SymTFT for Flavor Symmetries

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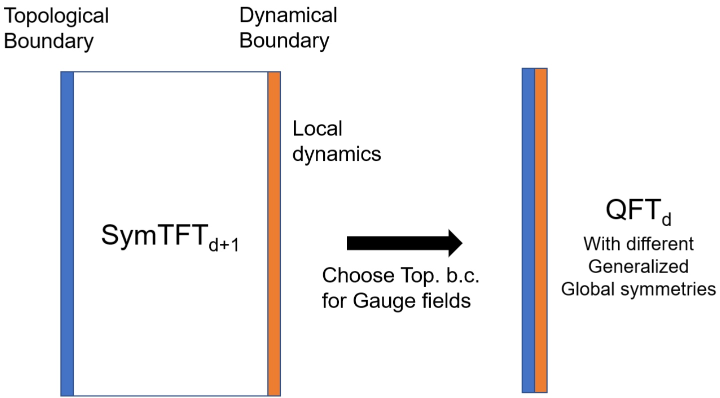
Topological Holography/SymTFT



- SymTFT also contains the anomaly polynomial information
- In AdS/CFT: SymTFT$_{d+1}$ - topological terms in the bulk (Witten 98')
- In CMT: SymTFT$_{d+1}$ - topological order (Kong, Levin, Wen etc.)

Topological Holography/SymTFT

- Examples (many more):

Bulk	Boundary	Symmetries
$\mathbb{Z}_2$ toric code	Kitaev chain	$\mathbb{Z}_2^{(0)}$
IIB on $AdS_5 \times S^5$	4d $\mathcal{N}=4$ $SU(N)$ SYM	$\mathbb{Z}_N^{(1)}$ (Witten 98')
4d TQFT	3d $\mathcal{N}=6$ ABJM	1,0-form sym. (Bergman, Tachikawa, Zafrir 20') (Beest,Gould,Schafer-Nameki,YNW 22')
7d TQFT	6d (2,0)& (1,0) SCFTs	2,1,0-form sym. (Hubner,Morrison,Schafer-Nameki,YNW 22 (Apruzzi 22') (Tian, YNW 24') (Apruzzi, Schafer-Nameki, Warman 24') (Bonetti, del Zotto, Minasian 24')

Topological Holography/SymTFT

• In the CMT/topological order community (categorical language), typically only tackle finite symmetries, e.g. $\mathbb{Z}_N$, D_4 etc.

• What about continuous symmetries e.g. $U(1)$, $SU(2)$ which appear more frequently in high energy physics?

• Should be more careful about gauging & dual symmetries

• In our work, realize the SymTFT of non-abelian continuous symmetry G as a non-abelian BF theory, and derived the relevant symmetry operators

$$S_{BF} = \int_{M_{d+1}} \mathrm{Tr}(B \wedge F)\,.$$

p-form symmetry

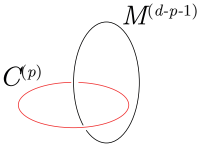
- A p-form symmetry with symmetry group G is generated by $(d-p-1)$-dimensional topological operators $U(g, M^{(d-p-1)})$ $(g \in G)$:

$$U(g_1, M^{(d-p-1)})U(g_2, M^{(d-p-1)}) = U(g_1 g_2, M^{(d-p-1)}) .$$

and acts on p-dimensional object(operator) $V^i(\mathcal{C}^{(p)})$.

- $U(g, M^{(d-p-1)})$ has non-trivial action on $V^i(\mathcal{C}^{(p)})$ when $M^{(d-p-1)}$ and $\mathcal{C}^{(p)}$ are non-trivially linked.

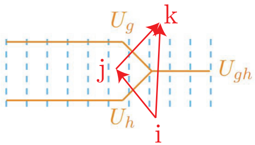
$$U(g, M^{(d-p-1)})V^i(\mathcal{C}^{(p)}) = R^i{}_j(g)V^j(\mathcal{C}^{(p)}) .$$



- $(p > 0)$-form symmetry is abelian

Gauging of p-form symmetry

- Gauging finite abelian p-form symmetry $G^{(p)}$:
- Flat connections $A_{(p+1)} \in C^{p+1}(M_d, G^{(p)})$, equivalent classes mod out gauge transformation: $[A_{(p+1)}] \in H^{p+1}(M_d, G^{(p)})$, $p = 0$ example:



$$A_{ij} A_{jk} = A_{ik}$$

Gauging: (1) coupling $\sim A \wedge J$

(2) summing over all configurations of $[A_{(p+1)}]$, i.e. the holonomies

Gauging of p-form symmetry

- Dual symmetry after the gauging $G^{(p)}$:

$$\widehat{G}^{(d-p-2)} = \mathrm{Hom}(G^{(p)}, U(1)) \sim \widehat{G}^{(d-p-2)} = \mathrm{Rep}(G^{(p)})$$

- $(d-p-2)$-form symmetry generated by the topological operator

$$W_{\mathbf{R}}(C^{(p+1)}) = \exp\left(i \int_{C^{(p+1)}} A_{(p+1),\mathbf{R}}\right) \ , \ \mathbf{R} \in \mathrm{Rep}(G^{(p)})$$

- Example: $G^{(p)} = \mathbb{Z}_N$, $\widehat{G}^{(d-p-2)} = \mathbb{Z}_N$,

$$W_{\hat{g}}(C^{(p+1)}) = \hat{g}^{\int_{C^{(p+1)}} A_{(p+1)}} \ , \ \hat{g} = e^{\frac{2\pi i \hat{n}}{N}} \ (\hat{n} \in \{0, 1, \ldots, N-1\})$$

- (Representation symmetry $\mathrm{Rep}(G^{(0)})$ also appears after gauging a non-abelian $G^{(0)}$)

Gauging of p-form symmetry

• Partition function of the theory after gauging w/ new background gauge field $[\widehat{A}^{(d-p-1)}] \in H^{d-p-1}(M_d, \widehat{G}^{(d-p-2)})$

$$\widehat{Z}(\widehat{A}^{(d-p-1)}) = \frac{1}{|H^{p+1}(M_d, G^{(p)})|} \sum_{[A_{(p+1)}]} Z(A^{(p+1)}) e^{i \int_{M_d} \langle \widehat{A}^{(d-p-1)}, \cup A^{(p+1)} \rangle}$$

• Gauging $\equiv$ Fourier transform, between sets of functions, i.e. moduli space of flat connections $H^{p+1}(M_d, G^{(p)})$ and $H^{d-p-1}(M_d, \widehat{G}^{(d-p-2)})$!

$$L^2(H^{p+1}(M_d, G^{(p)})) \overset{T}{\sim} L^2(H^{d-p-1}(M_d, \widehat{G}^{(d-p-2)}))\,.$$

Gauging with SymTFT

- SymTFT of $G^{(p)} = \mathbb{Z}_N$, without anomaly has a form of BF-theory

$$S_{sym} = \frac{2\pi}{N} \int_{M_{d+1}} a^{(p+1)} \cup \delta \hat{a}^{(d-p-1)}$$

- $a, \hat{a} \in \{0, 1, \ldots, N-1\}$
- Gauge invariant topological operators

$$U_g(\Sigma_{d-p-1}) = g^{\int_{\Sigma_{d-p-1}} \hat{a}^{(d-p-1)}} \ , \ g = e^{\frac{2\pi i n}{N}}$$

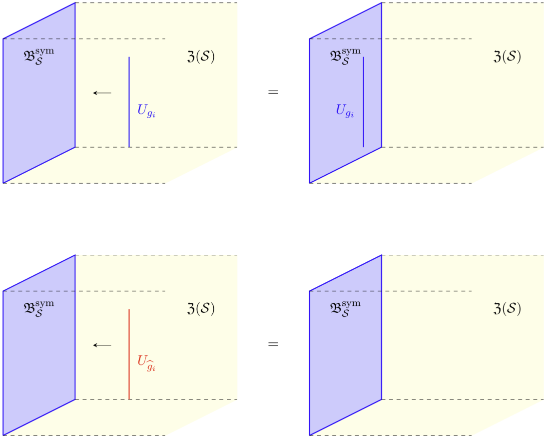
$$W_{\hat{g}}(C_{p+1}) = \hat{g}^{\int_{C_{p+1}} a^{(p+1)}} \ , \ \hat{g} = e^{\frac{2\pi i \hat{n}}{N}}$$

- Linking correlation function:

$$\langle U_g(\Sigma_{d-p-1}) W_{\hat{g}}(C_{p+1}) \rangle = \exp\left(\frac{2\pi i n \hat{n}}{N} \langle \Sigma_{d-p-1}, C_{p+1} \rangle \right)$$

Topological boundary conditions

(1) Neumann b.c. for U_g, $\hat{a}^{(d-p-1)}$, Dirichlet b.c. for $W_{\hat{g}}$, $a^{(p+1)}$



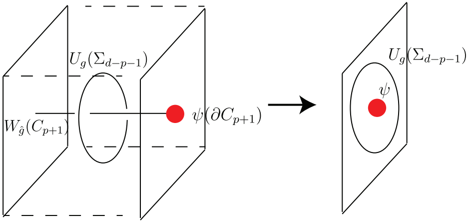
(Bhardwaj, Bottini, Fraser-Taliente, Gladden, Gould, Platschorre, Tillim 23')

Topological boundary conditions

(1) Neumann b.c. for U_g, $\hat{a}^{(d-p-1)}$, Dirichlet b.c. for $W_{\hat{g}}$, $a^{(p+1)}$

$W_{\hat{g}}(C_{p+1})$ can be attached to the boundary, no longer gauge invariant.

Should attach a charged operator $\psi_{\hat{g}}(\partial C_{p+1})$ on the physical boundary



$$U_g(\Sigma_{d-p-1})W_{\hat{g}}(C_{p+1})\psi_{\hat{g}} = \exp\left(\frac{2\pi i n\hat{n}}{N}\langle\Sigma_{d-p-1}, C_{p+1}\rangle_{M_{d+1}}\right)\psi_{\hat{g}}$$

- After squeezing

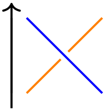
$$U_g(\Sigma_{d-p-1})\psi_{\hat{g}} = \exp\left(\frac{2\pi i n\hat{n}}{N}\langle\Sigma_{d-p-1}, \partial C_{p+1}\rangle_{M_d}\right)\psi_{\hat{g}}$$

- U_g generates p-form symmetry $G^{(p)} = \mathbb{Z}_N$ in M_d, acting on $\psi_{\hat{g}}$!

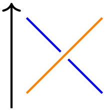
Topological boundary conditions

(2) Dirichlet b.c. for U_g, $\hat{a}^{(d-p-1)}$, Neumann b.c. for $W_{\hat{g}}$, $a^{(p+1)}$

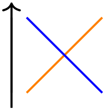
- $W_{\hat{g}}(C_{p+1})$ generates the dual $\widehat{G}^{(d-p-2)}$ symmetry
- Swapping b.c. $\equiv$ Gauging!
- Dirac quantization: U and W cannot both have Dirichlet b.c.
- Another perspective (x^{d+1} is perpendicular to the plane)



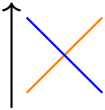
$= \exp\left(\frac{2\pi i n\hat{n}}{N}\right)$



- After squeezing to M_d:



$= \exp\left(\frac{2\pi i n\hat{n}}{N}\right)$



- Partition function not well defined when $\exp\left(\frac{2\pi i n\hat{n}}{N}\right) \neq 1$!

Gauging continuous symmetries

• For continuous symmetries G, two types of gauging

(1) Gauging w/ non-flat gauge field A: new local dynamics (not a SymTFT, see Sym-Th (Apruzzi, Bedogna, Dondi 24')

(2) Gauging w/ flat connection: SymTFT!

• Proposals for continuous SymTFT: (Brennan, Sun 24')(Bonetti, del Zotto, Minasian 24')(Antinucci, Benini 24')

• No satisfactory construction of operator spectrum, top. b.c. etc for continuous non-abelian 0-form symmetry G!

Gauging continuous symmetries

- Dual symmetry after gauging G w/ flat connection: Wilson loop $W_{\mathbf{R}}(\mathcal{C}) = \mathrm{Tr}_{\mathbf{R}}\mathcal{P}\exp i\oint_{\mathcal{C}} A_{\mathbf{R}}$ is topological, generates a dual $\mathrm{Rep}(G)^{(d-2)}$ symmetry!

- Fourier transformation perspective:
- Moduli space of flat G-connection:

$$\widetilde{\mathcal{A}} = \frac{\mathrm{Hom}(\pi_1(M_d), G)}{G}$$

- e.g. if $\pi_1(M_d) = \mathbb{Z}$, $\widetilde{\mathcal{A}} = Cl(G)$

Gauging continuous symmetries

- Tannaka duality between monoidal categories:

$$L^2(Cl(G)) \overset{T}{\cong} L^2(G^\vee)$$

- $G^\vee$ is the isomorphism classes of irreps of G

$$T[f](\lambda) = \frac{1}{d_\lambda} \int_G f(g)\chi_\lambda(g)dg$$

$$\begin{aligned} T^{-1} : L^2(G^\vee) &\to L^2(Cl(G)) \\ \phi &\mapsto T^{-1}[\phi] : g \mapsto \sum_{\lambda \in G^\vee} \phi(\lambda)\overline{\chi_\lambda(g)} d_\lambda \,. \end{aligned}$$

Non-abelian SymTFT

- Non-abelian BF-theory

$$S_{sym} = \int_{M_{d+1}} \mathrm{Tr}(B \wedge F)$$

- $F = dA - iA \wedge A$; $B \in \mathfrak{g}$ is a $(d-1)$-form, non-compact gauge field
- Gauge transformations:

$$A \to gAg^{-1} + igdg^{-1} \ , \ B \to gBg^{-1}$$

$$B \to B + D_A K$$

- Equation of motions:

$$F = 0 \ , \ D_A B = 0 \, .$$

- Gauge invariant topological operator generating $\mathrm{Rep}(G)^{(d-2)}$ symmetry with Neumann b.c. of A:

$$W_{\mathbf{R}}(\mathcal{C}) = \mathrm{Tr}_{\mathbf{R}} \mathcal{P} \exp \left(i \oint_{\mathcal{C}} A_{\mathbf{R}} \right)$$

Symmetry operator

- Difficulty in constructing non-abelian symmetry generator $U_\alpha(\Sigma_{d-1})$ in M_{d+1}:
- A topological cod-2 operator in M_{d+1} cannot be non-abelian!
- Let's try to write down an operator of the form

$$U_\alpha(\Sigma_{d-1}) = \exp\left(i \int_{\Sigma_{d-1}} (\alpha, B)\right)$$

- α, $B \in \mathfrak{g}$; $(.,.)$ is a bilinear inner product
- To prove it's topological

$$d(\alpha, B) = D_A(\alpha, B) = (D_A\alpha, B) + (\alpha, D_A B) = (D_A\alpha, B) = 0$$

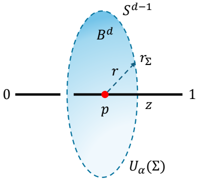
- Hence we require $D_A\alpha = 0$, i.e. α is a covariantly constant section of the adjoint bundle $\mathfrak{ad}(P)$ for an A, $U_\alpha(\Sigma_{d-1})$ is topological

Symmetry operator

- However, only true at the classical level!
- To compute the linking correlation function ($W_{\mathbf{R}}(\mathcal{C})$ is untraced)

$$\langle U_\alpha(\Sigma) W_{\mathbf{R}}(\mathcal{C})\rangle$$
$$= \int [DB][DA] \exp(i \int_{M_{d+1}} \mathrm{Tr}(B \wedge F)) \exp(i \int_\Sigma (\alpha, B)) \mathcal{P} \exp(i \int_{\mathcal{C}} A_{\mathbf{R}})$$

Integrate over all flat A where $D_A \alpha = 0$ no longer satisfies!

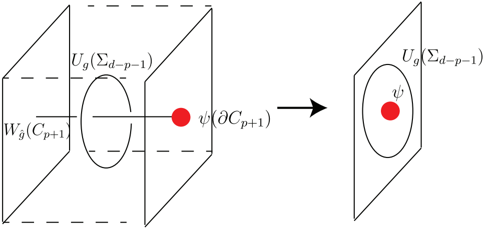


$$\langle U_\alpha(\Sigma) W_{\mathbf{R}}(\ell)\rangle = e^{-i\alpha_{\mathbf{R}}(p)} \langle W_{\mathbf{R}}(\ell)\rangle \quad (r_\Sigma \to 0)$$

is dependent on the position of p! $e^{-i\alpha_R(p)}$ is a Lie group element of G in the rep. R

Symmetry operator

- Hence $U_\alpha(\Sigma_{d-1})$ is not topological, as expected!



- After squeezing (taking $p = 0$), get the non-abelian symmetry action!

$$U_\alpha(\Sigma)\psi_{\mathbf{R}}(x) = e^{-i\alpha_{\mathbf{R}}(0)}\psi_{\mathbf{R}}(x)$$

Topological operator

- What if we want to construct a topological operator? Define

$$\tilde{U}_{\tilde{\alpha}}(\Sigma) = \int_G dg \exp\left(i \int_\Sigma (\tilde{\alpha}, B) \right)$$

Here $\tilde{\alpha}$ is the parallel transport of $g\alpha(p)g^{-1}$ by the holonomy of A:

$$\tilde{\alpha}(p') = e^{i \int_p^{p'} A} g\alpha(p)g^{-1} e^{-i \int_p^{p'} A}$$

- $\tilde{\alpha}$ lives in the same conjugacy class! $\tilde{\alpha} \in Cl(G)$ labels the $\tilde{U}_{\tilde{\alpha}}(\Sigma)$
- Can prove

$$\langle \tilde{U}_{\tilde{\alpha}}(\Sigma) W_{\mathbf{R}}(\ell) \rangle = \frac{\chi_{\mathbf{R}}(e^{-i\tilde{\alpha}})}{\dim(\mathbf{R})} \langle W_{\mathbf{R}}(\ell) \rangle$$

(Similar to (Cattaneo, Rossi 02'))

Topological operator

$$\widetilde{U}_{\widetilde{\alpha}}(\Sigma) W_{\mathbf{R}}(\ell) = \mathcal{P}\left(e^{i\int_p^1 A_{\mathbf{R}}} \left(\int dg \;\, g e^{-i\alpha_{\mathbf{R}}(p)} g^{-1} \right) \; e^{i\int_0^p A_{\mathbf{R}}} \right)$$

- For irreducible **R**, by Schur's lemma and using normalized Haar measure

$$\int dg \;\, g e^{-i\alpha_{\mathbf{R}}(p)} g^{-1} = \frac{\chi_{\mathbf{R}}(e^{-i\alpha(p)})}{\dim \mathbf{R}} \times \mathbb{I}_{\dim \mathbf{R} \times \dim \mathbf{R}}$$

- After taking VEV:

$$\langle \widetilde{U}_{\widetilde{\alpha}}(\Sigma) W_{\mathbf{R}}(\ell) \rangle = \frac{\chi_{\mathbf{R}}(e^{-i\alpha(p)})}{\dim \mathbf{R}} \langle W_{\mathbf{R}}(\ell) \rangle = \frac{\chi_{\mathbf{R}}(e^{-i\tilde{\alpha}})}{\dim \mathbf{R}} \langle W_{\mathbf{R}}(\ell) \rangle$$

Dirac pairing

- Hence in the non-abelian G SymTFT, the Dirac pairing function is the normalized character!

$$(e^{-i\tilde{\alpha}}, \mathbf{R}) = \frac{\chi_{\mathbf{R}}(e^{-i\tilde{\alpha}})}{\dim(\mathbf{R})}$$

- For example top. b.c. for $G = SU(2)$
- Top. operators: $W_{\mathbf{R}}$, $\mathbf{R} = j \in \frac{1}{2}\mathbb{Z}$; $\tilde{U}_\theta$, $\theta \in Cl(SU(2)) = [0, \pi]$
- 2×2 special unitary matrices can be diagonalized into

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \quad \theta \in [0, \pi] \,.$$

- Normalized character

$$\frac{\chi_j(e^{-i\theta})}{\dim(\chi_j)} = \frac{\sin(2j+1)\theta}{(2j+1)\sin\theta} \,.$$

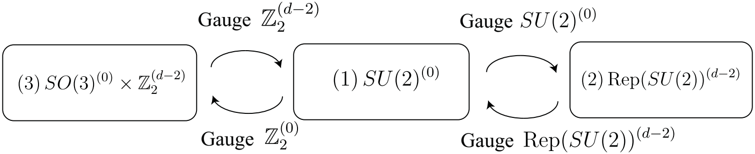
Topological b.c. for $G = SU(2)$

- All Dirac pairings between Dirichlet b.c. operators = 1

(1) Dirichlet b.c. to all W_j, Neumann b.c. to all $\tilde{U}_{\tilde{\alpha}}$: $SU(2)^{(0)}$ symmetry

(2) Neumann b.c. to all W_j, Dirichlet b.c. to all $\tilde{U}_{\tilde{\alpha}}$: $\mathrm{Rep}(SU(2))^{(d-2)}$ symmetry

(3) Dirichlet b.c. to all W_j ($j \in \mathbb{Z}$) and $\tilde{U}_0$, $\tilde{U}_\pi$: $SO(3)^{(0)} \times \mathbb{Z}_2^{(d-2)}$ symmetry



Anomalies

- Adding twist terms in the BF-action

$$S_{sym} = \int_{M_{d+1}} \mathrm{Tr}(B \wedge F) + I(A, B)$$

- Obstruct the gauging/ some topological boundary conditions
- e.g. $d = 4$, adding a Chern-Simons term

$$S_{sym} = \int_{M_5} \mathrm{Tr}(B \wedge F) + CS_5(A)\,,$$

- Action is no longer gauge invariant under

$$A \to gAg^{-1} + igdg^{-1}\ ,\ B \to gBg^{-1}$$

- g Gauge variation $\to$ new boundary term, does not vanish when giving A Neumann b.c.
- Hence the Neumann b.c. for A, i.e. gauging G is obstructed!

Interpretation in AdS/CFT

- Justification of BF-action in holography! ($U(1)$ case: (DeWolfe, Higginbotham, 20'))
- We start from AdS_5 YM action

$$S_{YM} = \int d^5x\sqrt{-g}\left(-\frac{1}{4}tr(F_{\mu\nu}F^{\mu\nu})\right)$$

- AdS metric: $ds^2 = \frac{L^2}{r^2}dr^2 + \frac{r^2}{L^2}\eta_{ij}dx^i dx^j$
- Expand the solution of e.o.m. for A near the AdS boundary $r \to \infty$

$$A_i(x^i, r) = \alpha_i(x^i)L + \gamma_i(x^i)\frac{L^5 \log r}{r^2} + \beta_i(x^i)\frac{L^5}{r^2} + \dots$$

- Define $a_i = \alpha_i L$, and covariant derivative $\mathcal{D}_i = \partial_i - i[a_i, \cdot]$,
- e.o.m. for β_i:

$$\mathcal{D}^i\beta_i = 0$$

Interpretation in AdS/CFT

- Evaluate δS_{YM} by expanding A with $\alpha, \beta \dots$

$$\delta S_{YM} = 2L^3 \int d^4x Tr(\delta\alpha_i \beta^i) + (\dots)\,,$$

- response of source α_i at the boundary

$$\langle J^i(x_i) \rangle = \frac{\delta S}{\delta \alpha_i} = 2L^3 \beta^i$$

- e.o.m. for β_i:

$$\mathcal{D}^i \beta_i = 0$$

- Covariant conservation eq. for non-abelian current J_i after assigning (1) a_i $\sim$source (2) $\beta_i \sim J_i$ =current!

Interpretation in AdS/CFT

- Same results can be derived starting from the BF action as well!

$$S_{BF} = \int_{M_5} \mathrm{Tr}(B \wedge F)$$

$$\delta S = \delta\left(\int \mathrm{Tr} B \wedge F\right) = \int d^5x \mathrm{Tr}\left(\delta \widetilde{B}^{\mu\nu} F_{\mu\nu} + \widetilde{B}^{\mu\nu} D_\mu \delta A_\nu\right)$$
$$= \int d^5x \mathrm{Tr}\left(\delta \widetilde{B}^{\mu\nu} F_{\mu\nu} - D_\mu \widetilde{B}^{\mu\nu} \delta A_\nu + \partial_\mu(\widetilde{B}^{\mu\nu} \delta A_\nu)\right)$$

where $\widetilde{B} = *B$ is the Hodge dual of B.

- Applying e.o.m.

$$\delta S = \int d^5x \, \partial_\mu \mathrm{Tr}(\widetilde{B}^{\mu\nu} \delta A_\nu) = \int d^4x \mathrm{Tr} \widetilde{B}^{ri} \delta A_i \, .$$

- Source/response relation on M_4:

$$\langle J^i(\vec{x}, t)\rangle = \frac{\delta S}{\delta A_i} = \widetilde{B}^{ri} \, , \; D_i J^i = 0 \, .$$

Outlooks

• We constructed the SymTFT for continuous non-abelian flavor symmetry, with a detailed computation of operators

• Match with categorical language $Z(\mathrm{Vec}_G)$ or $Z(\mathrm{Rep}(G))$? $Z(\mathrm{nVec}_G)$? (ongoing work w/ Jia, Luo, Tian, Zhang)

• Details of anomalies

• Derivation of SymTFT action from string theory/holography

• Find applications in CMT etc.

• Thanks!

Proof of correlation functions

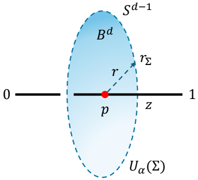
$$
\begin{aligned}
&\langle U_\alpha(\Sigma)\ W_{\mathbf{R}}(\ell)\rangle \\
&= \int [DB][DA] \exp\left(i\int_{M_{d+1}} \mathrm{Tr}(B\wedge F)\right) \exp\left(i\int_\Sigma (\alpha, B)\right) \mathcal{P}\exp\left(i\int_\ell A_{\mathbf{R}}\right) \\
&= \int [DB][DA] \exp\left(i\int_{M_{d+1}} \mathrm{Tr}(B\wedge (F+\alpha\delta_\Sigma))\right) \mathcal{P}\exp\left(i\int_\ell A_{\mathbf{R}}\right)
\end{aligned} \tag{1}
$$

We look for a shift $A \to A - \kappa$ for which:

$$
F \to d(A-\kappa) - i(A-\kappa)\wedge(A-\kappa) = F - (D_A\kappa + i\kappa\wedge\kappa)\,. \tag{2}
$$

and require $D_A\kappa + i\kappa\wedge\kappa = \alpha\delta_\Sigma$

Proof of correlation functions



We place p at the center of B^d and define the following differential form with δ-function support:

$$\mathscr{H}(r,\Sigma) = H(r-r_\Sigma)\delta(z-p)dz, \;\; H(r-r_\Sigma) = \begin{cases} 0, & r > r_\Sigma \\ 1, & r \le r_\Sigma \end{cases} \tag{3}$$

$$D_A\kappa = (D_A\alpha)\mathscr{H}(r,\Sigma) + \alpha\delta_\Sigma \text{ and } \kappa\wedge\kappa \propto dz\wedge dz = 0\,. \tag{4}$$

Proof of correlation functions

Thus under $A \to A - \kappa$ we have:

$$\int_{M_{d+1}} \mathrm{Tr}(B \wedge (F + \alpha\delta_\Sigma)) \to \int_{M_{d+1}} \mathrm{Tr}(B \wedge (F - (D_A\alpha)\mathscr{H}(r,\Sigma)))$$
$$= S_{BF} - \int_{M_{d+1}} \mathrm{Tr}(B \wedge D_A\alpha)\mathscr{H}(r,\Sigma) \quad (5)$$
$$= S_{BF} - \int_{M_d} \mathrm{Tr}(B \wedge D_A\alpha)H(r - r_\Sigma)$$

Since $\int_{M_d} \mathrm{Tr}(B \wedge D_A\alpha)H(r - r_\Sigma) \propto r_\Sigma^d$ (for well-behaved B and α), after $r_\Sigma \to 0$ (1) becomes:

$$\langle U_\alpha(\Sigma)\ W_{\mathbf{R}}(\ell)\rangle = \int \mathcal{D}B\ \mathcal{D}A\ \exp\left(i\int_{M_{d+1}} \mathrm{Tr}(B \wedge F)\right)\ \mathcal{P}\exp\left(i\int_\ell (A_{\mathbf{R}} - \kappa_{\mathbf{R}})\right)$$
$$= \langle \mathcal{P}\exp\left(i\int_\ell (A_{\mathbf{R}} - \kappa_{\mathbf{R}})\right)\rangle \quad (6)$$

Proof of correlation functions

Omitting $\langle\rangle$, using product of 1D δ-functions to simplify

$$
\begin{aligned}
&U_\alpha(\Sigma)W_{\mathbf{R}}(\ell) \\
&= \lim_{\Delta z\to 0}\prod_{z_i}\exp\left(i(A_{\mathbf{R}}(z_i)-\kappa_{\mathbf{R}}(z_i))\Delta z\right) \\
&= \lim_{\Delta z\to 0}\prod_{z_i>p}\exp\left(iA_{\mathbf{R}}(z_i)\Delta z\right)\exp\left(-i\alpha_{\mathbf{R}}(z)\delta(z-p)\Delta z\right)\prod_{z_i<p}\exp\left(iA_{\mathbf{R}}(z_i)\Delta z\right) \\
&= \mathcal{P}\left[\exp\left(i\int_p^1 A_{\mathbf{R}}\right)\exp\left(-i\alpha_{\mathbf{R}}(p)\right)\exp\left(i\int_0^p A_{\mathbf{R}}\right)\right]
\end{aligned}
\tag{7}
$$

for a partition $\{z_i\}$ of $\ell \cong [0,1]$ and $\Delta z = \max_i |z_{i+1}-z_i|$.