Strings in 6D hidden in 2D Virasoro block

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Goal

- I will present a generic and systematic way to analyse strongly-coupled CFTs with a continuous global symmetry – non-SUSY, SUSY, even non-Lagrangian.
- By using the method, the large-charge expansion, we can study correlation functions involving large-charge operators.
- We will fisrt study the operator dimension of ϕ^J at $J \gg 1$ in 3D $O(2)$ WF theory.
- I will then compute the two-point function of chiral ring operators Φ^n at $n \gg 1$ in 6D $\mathcal{N} = (2,0)$, A_1 theory.
- I will relate the result to some double-scaling limit of the vacuum Virasoro block in 2D – In the end I uncover a completely new structure in 2D chiral algebra.

Strongly-coupled theories

- Strongly-coupled systems are important in various areas of physics – string theory, particle physics, nuclear physics and condensed matter.
- They are difficult to solve, though.
- Obviously. You cannot use the perturbation theory. They usually lack completely generic and systematic methods to solve.
- We have developed various methods to study them – integrability, supersymmetry, duality, anomaly, bootstrap, lattice simulations, etc.
- There are many of them by now, but strongly-coupled theories are still difficult. Let's think of another way to solve them.

Strongly-coupled CFTs

- Let's now focus on scale-invariant theories, or more specifically, CFTs.
- They are important; they describe second-order phase transitions. They can also be used to study quantum gravity through AdS/CFT.
- They are easier to handle in some sense: they have more symmetries.
- But they are also typically strongly-coupled: they are prototypical examples of strongly-coupled systems.
- Some of them do not even have a Lagrangian description like the 6D SCFT – more about it later.

The Large-Charge Expansion

- It is a new method of analysing generic CFTs, which can be non-SUSY or SUSY.
- Here is the idea...
- Take a CFT with a continuous global symmetry $U(1)$. (Can be in general non-Abelian group G as well.)
- You have a charge Q: The $Q \gg 1$ limit is semiclassical.
- Generic CFTs therefore admits a perturbative expansion in $1/Q$.
- Let's see how this is actually implimented now.

$O(2)$ Wilson-Fisher theory in $D = 3$

- One of the simplest strongly-coupled CFTs with a continuous symmetry – $O(2)$ Wilson-Fisher theory in $D = 3$.
- The Lagrangian is the usual ϕ^4 Lagrangian

$$L = |\partial\phi|^2 - m^2 |\phi|^2 - \frac{\lambda}{4} |\phi|^4$$

- In the $D = 4 - \epsilon$ expansion, the fixed-point is given by

$$0 = \beta(\lambda) \equiv \mu \frac{d\lambda}{d\mu} = -\epsilon\lambda + \frac{5\lambda^2}{(4\pi^2)^2} \Leftrightarrow \frac{\lambda_*}{(4\pi^2)^2} = \frac{\epsilon}{5}$$

- Extrapolating to $\epsilon = 1$, we believe that we have a strongly-coupled CFT in three dimensions.

$O(2)$ Wilson-Fisher theory in $D = 3$

- It is usually difficult to analytically compute physical quantities for this theory, because the theory is strongly-coupled.
- BUT: there is a way to compute them.
- Not just one physical quantity, but infinitely many. We can for example compute the operator dimension $\Delta[\phi^J]$ at $J \gg 1$.

Step 1: State-operator map

- Let us compute $\Delta[\phi^J] \equiv \Delta(J)$ now.
- According to the state-operator mapping, $\Delta(J)$ is nothing but the energy of the corresponding state $|J\rangle$ on $S^2 \times \mathbb{R}$, at radius 1.
- ϕ^J is the lowest dimension operator at charge J. So $|J\rangle$ is the lowest energy state (ground state) at charge J.

Step 1: State-operator map

- We are computing the ground state energy at charge J.
- This can be computed from the low-temperature partition function,

$$Z(\beta, J) \equiv \mathrm{Tr}\left[\delta(\hat{J} - J)e^{-\beta H}\right]$$

$$\Delta(J) = -\lim_{\beta\to\infty} \frac{\partial}{\partial\beta} \log Z(\beta, J) = -\lim_{\beta\to\infty} \frac{1}{\beta} \log Z(\beta, J)$$

- $Z(\beta, J)$ can be computed from the path integral,

$$Z(\beta, J) = \int \mathcal{D}\phi\Big|_{\text{charge } J \text{ configurations}} e^{-L[\phi]} \quad \text{on } S^2 \times S^1$$

Step 2: Saddle-point approximation

- We use the saddle-point analysis to evaluate this:

$$Z(\beta, J) \equiv \mathrm{Tr}\left[\delta(\hat{J} - J)e^{-\beta H}\right], \quad \Delta(J) = -\lim_{\beta\to\infty}\frac{1}{\beta}\log Z(\beta, J)$$

$$Z(\beta, J) \approx \int \mathcal{D}\phi\Big|_{\text{charge } J \text{ configurations}} e^{-S[\phi]} \quad \text{on } S^2 \times S^1$$

$$L = |\partial\phi|^2 - m^2|\phi|^2 - \frac{\lambda}{4}|\phi|^4$$

- The dominant saddle-point configuration is the lowest energy configuration at charge $J \gg 1$:

$$\phi = A \times e^{i\mu t}$$

- This is semi-classical at $J \gg 1$, as $A \gg 1$.

Step 3: Effective field theory

- This saddle-point configuration $\langle\phi\rangle = A \times e^{i\mu t}$ breaks the $O(2)$ symmetry. We can write down the effective action around it.
- The NG boson is χ, where our vacuum is $\langle\chi\rangle = \mu t$. ($\phi = a \times e^{i\chi}$, $\mu \gg 1$)
- The EFT realises the conformal and $O(2)$ symmetry non-linearly.

Step 3: Effective field theory

- The radial model a is massive with mass $|\partial\chi| \equiv \sqrt{-\partial_\mu\chi\partial^\mu\chi} \sim \mu$.
- This can appear in the denominator of the EFT.
- The $O(2)$ symmetry: only $\partial\chi$ can appear.
- The scale invariance: no intrinsic scales in the system.
- The leading order (in J, or in μ) effective action is then

$$L = c_{3/2} |\partial\chi|^3 + \cdots$$

  which scales as $O(\mu^3)$.
- By Noether theorem, $J \propto |\partial\chi| \partial_0\chi = O(\mu^2)$.

Step 3: Effective field theory

- We go on to write down the EFT to subleading orders,

$$L = c_{3/2} \left|\partial\chi\right|^3 + c_{1/2} \mathtt{Ric}_3 \times \left|\partial\chi\right| + O(J^{-1/2})$$

- I have used $|\partial\chi| = O(\mu) = O(J^{1/2})$.
- Extremely important that there is no operator at $O(J^0)$.
- The coefficients $c_{3/2}$, $c_{1/2}$ are undetermined as they are Wilsonian coefficients.
- As $\left|\partial\chi\right|^3 = O(J^{3/2})$, the loop is suppressed by $J^{-3/2}$.
- $\Delta(J)$ is given by the ground state energy of the vacuum, $\langle\chi\rangle = \mu t$.

Step 3: Effective field theory

- Our EFT:

$$L = c_{3/2} \left|\partial\chi\right|^3 + c_{1/2} \mathtt{Ric}_3 \times \left|\partial\chi\right| + O(J^{-1/2})$$

- The energy of the saddle-point $\langle\chi\rangle = \mu t$, $\mu = O(J^{1/2})$ is given by

$$\Delta(J) = \Delta_{\rm cl}(J) + \Delta_{\rm one-loop}(J) + \cdots$$

- It is immediate to see

$$\Delta_{\rm cl}(J) = c_{3/2} J^{3/2} + c_{1/2} J^{1/2} + O(J^{-1/2})$$

- The leading order result could just have been guessed from dimensional analysis. BUT: very different from the weak-coupling intuition.

Step 3: Effective field theory

- Our EFT and 0-loop:

$$L = c_{3/2} \left|\partial\chi\right|^3 + c_{1/2}\mathtt{Ric}_3 \times \left|\partial\chi\right| + O(J^{-1/2})$$
$$\Delta_{\rm cl}(J) = c_{3/2} J^{3/2} + c_{1/2} J^{1/2} + O(J^{-1/2})$$

- The fluctuation Lagrangian is

$$L \ni \chi_{\rm fluc} \left(\partial_t^2 - \frac{1}{2} \triangle_{S^2} \right) \chi_{\rm fluc}$$

- And it takes you a one-loop computation to see

$$\Delta_{\rm one-loop} = \log\det \left(\partial_t^2 - \frac{1}{2} \triangle_{S^2} \right)$$
$$= \frac{1}{2} \sum_{\ell=0}^{\infty} (2\ell+1) \sqrt{\ell(\ell+1)} \xrightarrow[\text{renormalise}]{\text{regularise \&}} -0.094$$

Final result

- After all this, we see that the operator dimension of ϕ^J in $O(2)$ WF theory in $D = 3$ is

$$\Delta[Q] = c_{3/2} Q^{3/2} + c_{1/2} \mathtt{Ric}_3 Q^{1/2} - 0.094 + O(Q^{-1/2}).$$

- The undetermined coefficients cannot be determined with this method.
- OTOH: The argument works for any theories with the same symmetry breaking pattern at finite charge density.
- For example, large monopole number operator of $\mathbb{C}P^{N-1}$ behaves exactly the same:

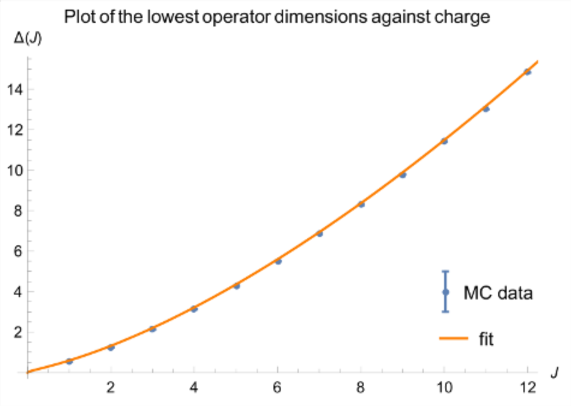
$$\Delta[Q] = b_{3/2} Q^{3/2} + b_{1/2} \mathtt{Ric}_3 Q^{1/2} - 0.094 + O(Q^{-1/2}).$$

- This is some sort of "large-charge universality". It would be interesting to check dualities using the idea.

Remark: Numerics

- There is some numerics for $\Delta(J)$ as well:

$$\Delta[Q] = \frac{1.195}{\sqrt{4\pi}} Q^{3/2} + 0.075\sqrt{4\pi} Q^{1/2} - 0.094 + O(Q^{-1/2}).$$



Non-Lagrangian theories at large charge

- The language of CFTs are so abstract that we don't need the notion of concrete Lagrangians.
- There are strongly-coupled theories which are not connected to any weakly-coupled field theories (as of yet).
- These are called non-Lagrangian.
- Why do we study them? My takes are:
    - Interesting in terms of String Theory, which somehow knows how to construct them.
    - It's really genuinely strongly-coupled, so its an advanced playground for methods of solving CFTs.
    - Might have interesting holography dual.

Interacting CFTs in $D > 4$?

- Known 6D interacting CFTs are all non-Lagrangian.
- First off: are there interacting unitary CFTs above four-dimensions?
- If you start from the Lagrangian, you start from the free theory and then add relevant interactions.
- If you have a field ϕ, its mass dimension is one.
- I want the interaction to have a minima, so ϕ^4 is the lowest power that I can think of.
- They are irrelevant, so most likely won't give you an interacting CFT.

Interacting SCFTs in $D = 6$?

- Same is true for SCFTs. Let's focus on $\mathcal{N} = (2, 0)$ theories (16 supercharges).
- The superconformal algebra: $\mathfrak{osp}(6, 2|4) \supset \mathfrak{so}(6, 2) \oplus \mathfrak{sp}(4)$.
- The R-symmetry is $\mathfrak{sp}(4) \cong \mathfrak{so}(5)$
- The only low-spin massless representation: the Abelian tensor multiplet:
  - five real scalars in the **5** of $SO(5)_R$
  - a chiral two-form potential with self-dual field strength
  - a pseudo-Majorana-Weyl spinor in the **4** of $SO(5)_R$
- Likewise there seems no hope of constructing Lagrangian SCFTs.

6d SCFT at rank one

- There are string theory constructions of 6d SCFTs.
- Let's focus on the simplest, the A_1 theory.
- Place two $M5$-branes in an eleven-dimensional space.
- The low-energy limit is the A_1 theory.
- The moduli space is (locally) $\mathbb{R}^5$, because you can seperate the two branes in the transverse direction.
- (If you take the rank to be large, you can use AdS/CFT.)

6d SCFT at rank one

- Now it is notoriously non-Lagrangian. It's genuinely strongly-coupled with no parameters connecting it to a free point.
- Not many things are known aside from the quantities protected by SUSY.
- Holography won't help us either because the rank is small.
- BUT: We can use the large-charge expansion to study physical quantities.

half-BPS operators

- Let's study the two-point functions of chiral ring operators in the A_1 theory.
- In terms of the CFT language, this is related to the OPE coefficient.
- The operator dimension is obviously easy to compute, but this is in general hard to compute.

half-BPS operators

- The half-BPS ring of the A_1 theory s generated by only one element.
- It is generated by the bottom component of the stress tensor, Φ^{IJ}. It is in the rank-2 symmetric traceless rep of the R-symmetry. It has a protected dimension of 4.
- The chiral ring operators are the symmetric product Φ^n, in rank-(2n) symmetric traceless rep and has a protected dimension, $4n$.
- We want to compute the two-point functions of Φ^n using the large-charge EFT:

$$\lambda_n \equiv x^{8n} \langle \Phi^n(0) \Phi^n(x) \rangle$$

- We can simply place two operators on the antipodal points on S_6, and then compute using the path-integral using the EFT.

Moduli EFT

- Let's write down the EFT at large R-charge. The existence of moduli means that the massless field is given by φ^I, the bottom component of the Abelian tensor multiplet.
- It has dimension 2 and in rank-1 symmetric traceless rep, so we have

$$\Phi^n \longleftrightarrow \varphi^{2n}$$

with some unknown prefactor.

- We can use the EFT to compute λ_n:

$$\langle \varphi^{2n}(x_1)\varphi^{2n}(x_2)\rangle = \frac{1}{\mathrm{Z}}\int \mathcal{D}\varphi\varphi^{2n}(x_1)\varphi^{2n}(x_2)\exp\left[-\int d^6x\mathcal{L}_{\mathrm{EFT}}[\phi]\right].$$

Moduli EFT

- The EFT is nothing but the EFT on the moduli: The derivative has dimension 1, so if it appears in the numerator, $|\varphi|^{1/2}$ needs to appear in the denominator.
- In other words, one more derivative means the scaling lower by $n^{1/2}$.
- It is non-trivial to construct the EFT but can be done:

$$\frac{1}{2}\sum_{I=1}^{5}\left(\partial_\mu\varphi^I\right)^2 - \sqrt{\frac{7}{98304\pi^3}\frac{\Delta a}{a_{U(1)}}}\frac{\partial^2(\varphi^I\varphi^J)\partial^2(\varphi^I\varphi^J)}{4\left|\varphi\right|^3} - \Delta a \times \tau E_6 + \cdots$$

  in the Euclidean signature.
- τ is the dilaton given by $\tau \equiv -\frac{1}{2}\log\left|\varphi^I\right|$.
- Δa is the a-anomaly difference between the A_1 theory and the low-energy Abelian tensor multiplet.

Operator dimension

- It is immediate to check that the leading order operator dimension of $\Phi^n \propto \varphi^{2n}$ is $4n$: just a free theory.
- Subleading contributions can be shown to vanish to a certain order, which is reassuring.
- It has got to be like this, as they have protected dimensions.

The OPE coefficient.

- Let's compute $\lambda_n \equiv x^{8n} \left\langle \varphi^{2n}(0)\varphi^{2n}(x) \right\rangle$.
- φ^{2n} is in the symmetric traceless rep, so it's the same as taking one $U(1)$ charge to be large.
- Replace $\phi = (\varphi^1 + i\varphi^2)/\sqrt{2}$, forget $\varphi^{3,4,5} = 0$.
- We can compute λ_n in the saddle-point approximation of

$$\langle \phi^{2n}(x_1)\overline{\phi}^{2n}(x_2) \rangle = \frac{1}{Z_{S^6}} \int \mathcal{D}\phi \phi^{2n}(x_1)\overline{\phi}^{2n}(x_2) \exp\left[- \int d^6x \mathcal{L}_{\mathrm{EFT}}[\phi] \right]$$

- Saddle-point approximation in terms of

$$\mathcal{L}_{\mathrm{EFT}} - 2n \log(\phi(x))\delta(x - x_1) - 2n \log(\bar{\phi}(x))\delta(x - x_2)$$

where $\phi = O(\sqrt{n})$.

Structure of the expansion

- We are computing

$$\langle\phi^{2n}(x_1)\overline{\phi}^{2n}(x_2)\rangle = \frac{1}{Z_{S^6}}\int \mathcal{D}\phi\,\phi^{2n}(x_1)\overline{\phi}^{2n}(x_2)\exp\left[-\int d^6x\mathcal{L}_{\mathrm{EFT}}[\phi]\right]$$

$$\frac{1}{2}\sum_{I=1}^{5}\left(\partial_\mu\varphi^I\right)^2 - \sqrt{\frac{7}{98304\pi^3}\frac{\Delta a}{a_{U(1)}}}\frac{\partial^2(\varphi^I\varphi^J)\partial^2(\varphi^I\varphi^J)}{4\left|\varphi\right|^3} - \Delta a\times\tau E_6 + \cdots$$

- What we do: compute the saddle-point and then sum all the vacuum diagrams.

The OPE coefficient.

- The contribution coming from the leading free action is just the Wick contraction, so

$$\log \lambda_n = \log \Gamma(2n+1) + (\text{higher-derivative}) \,.$$

- We can go on by computing the saddle-point from the free action, and then plug it into the EFT. This gives you all the contributions above $O(1)$. (I can tell you after the talk why this is the case.)
- Eventually we get

$$\log \lambda_n = \log \Gamma(2n+1) + 8\sqrt{n} - 2\log n + An + B + O(n^0)$$

where A and B are undetermined.
- A is convention-dependent and B, scheme-dependent, so we cannot compute them this time.

6d/2d correspondence

- One can compute λ_n from 2D chiral algebra, because of the 6d/2d correspondence.
- Rastelli and friends showed that, if one takes the cohomology class of certain nilpotent supercharges, the half-BPS operators are in one-to-one correspondence with the element of the 2D chiral algebra.
- For example, the A_1 theory corresponds to $c = 25$ chiral algebra, and Φ corresponds to $T(z)$ there.
- Interestingly, λ_n appear in the double-scaling limit of the vacuum Virasoro block (I can explain after the talk):

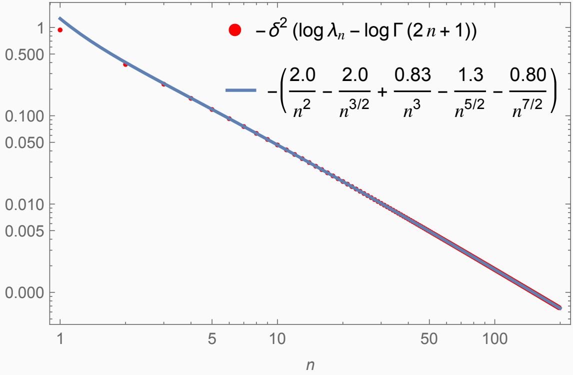
$$\mathcal{F}(c, h, z) = \sum_{n=0}^{\infty} \frac{(zh)^{2n}}{\lambda_n} \times \left(1 + O(h^{-1})\right)\left(1 + O(z)\right)$$

  where $h \to \infty$, $z \to 0$, while hz is fixed.
- This is related to the thermodynamic limit of 2d CFTs. Or the element of the inverse Shapovalov form in the $\{2, 2, \ldots, 2\}$.

6d/2d correspondence

- And then, there is a fast numerical algorithm to compute the expansion coefficients of the Virasoro block by using the Zamolodchikov recursion relation. We can therefore compute λ_n.
- We see that, at $c = 25$, we have a consistent result with

$$\log \lambda_n = \log \Gamma(2n+1) + 8\sqrt{n} - 2\log n + An + B + O(n^0).$$



6d/2d correspondence

- We can numerically compute the same λ_n for different c.
- We numerically found that

$$\log \lambda_n = \log(2n)! + \alpha n + 4\sqrt{\frac{2(c-1)}{12}}\sqrt{n} - \frac{c-1}{12}\log n + O(n^0)\ .$$

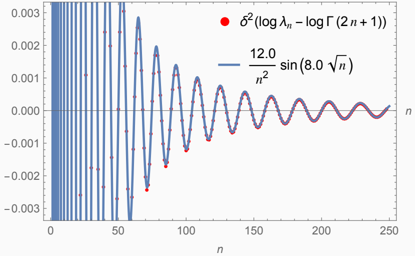
  which looks very tempting: But we do not know how to prove this just from 2D.
- Anyhow, the large charge expansion has really shown us a completely new structure for the Virasoro algebra, previously unknown.

Tensionless string in 6D

- In particular, at $c = 1$, the numerics is clean enough to see the subleading corrections.
- We numerically found that

$$\log \lambda_n = \log(2n)! + \frac{0.25}{n} \sin\left(8.0\sqrt{n}\right) + \cdots \ .$$

- This is crazy! This comes from the tension of the BPS-string on the moduli space. We have computationally seen an evidence for the tensionless string in 6D SCFT (although fictitious SCFT corresponding to $c = 1$, which is non-unitary).



Future directions

- To prove formulas just from 2D chiral algebra.
- Computing the tension of the BPS string.
- 4D Higgs branch v.s. 2D WZW model
- 3d/1d correspondence.
- And more......