Quasi-characters revisited

Sunil Mukhi
IISER Pune/ICTS-TIFR Bengaluru

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Based on:

Quasi-Characters Revisited,
Arpit Das and Sunil Mukhi,
To appear.

and previous work:

Towards a Classification of Two-Character Rational Conformal Field Theories,
A. Ramesh Chandra and Sunil Mukhi,
JHEP 04 (2019) 153, arXiv:1810.09472.

Curiosities above $c = 24$,
A. Ramesh Chandra and Sunil Mukhi,
SciPost Phys. 6 (2019) 5, 053, arXiv:1812.05109.

On the Classification of Rational Conformal Field Theories,
Samir D. Mathur, Sunil Mukhi and Ashoke Sen,
Phys.Lett. B 213 (1988) 303.

Outline

- The classification of 2d Rational Conformal Field Theories (RCFT) is a largely unsolved problem.
- 2d CFTs have an infinite-dimensional chiral algebra that includes the Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$$

  and a similar one for $\bar{L}_n$. Here c is the central charge.
- Often they have extended chiral algebras, including – but not limited to – Kac-Moody algebras:

$$[J_n^a, J_m^b] = if^{abc}J_{n+m}^c + \frac{k}{2}n\delta_{n+m,0}$$

  where k is a central element called the level.

- Two equivalent definitions of RCFT [Anderson-Moore 1988]:
    - Finite number of primaries under the full chiral algebra,
    - c and the conformal dimensions h_i are rational numbers.
- Here, "primary under the full chiral algebra" means:

$$L_n|\phi\rangle = J_n^a|\phi\rangle = W_n^{(3)}|\phi\rangle \cdots = 0, \quad n > 0$$

- In this talk I only consider bosonic CFT (no operators of half-integer spin).

- All unitary RCFT with $0 < c < 1$ are classified [Belavin-Polyakov-Zamoldchikov 1984]. There are finitely many for certain given c values.
- All unitary RCFT with $c = 1$ are conjecturally classified [Dijgraaf-Verlinde-Verlinde 1988, Ginsparg 1988] into an A-D-E series. There are infinitely many, all at the same c.
- There is no rational $c > 1$ for which all unitary RCFTs are known.
- We also don't know what c values are allowed/disallowed. But for each allowed $c = c_0$ there are infinitely many allowed CFTs at $c = c_0 + n$ for any positive integer n.

- Things get better if we specify the rank r of the CFT: the number of primaries under the full chiral algebra. Then:

  Theory of Modular Tensor Categories (MTC)

  $\rightarrow$ Allowed values of c, h_i as well as T, S matrices for given rank

  $\rightarrow$ Classify RCFT of given rank and MTC

- Unitary MTCs are classified up to rank 4. (Some information exists for higher rank).
- Hence this approach is practical, at least for small rank. However surprisingly little has been done....

- The known classifications of bosonic CFT by rank and range of central charge are as follows:

  (i) All rank-1 CFTs classified for $c < 32$ ([Schellekens 1992]): $\rightarrow$ 71 theories.

  (ii) All rank-2 CFTs classified for $c < 25$ ([Mukhi-Rayhaun 2022]): $\rightarrow$ 123 theories.

- There is also a recent parallel result for fermionic CFT (FCFT):

  - All rank-1 chiral fermionic CFTs classified for $c \leq 24$ ([Boyle Smith-Lin-Tachikawa-Zheng, Rayhaun, Höhn-Möller 2023]).

- An MTC is a collection of "isomorphism classes of simple objects $\phi_i, i = 0, 1, \cdots, r-1$", with $\phi_0 = \mathbf{1}$ being the identity, and several properties.
- We list only a few of them:
    - A category has morphisms $\sim$ which we can think of as identifications of objects with themselves.
    - Tensor product (fusion) of two objects is commutative and associative, and leads to a (finite) formal sum of other objects:

      $$\phi_i \otimes \phi_j = \sum_k N_{ij}^k \phi_k$$

      which satisfies $\phi \otimes \mathbf{1} \sim \mathbf{1} \otimes \phi \sim \phi$.

- There are concepts of braiding and fusion:

$$B: \quad \phi_i \otimes \phi_j \to \phi_j \otimes \phi_i$$
$$F: \quad (\phi_i \otimes \phi_j) \otimes \phi_k \to \phi_i \otimes (\phi_j \otimes \phi_k)$$

- There is an action of SL(2,Z) on the simple objects:

$$T: \phi_i \to e^{2\pi i\left(-\frac{c}{24}+h_i\right)}\phi_i$$
$$S: \phi_i \to S_{ij}\phi_j$$

with S unitary and symmetric, and $S^2 = (ST)^3 = \mathcal{C}$ (conjugation). Here,

$$c = \text{topological central charge, defined mod 8}$$
$$h_i = \text{topological dimensions, defined mod 1}$$

- What are the MTCs of rank $r = 1$? The only simple object is the trivial object $\mathbf{1}$.
- Then $B = F = S = 1$ and $T^3 = e^{-\frac{i\pi c}{4}} = 1$, so $c = 0$ modulo 8.
- Thus a CFT corresponding to such an MTC has $c = 8n$ for some positive integer n, and $T = e^{-\frac{2\pi i n}{3}}$.
- Similarly all MTCs have c defined only modulo 8.

- In particular, this means that at any rank r, for every allowed $c = c_0$ there will be a CFT of the same rank and $c = c_0 + 8n$ for any positive integer n.

- For rank $r = 2$, the simple objects are $\{\mathbf{1}, \phi\}$ and there there are just 2 MTCs up to certain conjugations:

  (i) Semion MTC.

  Fusion rules: $\phi \otimes \phi = \mathbf{1}$

  Topological central charge: $c = 1, 7$

  Topological $(c, h) : (1, \frac{1}{4}), (7, \frac{3}{4})$

  (ii) Fibonacci MTC.

  Fusion rules: $\phi \otimes \phi = \mathbf{1} \oplus \phi$

  Topological $(c, h) : (\frac{14}{5}, \frac{2}{5}), (\frac{26}{5}, \frac{3}{5})$

- In this talk I will focus on the rank-2 case.

- The partition function of a rational CFT is:

$$Z(q,\bar{q}) = \sum_{i=0}^{r-1} M_{ij}\chi_i(q)\bar{\chi}_j(\bar{q}), \qquad q \equiv e^{2\pi i\tau}$$

  where $\chi_i(q)$ are a set of r generalised characters:

$$\chi_i(q) = \mathrm{tr}_i\, q^{L_0-\frac{c}{24}}$$

  and M_{ij} is a constant integral matrix. Here, tr_i is the trace over holomorphic descendants of the ith primary under the full chiral algebra.
- Characters are holomorphic in the interior of moduli space but can diverge on the boundary $q \to 0$.

- From the (assumed) equivalence of operator and path-integral quantisations of QFT, it is argued that the partition function of a CFT should be modular invariant:

$$Z(q_\gamma, \bar{q}_\gamma) = Z(q, \bar{q})$$

where:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL(2,Z)},\ \ q_\gamma \equiv e^{2\pi i \frac{a\tau+b}{c\tau+d}}$$

- In RCFT this is true if and only if the characters are vector-valued modular functions (VVMF) of weight 0. They transform in an r-dimensional representation ϱ of SL(2,Z):

$$\chi_i\left(q_\gamma\right) = \sum_{j=0}^{p-1} \varrho_{ij}(\gamma)\chi_j(q), \quad \varrho^\dagger M \varrho = M$$

- VVMFs have a q-expansion:

$$\chi_i(q) = q^{\alpha_i}(a_{i,0} + a_{i,1}q + a_{i,2}q^2 + \cdots),\ \ i = 0, 1, \cdots, p-1$$

- Generically the coefficients $a_{i,n}$ can be non-integral and even negative. But to describe a CFT they must be positive integers.
- We define admissible characters as VVMF's for which:

$$\begin{aligned} a_{i,m} &= \text{ integer } \geq 0 \quad \forall\, i, m \\ a_{0,0} &= 1 \end{aligned}$$

- This restriction is highly non-trivial and serves as a bootstrap that constrains possible RCFT characters.

- It is very important to keep in mind that admissible characters may not be CFTs. In fact "most" are not.
- We should think in terms of two sequential tasks:

  (1) Classifying admissible characters.
  (2) Finding CFTs among them.

- In this talk I will mainly discuss task (1).

- As we will now see, all admissible VVMF can be constructed in principle using Modular Linear Differential Equations [Mathur-Mukhi-Sen 1988].
- These equations are classified by the rank r and one more non-negative integer ℓ called the Wronskian index.
- Importantly we do not assume the value of c, it is an output.
- Now I mainly restrict to rank-2 for the rest of the talk.

Outline

- The MLDE approach to classify admissible characters works as follows.
- We consider a 2nd order equation of the form:

$$\Big(D_\tau^2 + \phi_2(\tau)D_\tau + \phi_4(\tau)\Big)\chi(\tau) = 0$$

where

$$D_\tau \equiv \frac{1}{2\pi i}\frac{\partial}{\partial\tau} - \frac{k}{12}E_2(\tau)$$

is the Ramanujan-Serre derivative (= covariant derivative in τ) and k is the weight of the modular form on which it acts.

- Here ϕ_2, ϕ_4 are meromorphic modular functions of weight $2, 4$ respectively. Then the equation is SL(2,Z) invariant.

- For every modular pair (ϕ_2, ϕ_4) there is a solution of this equation transforming as a doublet under $SL(2, Z)$ so it is a VVMF.
- The solution is generically not admissible, but may be so for special choices of (ϕ_2, ϕ_4).
- So every MLDE gives a VVMF, but the converse is also true: we now show that every rank-2 VVMF arises from an MLDE.

- Suppose we are given a VVMF χ_0, χ_1. A general linear combination χ of these two functions satisfies:

$$\begin{vmatrix} \chi_0 & \chi_1 & \chi \\ D_\tau\chi_0 & D_\tau\chi_1 & D_\tau\chi \\ D_\tau^2\chi_0 & D_\tau^2\chi_1 & D_\tau^2\chi \end{vmatrix} = 0$$

- Expanding by the last column, we get:

$$\begin{vmatrix} \chi_0 & \chi_1 \\ D_\tau\chi_0 & D_\tau\chi_1 \end{vmatrix} D_\tau^2\chi - \begin{vmatrix} \chi_0 & \chi_1 \\ D_\tau^2\chi_0 & D_\tau^2\chi_1 \end{vmatrix} D_\tau\chi + \begin{vmatrix} D_\tau\chi_0 & D_\tau\chi_1 \\ D_\tau^2\chi_0 & D_\tau^2\chi_1 \end{vmatrix} \chi = 0$$

- This is an MLDE, with:

$$\phi_2 = -\frac{\begin{vmatrix} \chi_0 & \chi_1 \\ D_\tau^2\chi_0 & D_\tau^2\chi_1 \end{vmatrix}}{\begin{vmatrix} \chi_0 & \chi_1 \\ D_\tau\chi_0 & D_\tau\chi_1 \end{vmatrix}}, \ \phi_4 = \frac{\begin{vmatrix} D_\tau\chi_0 & D_\tau\chi_1 \\ D_\tau^2\chi_0 & D_\tau^2\chi_1 \end{vmatrix}}{\begin{vmatrix} \chi_0 & \chi_1 \\ D_\tau\chi_0 & D_\tau\chi_1 \end{vmatrix}}$$

- This shows that ϕ_2, ϕ_4 can have poles wherever the denominator (called the Wronskian) has zeroes.

- The number of such poles in the interior of moduli space is denoted $\frac{\ell}{6}$, where ℓ is an integer ≥ 0 called the Wronskian index.
- The factor of $\frac{1}{6}$ comes from the singular points of moduli space, $\tau = e^{\frac{2\pi i}{3}}, i$, where we can have fractional poles of order $\frac{1}{2}, \frac{1}{3}$.
- With two characters it can be shown that ℓ is even: $\ell = 0, 2, 4, \cdots$ [Naculich 1989].

- For any given ℓ there is a finite basis of functions of E_4, E_6 from which ϕ_2, ϕ_4 are built.
- Thus the MLDE has a finite number of parameters that grows with ℓ.
- For future reference, the Riemann-Roch theorem gives us the valence formula:

$$\begin{aligned} \ell &= \frac{rc}{4} - 6\sum_{i=1}^{r-1} h_i + \frac{r(r-1)}{2} && \text{for rank } r \\ &= \frac{c}{2} - 6h + 1 && \text{for rank } 2 \end{aligned}$$

- Digression: The Wronskian index is still mysterious!
- It is easy to calculate for any known RCFT of any rank. We find:
- All $c < 1$ minimal models, as well as the $\mathsf{A}_{1,k}$ and $\mathsf{A}_{n,1}, B_{n,1}, D_{n,1}$ WZW models have $\ell = 0$ [Mathur-Mukhi-Sen 1989].
- The same is true for the following WZW models based on simple Kac-Moody algebras [Das-Gowdigere-Santara 2021]:
  - $\mathsf{C}_{n,1}$,
  - $\mathsf{E}_{6,1}$, $\mathsf{E}_{6,2}$, $\mathsf{E}_{7,1}$, $\mathsf{E}_{7,2}$, $\mathsf{E}_{8,2}$,
  - $\mathsf{G}_{2,1}$, $\mathsf{G}_{2,2}$, $\mathsf{F}_{4,1}$, $\mathsf{F}_{4,2}$
- But $\ell > 0$ for all other WZW models.
- Rank-1 CFTs of central charge $8n$ have Wronskian index $2n$. The simplest one, $\mathsf{E}_{8,1}$, has $c = 8$ and $\ell = 2$.
- What is the pattern? Does the Wronskian index have a "physical" meaning?

- In [Mathur-Mukhi-Sen 1988] the Wronskian index was chosen to be 0 for simplicity.
- From the theory of holomorphic modular forms it follows that $\phi_2 = 0$ and $\phi_4(\tau) = \mu E_4(\tau)$, where E_4 is an Eisenstein series and μ is an arbitrary real parameter.
- This leads to the "MMS equation":

$$\Big(D_\tau^2 + \mu E_4(\tau)\Big)\chi = 0$$

- The entire solution is determined by the single real number μ in the MLDE (except for overall normalisations).
- This simplification is due to modular invariance and the choice $\ell = 0$.

- Insert a power series solution:

$$\chi_i(q) = q^{\alpha_i}(a_{i,0} + a_{i,1}q + a_{i,2}q^2 + \cdots), \; i = 0, 1$$

- We parametrise the leading exponents as:

$$\alpha_0 = -\frac{c}{24}, \quad \alpha_1 = -\frac{c}{24} + h$$

  This is just notation, but if the solutions describe a CFT then (c, h) will be the central charge and the conformal dimension of the primary.

- The equation is now solved recursively by the Frobenius method to any desired order. We scan over values of μ and from time to time find a “miracle” where an admissible solution comes out.

- Let us see an example.

- MMS equation with $\mu = -\frac{119}{3600}$ gives admissible characters:

$$\begin{aligned}\chi_0(q) &= q^{-7/60}(1 + 14q + 42q^2 + 140q^3 + 350q^4 + 850q^5 \\ &\qquad + 1827q^6 + 3858q^7 + 7637q^8 + 14756q^9 + \cdots) \\ \chi_1(q) &= q^{17/60}(1 + \tfrac{34}{7}q + 17q^2 + 46q^3 + 117q^4 + 266q^5 \\ &\qquad + 575q^6 + 1174q^7 + 2311q^8 + 4380q^9 + \cdots)\end{aligned}$$

$c = \frac{14}{5}, h = \frac{2}{5}$. Normalising second character by 7 $\rightarrow$ it becomes admissible. In fact this is the CFT $\mathsf{G}_{2,1}$.

- MMS equation with $\mu = -\frac{143}{4800}$ gives non-admissible characters:

$$\begin{aligned}\chi_0(q) &= q^{-13/120}(1 + \tfrac{455}{37}q + \tfrac{121784}{3589}q^2 + \tfrac{60836763}{563473}q^3 + \tfrac{4525367613}{17467663}q^4 \\ &\qquad + \tfrac{2893074116179}{4838542651}q^5 + \tfrac{2046920234847579}{1630588873387}q^6 + \cdots) \\ \chi_1(q) &= q^{11/40}(1 + \tfrac{363}{83}q + \tfrac{15849}{1079}q^2 + \tfrac{90512}{2407}q^3 + \tfrac{58528917}{633041}q^4 \\ &\qquad + \tfrac{128150964}{633041}q^5 + \tfrac{102972265445}{242454703}q^6 + \cdots)\end{aligned}$$

Formally $c = \frac{13}{5}, h = \frac{23}{60}$, but cannot be a CFT.

- We found a finite set of admissible characters, all with $0 < c < 8$, and guessed their identification with various known RCFT:

m_1	c	h	Identification
1	$\frac{2}{5}$	$\frac{1}{5}$	$c = -\frac{22}{5}$ minimal model $(c \leftrightarrow c - 24h)$
3	1 *	$\frac{1}{4}$	$k=1$ SU(2) WZW model
8	2	$\frac{1}{3}$	$k=1$ SU(3) WZW model
14	$\frac{14}{5}$ *	$\frac{2}{5}$	$k=1$ G_2 WZW model
28	4	$\frac{1}{2}$	$k=1$ SO(8) WZW model
52	$\frac{26}{5}$ *	$\frac{3}{5}$	$k=1$ F_4 WZW model
78	6	$\frac{2}{3}$	$k=1$ E_6 WZW model
133	7 *	$\frac{3}{4}$	$k=1$ E_7 WZW model
190	$\frac{38}{5}$	$\frac{4}{5}$	?
248	8	$\frac{5}{6}$	$\supset k=1$ E_8 WZW model

- The ones with a star are true rank-2 theories, while the rest have ranks 1, 3 or 4 but are "masquerading" as rank-2.

- Here we will be interested only in rank-2 CFT so we get exactly four CFTs:

$$\mathsf{A}_{1,1}, \mathsf{E}_{6,1}, \mathsf{G}_{2,1}, \mathsf{F}_{4,1}$$

with:

$$(c,h) = (1, \tfrac{1}{4}), (7, \tfrac{3}{4}), (\tfrac{14}{5}, \tfrac{2}{5}), (\tfrac{26}{5}, \tfrac{3}{5})$$

- This perfectly matches the four allowed MTC at rank 2! And here (c,h) are actual numbers, not just modulo 8 or 1.
- But remember we arbitrarily chose $\ell = 0$ (vanishing Wronskian index).
- Are there rank-2 CFTs with $\ell > 0$?
- They are plentiful, and the rest of this talk is about them.

Outline

$\ell > 0$

- Each of the four admissible $\ell = 0$ VVMFs at rank-2 can be uniquely identified with a known CFT.
- For $\ell = 2$, the MLDE is:

$$\left(D_\tau^2 + \frac{1}{3}\frac{E_6}{E_4}D_\tau + \mu E_4\right)\chi = 0$$

- This equation again has four admissible rank-2 solutions.
- Again all can be identified with CFT, but there are multiple (isospectral) CFTs for the same characters [Gaberdiel-Hampapura-Mukhi 2016].

- For $\ell = 4$ one can show that:

$$\ell(\chi_{\mathsf{E}_{8,1}}\psi) = 4 + \ell(\psi)$$

  so any admissible solution ψ of the MMS equation ($\ell = 0$) is an admissible solution of the $\ell = 4$ MLDE.

- There is just one more admissible rank-2 solution with $c = 33$.
- This takes the same form $\chi_{\mathsf{E}_{8,1}}\psi$ but ψ is a non-admissible solution of the MMS equation [Chandra-Mukhi 2018].
- Later it was famously shown to be a CFT [Grady, Lam, Tener, Yamauchi 2020].

- $\ell = 0, 2, 4$ are the easy cases for finding admissible characters. This is because the MLDE has only one free parameter and the admissible solutions are easily found.
- From $\ell = 6$ onwards the equation develops more parameters due to the movable location of zeroes of the Wronskian. Correspondingly the solutions also have free parameters.
- In fact for $\ell = 6n$ there are $2n + 1$ free parameters in the MLDE and the bootstrap becomes very difficult.
- The rank-2 MLDE for general ℓ was formulated in [Das-Gowdigere-Mukhi-Santara 2023] and with some difficulty, the admissible solutions for $\ell = 6$ were found. But this is not a promising approach for higher ℓ.

- The method of quasi-characters [Chandra-Mukhi 2018] bypasses the need to study these more difficult MLDEs.
- It provides a way to classify admissible characters for $\ell = 6n, \ell = 6n+2, \ell = 6n+4$ using only solutions to the "easy" MLDEs with $\ell = 0, 2, 4$.
- Here I will describe the case of $\ell = 6n$.

- This method uses more general non-admissible solutions of the original MMS equation. They have integral, but not always positive coefficients.
- We called them quasi-characters. They were found by adapting results of [Kaneko 2003, Kaneko-Koike 2006] which in turn are based on [Kaneko-Zagier 1998].
- Our key result is that all admissible characters with $\ell \geq 6$ are suitable linear combinations of quasi-characters.
- All rank-2 quasi-characters are known, so the only remaining problem is which linear combinations give admissible characters.

- There are four infinite series of rank-2 quasi-characters with $\ell = 0$. They have "central charge":

$$c = 24M + j, \quad j \in \{1, 7, \tfrac{14}{5}, \tfrac{26}{5}\}, \quad 0 \neq M \in \mathbb{Z}$$

  and from the valence formula, $h = 2M + \frac{j+2}{12}$.

- The above data determines the MMS parameter to be:

$$\mu = -(M + \tfrac{j}{24})(M + \tfrac{j+4}{24})$$

  Hence the full solution is fixed.

- There are infinitely many such quasi-characters, one for each $M \neq 0$. Though we label them by (c, h), these labels do not directly refer to any CFT.

- The excluded case $M = 0$ is an admissible character: the $\mathsf{A}_{1,1}$ WZW CFT. Its identity character is:

$$\chi_0 = q^{-\frac{1}{24}}(1 + 3q + 4q^2 + 7q^3 + 13q^4 + \cdots)$$

- For all $M \neq 0$ one gets quasi-characters. For example, for $M = 1$ the identity character is:

$$\chi_0 = q^{-\frac{25}{24}}(1 - 245q + 142640q^2 + 18615395q^3 + 837384535q^4 + \cdots)$$

  and all higher coefficients are positive integers.
- This cannot directly describe a CFT: a degeneracy of -245 is not physically sensible.

- Now the modular representation ϱ_n of the quasi-characters is known, and it is periodic under $M \to M+1$ ($c \to c+24$).
- So we can add the $M=0$ and $M=1$ quasi-characters and still get a VVMF with the same modular representation:

$$\chi_i = \mathcal{N}\chi_i^{M=0} + \chi_i^{M=1}, \quad i = 0, 1$$

- The sum has the identity character:

$$\chi_0 = q^{-\frac{25}{24}}(1 + (\mathcal{N} - 245)q + (3\mathcal{N} + 142640)q^2 + \cdots)$$

which is admissible for all integers $\mathcal{N} \geq 245$.

- One can read off that the sum has $(c, h) = (25, \frac{5}{4})$. From the Riemann-Roch theorem quoted earlier, we have:

$$\ell = \frac{c}{2} - 6h + 1 = 6$$

- If we add $n+1$ consecutive quasi-characters, we generate admissible characters for all $\ell = 6n$.
- One can similarly find quasi-characters with Wronskian index $= 2, 4$.
- Their linear combinations give, respectively, the admissible characters for $\ell = 6n+2, 6n+4$.
- We proved that in this way, all admissible two-characters sets are generated.

- The remaining series of quasi-characters at $\ell = 0$ are:

$$\begin{aligned} \mathsf{E}_7: &\quad c = 24M + 7,\ h = 2M + \tfrac{3}{4} \\ \mathsf{G}_2: &\quad c = 24M + \tfrac{14}{5},\ h = 2M + \tfrac{2}{5} \\ \mathsf{F}_4: &\quad c = 24M + \tfrac{26}{5},\ h = 2M + \tfrac{3}{5} \end{aligned}$$

  with $0 \neq M \in \mathbb{Z}$.
- Notice that this covers all rank-2 MTCs.

- All four series are experimentally found to have the following properties:
  - First consider $M > 0$. Then:

$$\chi_0 = q^{\alpha_0}\big(a_0 - |a_1|q + |a_2|q^2 - \cdots + |a_{2M}|q^{2M} + |a_{2M+1}|q^{2M+1} + \cdots\big)$$

$$\chi_1 = q^{\alpha_1}\,\big(a_0 + |a_1|a + |a_2|q^2 + \cdots\big)$$

    (we have suppressed the label i on the a_n).
  - Next consider $M < 0$. Then:

$$\chi_0 = q^{\alpha_0}\,\big(a_0 + |a_1|q + |a_2|q^2 + \cdots\big)$$

$$\chi_1 = q^{\alpha_1}\big(a_0 - |a_1|q + |a_2|q^2 - \cdots - |a_{2|M|-1}|q^{2|M|-1} - a_{2|M|}q^{2|M|} - \cdots\big)$$

- Thus one of the characters has coefficients that alternate in sign, but the alternation stops at $n = 2|M|$ and the sign is stable after that.
- Also the asymptotic behaviour is:

$$
\begin{aligned}
M > 0: &\quad \chi_0 \to +, \quad \chi_1 \to + \\
M < 0: &\quad \chi_0 \to +, \quad \chi_1 \to -
\end{aligned}
$$

- These are "experimental" facts, verified in many examples.
- I will now discuss our attempts to prove them [Das-Mukhi, to appear]. This is an interesting exercise in itself.
- It is also relevant to classification. This alternation gives us constraints on what linear combinations are admissible.

- Recall that the MMS equation is:

$$(D_\tau^2 + \mu E_4)\chi = 0$$

- The Frobenius solution for the identity character is given by the recursion relation:

$$a_n = \sum_{i=1}^{n} f_M(n,i)\, a_{n-i}$$

where:

$$f_M(n,i) \equiv \frac{-4\big(n-i-M-\frac{j}{24}\big)\sigma_1(i) + \frac{5(24M+j)(24M+j+4)}{12}\sigma_3(i)}{n(n-2M-\frac{j+2}{12})}$$

- We now want to solve this approximately.

- Approximate solutions are useless for determining admissibility, but useful for estimating the signs and growth of the coefficients.
- As a warm-up exercise let us consider two asymptotic limits:
    - $n \gg M$: Here we don't expect to see alternating signs, but only asymptotic behaviour.
    - $|M| \gg n$: Here we expect to see alternating behaviour but it shouldn't stop, since we can't reach $n = 2|M|$.

- For $n \gg M$ we use the Rademacher formula:

$$a_{j,n} \sim \sum_{i=0}^{1} \sum_{\alpha_i + m < 0} \left(S^{-1}_{ji} e^{4\pi\sqrt{|\alpha_i+m|(\alpha_0+n)}} + \cdots \right) a_{i,m}$$

- This requires knowledge of the modular S-matrix, which is known from MTC (or even earlier). We have:

$$\mathsf{A}_1, \mathsf{E}_7 : \; S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathsf{G}_2, \mathsf{F}_4 : \; S = \begin{pmatrix} \alpha_+ & \alpha_- \\ \alpha_- & -\alpha_+ \end{pmatrix}$$

where $\alpha_\pm = \sqrt{\frac{2}{5\pm\sqrt{5}}}$.

- This gives:

$$
\begin{aligned}
M > 0: \qquad & a_{0,n} \sim \exp\left(4\pi\sqrt{\left(M+\tfrac{j}{24}\right)\left(n-M-\tfrac{j}{24}\right)}\right) a_{0,0} \\
& a_{1,n} \sim \exp\left(4\pi\sqrt{\left(M+\tfrac{j}{24}\right)\left(n+M+\tfrac{j+4}{24}\right)}\right) a_{0,0} \\
M < 0: \qquad & a_{0,n} \sim \exp\left(4\pi\sqrt{\left(M-\tfrac{j+4}{24}\right)\left(n+M-\tfrac{c}{24}\right)}\right) a_{1,0} \\
& a_{1,n} \sim -\exp\left(4\pi\sqrt{\left(M-\tfrac{j+4}{24}\right)\left(n-M+\tfrac{j+4}{24}\right)}\right) a_{1,0}
\end{aligned}
$$

- This correctly explains the observed asymptotic behaviour and also estimates the growth with n at fixed central charge.

- For $|M| \gg n$, we take the limit of the Frobenius recursion coefficients to get:

$$a_{0,n} \sim -\tfrac{120M}{n} \sum_{i=1}^{n} \sigma_3(i) a_{0,n-i}$$

- This implies that each $a_{0,r}$ scales to leading order as M^r, so in the above sum one can neglect all but the $i = 1$ term. Since $\sigma_3(1) = 1$, this leads to:

$$a_{0,n} \sim \left(-\tfrac{120M}{n}\right)^n + \mathcal{O}\left(M^{n-1}\right)$$

- We see that for the identity character, the coefficients alternate for $M > 0$ and are positive for $M < 0$, exactly as observed in explicit calculations. The non-identity character similarly works out.

- To see the "stabilisation" at $n = 2|M|$, we need to consider the region $n \sim |M| \gg 1$.
- Recall the recursion relation:

$$a_n = \sum_{i=1}^{n} f_M(n,i)\, a_{n-i}$$

- In the above limit, the coefficients become:

$$f_M(n,i) \overset{|M|\to\infty}{\simeq} \tfrac{240M^2}{n(n-2M)}\sigma_3(i) \qquad \text{(identity)}$$

$$f_M(n,i) \overset{|M|\to\infty}{\simeq} \tfrac{240M^2}{n(n+2M)}\sigma_3(i) \qquad \text{(non-identity)}$$

- Let's focus on the identity character, for which:

$$f_M(n,i) \overset{|M|\to\infty}{\simeq} \tfrac{240M^2}{n(n-2M)}\sigma_3(i)$$

- Notice that:
  - For $M > 0$, this is $-$ve for $n < 2M$, $+$ve for $n > 2M$.
  - For $M < 0$ this is always $+$ve.
- Also for $M > 0$ it appears to diverge exactly at $n = 2M$. But actually we dropped a small finite piece, and its correct value is:

$$f_M(2M,i) \sim -\tfrac{1440M}{j+2}\sigma_3(i)$$

- Now let's take $M > 0$ and make a crude approximation: in the recursion relation we take the leading term in each step, to get:

$$a_n \sim \prod_{p=1}^{n} f_M(p,1) \sim \prod_{p=1}^{n} \tfrac{240M^2}{p(p-2M)}, \quad n < 2M$$

$$a_{2M} \sim \prod_{p=1}^{2M} f_M(p,1) \sim \left(-\tfrac{1440M}{j+2}\right) \prod_{p=1}^{2M-1} \tfrac{240M^2}{p(p-2M)}$$

$$a_n \sim \prod_{p=1}^{n} f_M(p,1) \sim \prod_{p=2M+1}^{n} \tfrac{240M^2}{p(p-2M)} \left(-\tfrac{1440M}{j+2}\right) \prod_{p=1}^{2M-1} \tfrac{240M^2}{p(p-2M)},$$
$$n > 2M$$

- Thus in this approximation, a_n alternates in sign until $n = 2M$ and then settles down to be positive, exactly as expected.
- The remaining cases (non-identity character and/or $M < 0$) work out similarly.

- We can also estimate the growth of the coefficient a_{2M} at which the transition from alternating to positive behaviour takes place.
- Taking $M \to \infty$, we get:

$$a_{2M} \sim \frac{e^{2M \log 240M^2}}{e^{4M \ln(2M/e)}}$$

$$\sim e^{12.19M}$$

which compares very well with extrapolation from explicit calculations, which gives $e^{12.13M}$.

- It seems we have proved what we set out to prove! But that's not really the case.
- The approximation made for $n \sim |M| \gg 1$ is not as good as for $M \gg n$.
- In the limit of large $|M|, n$ we find that $f_M(n, i) \sim 240$ which is "large" but not parametrically large.

- Indeed, by explicit computation we found that the first sub-leading term can dominate over the "leading" expression for large enough $|M|$.
- Without going into details here, we argue that a set of sub-leading terms can be summed and approximated by an exponential.
- Incorporating this contribution, the estimated growth becomes $\sim e^{12.14M}$, even closer to the explicit calculation!
- Even then everything is not perfect. The same procedures don't work as well for the $\ell = 2$ quasi-characters which have a different alternating behaviour.

- In an example I showed how two quasi-characters can be combined into an admissible character with $\ell = 6$.
- The general theory of what combinations are admissible has not yet been worked out.
- Let's see another example that gives a new insight: we add three different quasi-characters in the A_1 series.
- Recalling that this series has $c = 24M + 1$, we choose $M = 1, 0, -1$ corresponding to quasi-characters with $c = 25, 1, -23$.
- Then we sum them to get:

$$\chi_i = \chi_i^{M=1} + \mathcal{N}_1\, \chi_i^{M=0} + \mathcal{N}_2\, \chi_i^{M=-1}$$

- Calculating everything we find:

$$\chi_0 = q^{-\frac{25}{24}}\Big(1 + (-245+\mathcal{N}_1)q + (142640 + 3\,\mathcal{N}_1 + 26752\,\mathcal{N}_2)q^2$$
$$+ (18615395 + 4\,\mathcal{N}_1 + 1734016\,\mathcal{N}_2)q^3$$
$$+ (837384535 + 7\,\mathcal{N}_1 + 46091264\,\mathcal{N}_2)q^4 + \dots\Big)$$
$$\chi_1 = q^{-\frac{19}{24}}\Big(\mathcal{N}_2 + (2\,\mathcal{N}_1 - 247\,\mathcal{N}_2)q$$
$$+ (565760 + 2\,\mathcal{N}_1 - 86241\,\mathcal{N}_2)q^2$$
$$+ (51745280 + 6\,\mathcal{N}_1 - 4182736\,\mathcal{N}_2)q^3$$
$$+ (1965207040 + 8\,\mathcal{N}_1 - 96220123\,\mathcal{N}_2)q^4 + \dots\Big)$$

- We immediately read off $c = 25$ and $h = \frac{1}{4}$, and the valence formula then gives $\ell = 12$.
- But when is this admissible? The requirement to $\mathcal{O}(q^2)$ is:

$$\mathcal{N}_1 \geq 245, \quad -\frac{8915}{1672} - \frac{3\mathcal{N}_1}{26752} \leq \mathcal{N}_2 \leq \frac{2\mathcal{N}_1}{247}$$

- So we have a two-dimensional parameter space (of integers) with a collection of linear bounds.
- In principle there will be infinitely many bounds, one from each coefficient. That all these bounds have any common region at all is not obvious!
- In general, we need to know the signs and asymptotic growth estimates of the different terms to be sure there is any non-empty admissible region. This was the motivation for the present work.
- We now show that there is at least one genuine CFT in this space (actually there are at least 25). This implies that in this case, all the bounds do have a common intersection.

- We start with a Kervaire lattice Γ_{32} in 32 dimensions, with root system $\mathsf{A}_{24}\mathsf{E}_8$.
- This is an even self-dual lattice, so it automatically gives a rank-1 CFT. The central charge is 32.
- Now take the coset $\Gamma_{32}/\mathsf{E}_{7,1}$. Since the denominator is a rank-2 CFT, the quotient is also a rank-2 CFT and has $c = 25$. It also has $\ell = 12$ so it must lie in our two-parameter family.
- This quotient theory has a Kac-Moody algebra $\mathsf{A}_{24,1}\mathsf{A}_{1,1}$ of dimension 627. Equating this to $\mathcal{N}_1 - 245$ we find $\mathcal{N}_1 = 872$.
- The primary has dimension $\frac{1}{4}$ so it has to be the primary of $\mathsf{A}_{1,1}$ (times the identity of $\mathsf{A}_{24,1}$) which has a degeneracy of 3. Hence $\mathcal{N}_2 = 3$. These can be checked to satisfy our bounds.

Outline

- It is quite remarkable that we have a basis for all rank-2 admissible vector-valued modular functions, since the characters of any rank-2 CFT must be a subset of these.
- The basis is surprisingly simple: any Wronskian index ℓ is written as $\ell = 6n, 6n+2$ or $6n+4$ and then we only need the simple MLDEs for $\ell = 0, 2, 4$.
- I did not discuss the basis for $\ell = 2$ which has quite different features from the $\ell = 2$ case. The quasi-character coefficients a_n alternate in sign until $n = M$, then they flip sign once and continue alternating till $n = 2M$, and then stabilise!

- We do not have a complete basis of quasi-characters for any rank > 2 although there are partial results for rank-3 in [Mukhi-Poddar-Singh 2020].
- There is an alternative, and more popular, basis for all VVMFs: the Bantay-Gannon basis. However (i) the elements of this basis do not come from MLDEs, and (ii) the question of admissibility is not settled as far as I know.
- One amusing point that I only found out yesterday: after a long chat with ChatGPT, I learned that no modular form of SL(2,Z) has coefficients that alternate in sign and then stabilise to a single sign.
- So this phenomenon is unique to VVMFs. It might have applications to some other counting problems in physics.