ABJM and BL

**Yu Nakayama (UC Berkeley),**

**arxiv:0807.3344[hep-th]**

**Also work in progress with**

**S. Ryu, M. Sakaguchi and K. Yoshida**

# Index in theoretical physics

- Index bridges between math and physics
  - Worldsheet string theory
    - Gauss-Bonnet
    - Riemann-Roch
  - Gauge theory
    - Atiya-Singer
    - Fermion zero mode, flux vacua, D-brane…

Most successful marriage is Witten index

$$I = \mathrm{Tr}(-1)^F e^{-\beta H} \qquad H = \{Q, Q^\dagger\}$$

- Unification of mathematical formulae
  - Path integral of SUSY non-linear sigma model…

# Index in mathematics

- Gauss-Bonnet theorem

$$
\begin{aligned}
\chi(M) &= \frac{1}{2\pi}\int d^2x\sqrt{g}R \\
&= \mathrm{vertex} - \mathrm{edge} + \mathrm{face} \\
&= \sum_{p=0}(-1)^p \mathrm{dim} H^p(M) \\
&= \sum_{p-\mathrm{form}} (-1)^p e^{-\beta\Delta} \\
&= 2g-2
\end{aligned}
$$

$$\Delta = d^\dagger d + d d^\dagger$$

$$d^2 = (d^\dagger)^2 = 0$$

$$d^\dagger = *d*$$

Euler characteristic is a topological invariant!

- $(-1)^p$ is important!

# SUSY and index

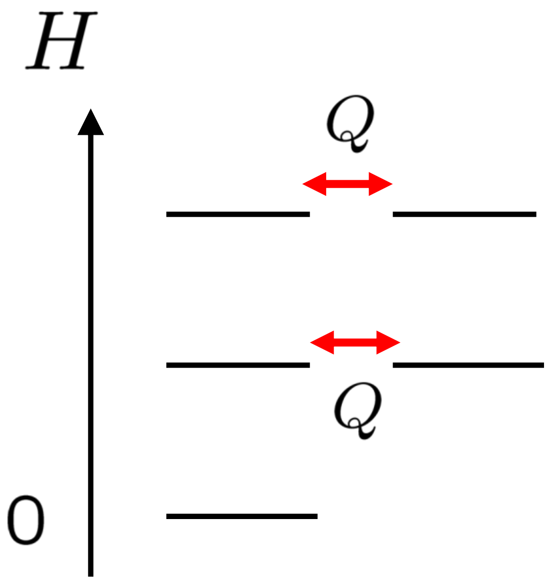
- SUSY algebra

$$\{Q^\dagger, Q\} = Q^\dagger Q + Q Q^\dagger = 2H \;,\; Q^2 = (Q^\dagger)^2 = 0$$

$$d^\dagger d + d d^\dagger = \Delta \;,\; d^2 = (d^\dagger)^2 = 0$$

- Witten Index

$$I_w = \mathrm{Tr}(-1)^F e^{-\beta H}$$




$\boxed{\text{Bose Fermi cancellation}}$

when $H|\Psi\rangle = E|\Psi\rangle$

then $Q|\Psi\rangle$

has same E but different F

# Gauss-Bonnet from Witten index

Physicists can derive index theorem from path integral

- Consider supersymmetric quantum mechanics (non-linear sigma model)

$$S = \int dt G_{ab}(\phi)\dot{\phi}^a\dot{\phi}^b + G_{ab}(\phi)\dot{\Psi}^a\Psi^b + \cdots$$

- One can show

$$\begin{aligned} I_w &= \mathrm{Tr}(-1)^F e^{-\beta H} \\ &= \chi(M) \end{aligned}$$

- Doing path integral by localization method

$$\begin{aligned} I_w &= \int \mathcal{D}\Phi\mathcal{D}\Psi e^{-S(\Phi,\Psi)} \\ &= \int e(M) \end{aligned}$$

e(M): Euler Class

# Index for superconformal field theory

- Consider SCFT with superconformal charge S

$$\{S, Q\} = \Delta$$

$$I_{SCFT} = \mathrm{Tr}(-1)^F e^{-\beta\Delta}$$

- Ex1. 2D NLSM with Calabi-Yau target space
  → Index is nothing but elliptic genus

$$I(\tau, z) = \mathrm{Tr}(-1)^F e^{2\pi i z J_L} q^{L_0} \bar{q}^{\bar{L}_0} \qquad \Delta = \bar{L}_0$$

$$I(\tau, z=0) = \chi(M)$$ Euler characteristic

$$I(\tau, z=1/2) = \sigma(M) + O(q)$$ Hirzebruch signature

$$I(\tau, z=(\tau+1)/2) = \hat{A} q^{-1/4} + O(q^{1/4})$$ A-roof genus

# Index for superconformal field theory II

- Ex2 SCFT in 4D (Romelsberger, Kinney-Maldacena-Minwalla-Raju, Nakayama)

$$I(t,y) = \mathrm{Tr}(-1)^F e^{-\beta\Delta} t^{2(E+j_2)} y^{2j_1}$$

$$\Delta = 2\{Q^\dagger, Q\} = E - 2j_2 - \frac{3}{2}r$$

- Counting short multiplets annihilated by Q

Encode geometric information of conical $CY_3$ probed by D3-brane (base Sasaki-Einstein space by AdS/CFT)

- Ex. CY = $C^3$ (N=4 SYM)

$$I(t,y) = \prod_n \frac{(1-t^{3n}y^n)(1-t^{3n}y^{-n})}{(1-t^{2n})^3}$$

- Novel mathematical inv? (counts holomorphic function etc…)

# Index for non-relativistic SCFT

But what is the condition to define such an index?
Any specific propertiy of SUSY algebra?

- Yes! Non-trivial anti-automorphism of algebra.

What is the non-relativistic SCFT?

- NR-limit of M2-brane gauge theory
  - M2-brane mini revolution
  - Chern-Simons matter theory in 1+2 dim

- NR CS-matter theory is used in quantum Hall effects

$$I = \mathrm{Tr}(-1)^F e^{-\beta\Delta} x^{R-2J} \qquad \Delta = -\frac{1}{2}(iD - J + \frac{3}{2}R)$$

- Yet another novel mathematical invariant for $CY_4$

# What you will hear in the second part of the talk

- Precise definition of my index

- Representation theories of non-relativistic superconformal algebra

- Explicit computation of index for NR CS-matter theory
  → Applications to condensed matter physics

- NR-limit of M2-brane gauge theory on $CY_4$
  → new mathematical invariant for $CY_4$

- What is the RHS of the index theorem

$$I(x) = \int d\Phi d\Psi e^{-S_{eff}(\Phi,\Psi)} \qquad \Phi, \Psi : \ \mathrm{matrix}$$

Due to localization

# Index for Non-relativistic superconformal algebra

Yu Nakayama (UC Berkeley)

now I finished 1/6 of the talk

# Index in theoretical physics I

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  - Gauge theory
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# Index in theoretical physics II

- Index is robust and first check of any duality.
- From AdS-CFT duality to microstate counting of black hole.
- Index (almost) does not depend on any continuous parameter of the theory
  - Exception: wall crossing etc…

Can we do better than Witten index?

- Yes, we can! Superconformal index

(Romelsberger, Kinney-Maldacena-Minwalla-Raju, Nakayama)

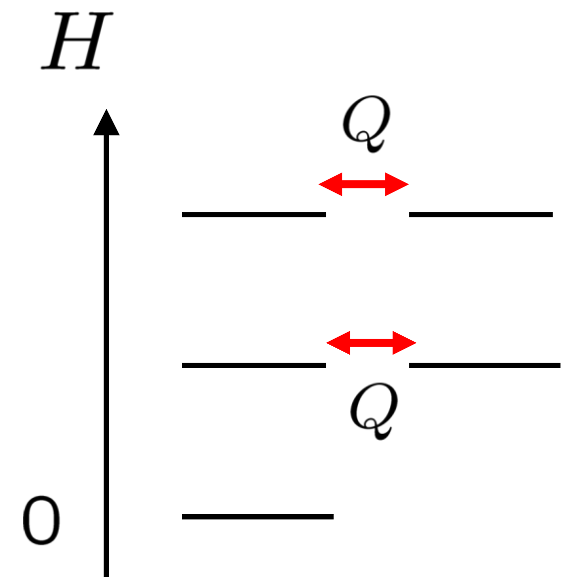
- Any other? When? Which SUSY algebra?
  - Index for non-relativisitic superconformal theory

# Witten index for supersymmetric field theory

- Witten Index on $R^4$ (or $T^3 \times R$) captures vacuum structure of the supersymmetric (field) theories

$$\begin{aligned} \mathcal{I}^W &= \mathrm{Tr}(-1)^F e^{-\beta H} = \mathrm{Tr}_{H=0}(-1)^F e^{-\beta H} \\ H &= \{Q^\dagger, Q\} \end{aligned}$$

  - Bose-Fermi cancellation
    - Only vacuum (H=0) states contribute
    - Does not depend on $\beta$
  - Many applications
    - Study on vacuum structure
    - Implication for SUSY breaking
    - Derivation of index theorem (geometry)



# Non-relativistic (NR) SCFT

- (1+2) dim non-relativistic super conformal field theory
  - Lorentz invariance → Galilean invariance
  - Scale invariance appears in massive theory:
    - Schrodinger equation is conformally invariant!

$$-\frac{\partial}{\partial t}\Psi = \frac{\partial_i^2}{2m}\Psi$$

- Construction from String theory
  - DLCQ of N=4 SYM in (1+3)d → (1+2)d NRSCFT
  - NR-limit of CS-matter theory (1+2)d NRSCFT
    - N=2 LLM (Lablanc-Lozano-Min hep-th/9206039
    - N=6 ABJM (Aharony-Bergman-Jafferis-Maldacena)
    - N=8 BL (Bagger-Lambert)

# Non-relativistic Superconformal Algebra I

- Bosonic part (Schrodinger algebra)

$$-\frac{\partial}{\partial t}\Psi = \frac{\partial_i^2}{2m}\Psi$$

  - Galilean algebra (H,P,G,J,M)

$$\begin{aligned} i[J,P] = -iP \ , \ i[J,\bar{P}] = i\bar{P} \\ i[J,G] = -iG \ , \ i[J,\bar{G}] = i\bar{G} \\ i[H,G] = P \ , \ i[H,\bar{G}] = \bar{P} \\ i[P,\bar{G}] = 2M \end{aligned}$$

  - +Dilatation(D)

$$\begin{aligned} i[D,P] &= \underline{-P} \ , \ i[D,\bar{P}] = -\bar{P} \ , \ i[D,G] = G \ , \ i[D,\bar{G}] = \bar{G} \\ i[D,H] &= \underline{-2H} \end{aligned}$$

  - +Special conformal transformation (K) $\delta t = \epsilon t^2 \ , \ \delta r_i = \epsilon t r_i$

$$i[K,\bar{P}] = -\bar{G}$$

$$i[H,K] = D \ , \ i[D,K] = 2K$$

# Non-relativistic Superconformal Algebra II

- Fermionic Part
  - SUSY algebra ($Q_1$, $Q_2$)

$$
\begin{aligned}
\{Q_1, Q_1^*\} &= 2M \ , \quad \{Q_2, Q_2^*\} = H \ , \quad \{Q_1, Q_2^*\} = \bar{P} \ , \quad \{Q_2, Q_1^*\} = P \ , \\
i[J, Q_1] &= \frac{i}{2} Q_1 \ , \quad i[J, Q_1^*] = -\frac{i}{2} Q_1^* \ , \quad i[J, Q_2] = -\frac{i}{2} Q_2 \ , \quad i[J, Q_2^*] = \frac{i}{2} Q_2^* \ , \\
i[\bar{G}, Q_2] &= -Q_1 \ , \quad i[G, Q_2^*] = -Q_1^* \ , \quad i[D, Q_2] = -Q_2 \ , \quad i[D, Q_2^*] = -Q_2^* \ ,
\end{aligned}
$$

  - + Superconforal (S)

$$
\begin{aligned}
i[K, Q_2] &= S \ , \quad i[H, S^*] = -Q_2^* \ , \quad i[\bar{P}, S] = -Q_1 \ , \quad i[J, S] = -\frac{i}{2} S \ , \\
\{S, S^*\} &= K \ , \quad \{S, Q_1^*\} = -G \ , \quad i[D, S] = S \ , \quad \{S, Q_2^*\} = \frac{i}{2}(iD - J + \frac{3}{2} R) \ , \\
i[R, Q_a] &= -iQ_a \ , \quad i[R, S] = -iS \ ,
\end{aligned}
$$

  - Grading structure w.r.t. D

H: +2, P, $Q_{2:}$ : +1

J,D,$Q_1$: 0,

G,S: -1, K: -2

# Involutive Anti-automorphism of algebra

There are two different conjugation in SSch

- Quantum mechanics (Dirac conjugation)

$$w_0(J) = J^\dagger = J, w_0(P) = P^\dagger = \bar{P}, w_0(G) = G^\dagger = \bar{G},$$
$$w_0(H) = H^\dagger = H, w_0(D) = D^\dagger = D, w_0(K) = K^\dagger = K, w_0(M) = M^\dagger = M$$
$$w_0(Q_1) = Q_1^*, w_0(Q_2) = Q_2^*, w_0(S) = S^*$$

- Conformal conjugation (BPZ conjugation)

$$w(J) = J, w(P) = \bar{G}, w(G) = \bar{P}, w(H) = -K,$$
$$w(K) = -H, w(D) = -D, w(M) = -M, w(R) = R,$$
$$w(Q_1) = iQ_1^*, w(Q_2) = iS^*, w(S) = iQ_2^*.$$

  - Different conjugation leads to different inner product.

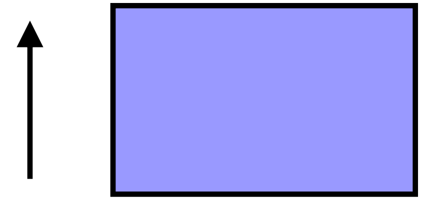
$$\langle O\Psi|\Psi\rangle = \langle\Psi|w(O)|\Psi\rangle$$

# $w_0(O)$ vs $w(O)$: Two conjugation?

Two conjugations are related by two different (but equivalent) quantization scheme

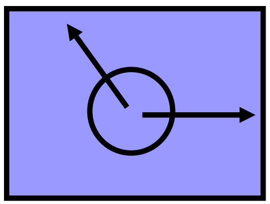
- Recall in 2D relativistic CFT on plane
  - Quantization w.r.t. "time" Hamiltonian

$$P^\dagger = P \ , K^\dagger = K \cdots$$



  - Radial quantization w.r.t. "radial" Hamiltonian

$$P = L_{-1} \ , K = L_{+1} \ , P^\dagger = K$$



- Similar in NRSCA

# Index for NRSCA

Two conjugation leads to two different index

- Index from $w_0$ → Witten index

$$I_0 = \text{Tr}(-1)^F e^{-\beta H} \ ,$$

$$\text{where } H = \{w_0(Q_2^*), Q_2^*\} = \{Q_2, Q_2^*\}.$$

  - Does not depend on $\beta$
  - Counts vacua (H=0)
- Index from w → NEW index for NRSCA!

$$I(x) = \text{Tr}(-1)^F e^{-\beta\Delta} x^{R-2J} \ ,$$

$$\text{where } \Delta = i\{S, Q_2^*\} = -\tfrac{1}{2}(iD - J + \tfrac{3}{2}R)$$

  - Does not depend on $\beta$
  - Counts BPS operators ($\Delta = 0$)
  - x is chemical potential to distinguish BPS operators

# Summary of representation of NRSCA

There are four types of representation in NRSCA

- Vacuum representation:

  Annihilated by $Q_2$ , $Q_2^*$ $\quad |d_0 = 0, j_0 = \frac{3}{2}r_0, m\rangle$

- Chiral representation

  Annihilated by $Q_2^*$ → $d_0 = -j_0 + \frac{3}{2}r_0$

  - Contribute to the index

- Anti-chiral representation

  Annihilated by $Q_2$ → $d_0 = j_0 - \frac{3}{2}r_0$

  - Complex conjugation of chiral representation

- Long representation

  Not annihilated either by $Q_2$ , $Q_2^*$

  - Two chiral representations make one long representation

# Some properties of index for NRSCA

$$I(x) = \mathrm{Tr}(-1)^F e^{-\beta\Delta} x^{R-2J} \ ,$$

$$\text{where } \Delta = i\{S, Q_2^*\} = -\frac{1}{2}(iD - J + \frac{3}{2}R)$$

- Index does not depend on $\beta$
- Protected by any exactly marginal deformation
  - Two chiral multiplets combine into one long multiplet → does not change index.
  - All the BPS states (actually infinitely many!) contribute. Know a lot about the NRSCFT (than just vacua)!
  - Independence of the exactly marginal deformation → the first step to check any duality.

# Example of Index for NRCFTs

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now I finished 2/3 of my talk

# N=2 Abelian CS-matter theory (LLM)

$$S = \int dtd^2x \left[\frac{\kappa}{2}\partial_t A_i \epsilon_{ij} A_j - A_0\left(\kappa B + e|\Phi|^2 + e|\Psi|^2\right) + i\Phi^*\partial_t\Phi + i\Psi^*\partial_t\Psi - \frac{1}{2m}|D_i\Phi|^2 - \frac{1}{2m}|D_i\Psi|^2 + \frac{e}{2m}B|\Psi|^2 + \frac{e^2}{2m\kappa}|\Phi|^4 + \frac{3e^2}{2m\kappa}|\Phi|^2|\Psi|^2\right]$$

- Construction

Begin with N=2 massive relativistic CS-matter theory

- Take NR-limit

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0$$

$$\phi = \frac{1}{\sqrt{2m}}(e^{-imt}\Phi + e^{imt}\hat{\Phi}^\dagger)$$

- We only keep particle $\Phi(x,t)$
  - → Schrodinger equation $-i\frac{\partial}{\partial t}\Phi = \frac{\partial_i^2}{2m}\Phi$

# More on the construction

- No direct relation between relativistic SCFT in (1+2) dim and NRSCFT in (1+2) dim!
  - No non-relativistic limit for “unparticle” traveling at the speed of light!
  - Add mass term → take NR limit
- Group theoretically $Sch_{2+1}$ is a subgroup of SO(2,4) ← (1+3) dim relativistic conformal algebra
  - Starting point of the AdS/NRCFT correspondence
  - (1+2) dim NRCFT may be dual to 5-dimensional gravity theory with light cone-compactification (Son, MIT group)
    - DLCQ of N=4  SYM as (1+2) NRSCFT?

# Applications

- M2-brane mini revolution
  - Add mass term ⇔ Add background 4-from flux
  - NR limit ⇔ near horizon (Penrose) limit? (contraction of algebra)

New arena for solvable(?) string theory, gauge gravity duality! Note: non-relativistic system is easier!

- Quantum Hall effect
  - CS-matter theory is effective field theory for "anyon": theoretical basis of (fractional) quantum hall effect.
  - Derivation of Laughlin's wavefunction

# N=2 Abelian CS-matter theory (continued)

$$S = \int dt d^2x \left[\frac{\kappa}{2}\partial_t A_i \epsilon_{ij} A_j - A_0\left(\kappa B + e|\Phi|^2 + e|\Psi|^2\right) + i\Phi^*\partial_t\Phi + i\Psi^*\partial_t\Psi \right.$$
$$\left. - \frac{1}{2m}|D_i\Phi|^2 - \frac{1}{2m}|D_i\Psi|^2 + \frac{e}{2m}B|\Psi|^2 + \frac{e^2}{2m\kappa}|\Phi|^4 + \frac{3e^2}{2m\kappa}|\Phi|^2|\Psi|^2\right]$$

- Conformally invariant (all order in perturbation theory)
- SUSY generator

$$Q_1 = \int d^2x \Phi^*\Psi$$
$$Q_2 = \int d^2x \Phi^* D_+\Psi$$
$$S = \int d^2x (t\Phi^* D_+\Psi + x_+\Phi^*\Psi)$$

- Index does not depend on exactly marginal deformation
  - we can take e → 0 limit (free theory)
- Count BPS states!

$$1, \Psi^*\Phi, D\Psi^*\Phi, \Phi^2\Psi^* D\Psi^* \cdots$$

| Letters | $U(1)$ | $J$ | $R$ | $iD$ | $R-2J$ |
|---|---|---|---|---|---|
| $\Phi$ | $1$ | $0$ | $2/3$ | $-1$ | $2/3$ |
| $\Psi^*$ | $-1$ | $-1/2$ | $1/3$ | $-1$ | $4/3$ |
| $P$ | $0$ | $-1$ | $0$ | $-1$ | $2$ |

# Computation of index

$$I(x) = \mathrm{Tr}(-1)^F e^{-\beta\Delta} x^{R-2J} \ ,$$

where $\Delta = i\{S, Q_2^*\} = -\frac{1}{2}(iD - J + \frac{3}{2}R)$

How to collect gauge invariant BPS ($\Delta = 0$) state efficiently?

- Idea: integrate over the gauge orbit (U(1) holonomy)

$$\Phi \to \Phi e^{i\theta}$$
$$\Psi^* \to \Psi^* e^{-i\theta}$$

| Letters | $U(1)$ | $J$ | $R$ | $iD$ | $R-2J$ |
|---|---|---|---|---|---|
| $\Phi$ | 1 | 0 | 2/3 | $-1$ | 2/3 |
| $\Psi^*$ | $-1$ | $-1/2$ | 1/3 | $-1$ | 4/3 |
| $P$ | 0 | $-1$ | 0 | $-1$ | 2 |

- Composite operator has charge:

$$\Phi^2 \propto e^{2i\theta}, \Phi\Psi^* \propto 1, \Psi^2 D\Psi^*\Psi^* \propto 1 \cdots$$

- Integration over U(1) holonomy only picks up singlets!

$$\int d\theta \Phi^2 e^{2i\theta} + \Phi\Psi^* = \Phi\Psi^*$$

$$\begin{aligned} I(x) = \mathrm{Tr}(-1)^F x^{R-J} &= \int \frac{d\theta}{2\pi} \prod_{m=0}^{\infty} \frac{1 - x^{\frac{4}{3}+2m} e^{i\theta}}{1 - x^{\frac{2}{3}+2m} e^{-i\theta}} \\ &= 1 - x^2 - 2x^4 - 2x^6 - 2x^8 + x^{12} + 5x^{14} + 7x^{16} + \cdots \ . \end{aligned}$$

# Non-relativistic limit of N=6 ABJM

More nontrivial example: NR-limit of ABJM theory

- ABJM model is M2 brane at $C_4/Z_k$ orbifold.
- Massive deformation and take NR-limit
- Gauge group is $U(N) \times U(N)$
- Several possible non-relativistic limit (choice of particle – anti-particle).
  - → Most supersymmetric limit has N=14 SUSY (with M. Sakaguchi and K.Yoshida)
    - $Q_2$ : 2
    - $Q_1$ : 10
    - S : 2
- Today I focus on N=2 NRSCA subalgebra

# Index for Non-relativistic ABJM

- We take $N \to \infty, k \to \infty$ limit to compute index.
- Index DOES depend on N (and k for finite N), but in the large N limit k dependence must vanish

Counting gauge invariant BPS states reduces to matrix integral (see Sundborg, Nakayama, KMMR)

$$I(x) = \int dU_1 dU_2 \exp\left(\sum_n \frac{1}{n} f_{12}(x^n) \mathrm{Tr} U_1^n \mathrm{Tr}(U_2^\dagger)^n + \frac{1}{n} f_{21}(x^n) \mathrm{Tr}(U_1^\dagger)^n \mathrm{Tr} U_2^n\right)$$

$$f_{12} = f_{21} = \frac{2x^{2/3} - 2x^{4/3}}{1 - x^2}$$

| Letters | $U(N) \times U(N)$ | $J$ | $R$ | $iD$ | $R-2J$ |
|---|---|---|---|---|---|
| $\Phi^a_{12}$ | $N \times \bar{N}$ | 0 | 2/3 | −1 | 2/3 |
| $\Phi^a_{21}$ | $\bar{N} \times N$ | 0 | 2/3 | −1 | 2/3 |
| $\Psi^{a*}_{21}$ | $\bar{N} \times N$ | −1/2 | 1/3 | −1 | 4/3 |
| $\Psi^{a*}_{12}$ | $N \times \bar{N}$ | −1/2 | 1/3 | −1 | 4/3 |
| $P$ | 0 | −1 | 0 | −1 | 2 |

# Matrix integral for index

- Index computation for NR ABJM reduces to matrix integral.
- The origin of this matrix integral is 1-loop computation on the cylinder (only remaining zero mode is Polyakov loop!).

Large N matrix integral is possible by saddle point approx.

$$I(x) = \int dU_1 dU_2 \exp\left(\sum_n \frac{1}{n} f_{12}(x^n)\mathrm{Tr}U_1^n \mathrm{Tr}(U_2^\dagger)^n + \frac{1}{n} f_{21}(x^n)\mathrm{Tr}(U_1^\dagger)^n \mathrm{Tr}U_2^n\right)$$

$$\begin{aligned} I(x) &= \prod_{n=1}^{\infty} \frac{(1+x^{\frac{2}{3}n}+x^{\frac{4}{3}n})^2}{(1+x^{\frac{2}{3}n}-x^{\frac{4}{3}n})(1+x^{\frac{2}{3}n}+3x^{\frac{4}{3}n})} \\ &= 1+4x^{4/3}-8x^2+24x^{8/3}-56x^{10/3}+156x^4-408x^{14/3}+1076x^{16/3}+\cdots \end{aligned}$$

- Interesting in number theory? Call for Ramanujan…

# Discussion

Yu Nakayama (UC Berkeley)

Now I’m in additional time…

# Further investigation

- More examples?
  - What is maximally non-relativistic SCA?
  - NR limit of ABJM has 14 SUSY
  - Known NRSCA with 24 SUSY
  - More indices for extended SUSY!
- NR limit of N=8 BL theory?
  - Problem with Majorana fermions…

AdS/NRCFT is the obvious next step

- Can we compute index for DLQC of N=4 SYM?
- Gravity side? KK decomposition of SUGRA background
- Non-relativistic limit of M2-brane background?

- More applications in condensed matter?

# A conjecture

- Wave-function of the universe = Laughlin's wave-function

$$\Psi = \prod_{i<j} (z_i - z_j)^{2k+1} \exp\left[-\frac{1}{4}eB(|z_1|^2 + \cdots + |z_N|^2)\right]$$

- U(1) theory: very strongly coupled universe
- k: Fraction of Landau level $\nu = \dfrac{1}{2k+1}$
- k = 0: free-fermion wave-function…

- Maybe exactly obtainable in SUSY CS theory…