Crossing symmetric dispersion relations for string amplitudes

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2401.05733 (PRL), **APS, Aninda Sinha**

2409.18259 (PRD), **Faizan Bhat, Debapriyo Chowdhury, APS, Aninda Sinha**

2506.03862, **Faizan Bhat, APS, Aninda Sinha**

Introduction

Scattering amplitudes in QFT satisfy some properties:

- Locality: Interactions are localized in space and time.
- Analyticity: 4-point amplitudes are analytic function of s, t and u.
- High energy behaviour: Regge bounds, Froissart bounds, etc.
- Crossing symmetry: Holds when external states are identical.

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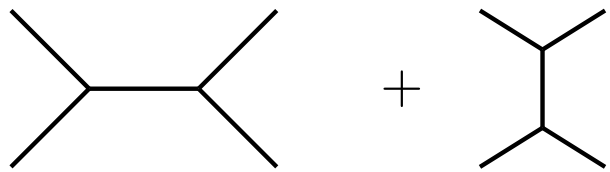
$\mathcal{M}(s,t) =$  $+$

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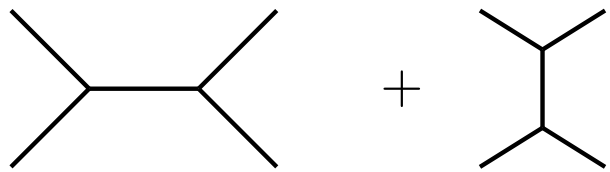
$\mathcal{M}(s,t) =$  $+$

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Can there be scattering amplitudes which are given by only single channel expansion?

Sum over s-channel diagrams $\stackrel{?}{=}$ sum over t-channel diagrams

Dual resonance

Amplitudes in string theory exhibit the unique features of dual resonance model.

$$\int_0^1 \mathrm{d}z\, z^{-s-1}(1-z)^{-t-1} = \frac{\Gamma(-s)\,\Gamma(-t)}{\Gamma(-s-t)}$$

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$$\frac{\Gamma(-s)\Gamma(-t)}{\Gamma(-s-t)} = \begin{cases} -\sum_{n=0}^{\infty} \frac{1}{n!}\frac{(1+t)_n}{s-n}, & t<0 \\ -\sum_{n=0}^{\infty} \frac{1}{n!}\frac{(1+s)_n}{t-n}, & s<0 \end{cases}$$

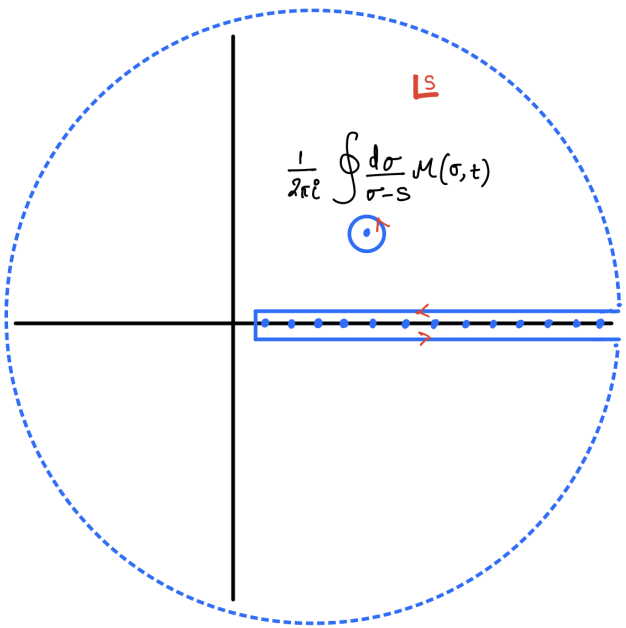
$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$$

Presence of infinite tower of massive higher spin states is crucial for the single channel expansion.

Dispersion relation

Kramers-Kronig relations in complex analysis relates real and imaginary parts of a function which is analytic in the upper-half plane.

A function can be reconstructed from its discontinuity.



$$\mathcal{M}(s,t) = \frac{1}{\pi}\int_{\mu_0}^{\infty} \frac{\mathrm{d}\sigma}{\sigma - s}\mathrm{Disc}\left[\mathcal{M}(\sigma,t)\right] \quad t < 0$$

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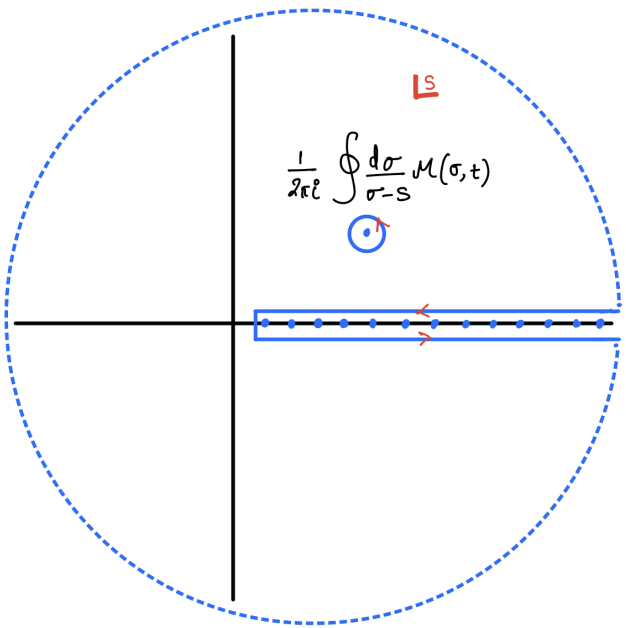

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Contrary to QFT amplitudes, string amplitudes decay with increasing energy

$$\frac{\Gamma(-s)\Gamma(-t)}{\Gamma(-s-t)} \sim s^t, \quad s \to \infty, \quad t \text{ fixed}$$

Phenomenological consequences

Dual resonance models became popular in the sixties. They were used to explain phenomenology of hadron scatterings involving pions.

Dolen, **Horn**, **Schmid** (1967) used finite energy sum rule to explain high energy pion scattering. They observed single channel expansion (interference model) requires modification of the form $M = M_{Regge} + M_{Res} - \langle M_{Res} \rangle$.

"Is duality an approximation or principle? ... The duality hypothesis never had more than slender experimental support, and the Veneziano model was merely an *ad hoc* way of satisfying this not-so-well motivated hypothesis."

Green, Schwarz, Witten; Superstring Theory, Vol 1

Objectives

Can we obtain a field-theoretic representations of tree-level string amplitudes? These representations satisfy the following properties:

- Crossing symmetry is manifest. Sum over poles in all the channels should be included.
- Except the physical singularities, amplitude should be analytic and convergent for all values of kinematic variables.
- Regular terms in the representations, referred to as contact terms, are finite.
- For certain range of kinematic variables, say $|s_i| < R$, infinite series representation can be truncated to retain reasonably finite number of terms, $N \sim R$, such that the truncated sum exhibits important features of the actual representation.

Open string amplitude

Crossing symmetric dispersion relation

S-matrix bootstrap

Closed string amplitude

Conclusion

Euler-Beta function

$$\frac{\Gamma(-s_1)\Gamma(-s_2)}{\Gamma(-s_1-s_2)} = \int_0^1 \mathrm{d}z\, z^{-s_1-1}(1-z)^{-s_2-1} = -\sum_{n=0}^{\infty} \frac{1}{n!}\frac{(s_2+1)_n}{s_1-n}.$$

Beta function has integral representation

$$\int_0^1 dz\, \frac{z^{-s_1-1}+z^{-s_2-1}}{(1+z)^{-s_1-s_2}}, \qquad \text{Gradshteyn, Ryzhik}$$

which converges for $\mathrm{Re}\,(s_1) < 0$ and $\mathrm{Re}\,(s_2) < 0$.

Taylor series expansion of the integrand around $z = 0$ gives a series representation:

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Above series converges for $\mathrm{Re}(s_1+s_2) > -1$.

Let us see another example due to **Giddings (1986)**:

$$\frac{\Gamma(-s_1)\,\Gamma(-s_2)}{\Gamma(-s_1-s_2)} = \underbrace{\int_0^{1/2} dz\, z^{-s_1-1}(1-z)^{-s_2-1}}_{s_1 \text{ channel}} + \underbrace{\int_{1/2}^{1} dz\, z^{-s_1-1}(1-z)^{-s_2-1}}_{s_2 \text{ channel}}$$

We expand s_1 and s_2 channel integrands around $z = 0$ and $z = 1$ respectively. Integrating each term we will get

$$\sum_{n=0}^{\infty} \frac{(-\frac{1}{2})^n}{n!} \left(\frac{2^{s_1}(-n-s_2)_n}{n-s_1} + \frac{2^{s_2}(-n-s_1)_n}{n-s_2} \right), \qquad \mathrm{Re}\,(s_1) < 0,\ \mathrm{Re}\,(s_2) < 0$$

We can try to analytically continue the above series to other domains.

This series has poor convergence for positive real values of s_1, s_2.

New representation

We use crossing-symmetric dispersion relation to obtain the following series

$$\frac{\Gamma(-s_1)\Gamma(-s_2)}{\Gamma(-s_1-s_2)} = -\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{s_1-n}+\frac{1}{s_2-n}+\frac{1}{\lambda+n}\right)\left(1-\lambda+\frac{(s_1+\lambda)(s_2+\lambda)}{\lambda+n}\right)_n$$

APS, Aninda Sinha; 2401.05733

λ is a free positive parameter.

$$\lambda\to\infty \quad : \quad -\frac{1}{n!}\left(\frac{1}{s_1-n}+\frac{1}{s_2-n}\right)(s_1+s_2-n+1)_n$$

$$\lambda\to -s_2 \quad : \quad -\frac{1}{n!}\frac{(1+s_2)_n}{s_1-n}$$

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Crossing-symmetric dispersion relations were studied in the seventies by Auberson, Khuri, Mahoux, Roy, Wanders, ... We have used a refined version of CSDR.

Comparisons of different representations

Parameters	Actual answer	Gradshteyn Ryzhik	Giddings	New representation ($\lambda = 8.5$)
$s_1 = 17.5$ $s_2 = -15.2$	-0.00021590	-0.00021454	-0.028595	-0.00021590
$s_1 = 7.1$ $s_2 = -1.1$	0	0	-0.0032111	9.9293×10^{-8}
$s_1 = -7.6$ $s_2 = 0.5$	-9.2810	-33543	-9.2810	-9.2810

Table: Number of terms in each series is 25.

Open string amplitude

$$\frac{\Gamma(-s_1)\Gamma(-s_2)}{\Gamma(1-s_1-s_2)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{s_1-n}+\frac{1}{s_2-n}+\frac{1}{\lambda+n}\right)\left(1-\lambda+\frac{(s_1+\lambda)(s_2+\lambda)}{\lambda+n}\right)_{n-1}$$

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Four-gluon amplitude is obtained with the prefactor $\mathrm{tr}\mathcal{F}^4$.

$$\mathrm{tr}\mathcal{F}^4 R_n = \sum_j c_j^{(n)} \left\{ \odot + \square\square + \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array} + \right\} \otimes j$$

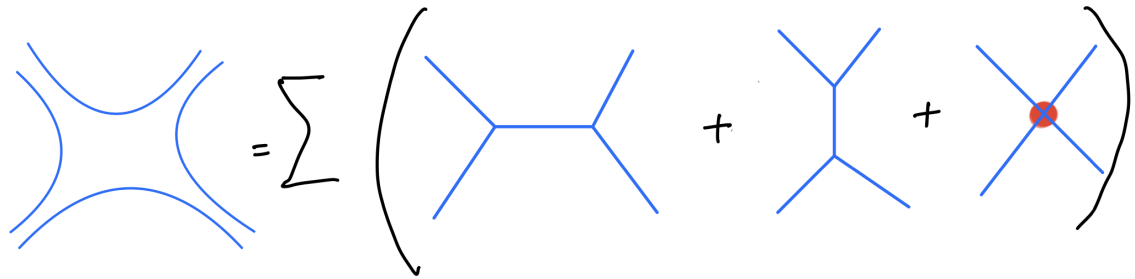
- Series has large domain of convergence in s_1 and s_2 variables.
- λ is attributed to field redefinition ambiguities present in the Lagrangian.
- $\mathrm{Re}(\lambda) > -1$ is required for convergence of the series.
- λ improves convergence of the series $\left(n^{-2-\lambda}\right.$ for large $\left. n\right)$.
- Residues are independent of λ.
- Contact terms depend on λ. Sum of the contact terms is convergent.

Mass-levels

$$
\begin{aligned}
t_0 &= \frac{1}{s_1 s_2} \\
t_1 &= \left\{ \frac{1}{s_1 - 1} + \frac{1}{s_2 - 1} \right\} + \frac{1}{1+\lambda} \\
t_2 &= \frac{1}{2} \left\{ \frac{1+s_2}{s_1 - 2} + \frac{1+s_1}{s_2 - 2} \right\} + \frac{2\lambda^2 + 3\lambda + 2 + 2(\lambda+1)x + y}{2(\lambda+2)^2} \\
t_3 &= \frac{1}{6} \left\{ \frac{10-d}{4(d-1)} \left(\frac{1}{s_1 - 3} + \frac{1}{s_2 - 3} \right) \right. \qquad \text{Unitarity holds for } d \leq 10 \\
&\quad \left. + \frac{\frac{9(d-2)}{4(d-1)} + 3s_2 + s_2^2}{s_1 - 3} + \frac{\frac{9(d-2)}{4(d-1)} + 3s_1 + s_1^2}{s_2 - 3} \right\} \\
&\quad + \frac{1}{6} \left(3x + \frac{3x^2 - 18x + 20}{\lambda + 3} + \frac{3\left(-3x^2 + xy + 18x - 3y - 27\right)}{(\lambda+3)^2} \right. \\
&\quad \left. + \frac{9x^2 + y^2 - 6xy - 54x + 18y + 81}{(\lambda+3)^3} \right)
\end{aligned}
$$

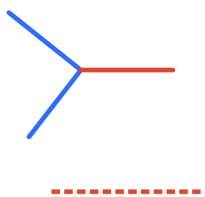
Here $x = s_1 + s_2$ and $y = s_1 s_2$.

Level truncation



Consider a Lagrangian

$$\mathcal{L} = -\frac{1}{2}\sum_{i=1}^{2} \partial_\mu \phi_i \partial^\mu \phi_i - \frac{1}{2}\sum_{j=1}^{n} \left(\partial_\mu \psi_j \partial^\mu \psi_j - m_j^2 {\psi_j}^2\right) - \sum_{j=1}^{n} g_j \phi_1 \phi_2 \psi_j$$



$$\mathcal{M}\left(\phi_1\phi_2\phi_1\phi_2\right) = \sum_{j=1}^{n} g_j^2 \left(\frac{1}{s - m_j^2} + \frac{1}{u - m_j^2}\right)$$

Field redefinition $\psi_k \to \psi_k + \lambda\phi_1\phi_2$ generates contact terms.

If some massive exchanges are omitted then the amplitude depends on λ through the contact terms.

Field theory expansion

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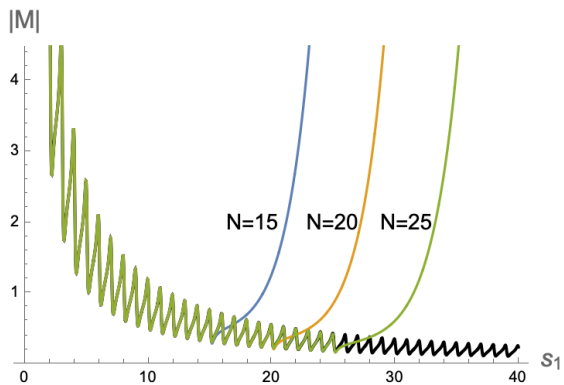
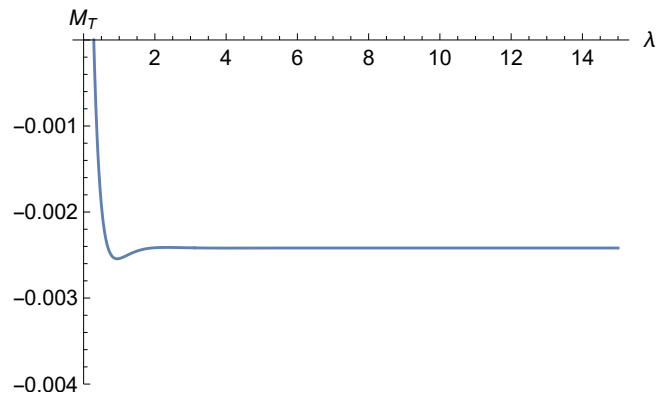
Although we could find a manifestly crossing symmetric and local representation for open string amplitude yet having infinite terms in the series is antithetical to QFT expansion.

In our representation of string amplitude, truncated sum exhibits a one-parameter ambiguity. This ambiguity goes away as more terms are retained in the sum.



Absolute value of truncated amplitude with $s_2 = -0.2$ for different values of N plotted as function of $s_1 + 0.1i$.



$M_T(s_1, s_2)$ with $N = 20$ plotted against λ for $s_1 = 7.5$, $s_2 = -2.2$.

Truncated amplitude depends on field redefinition ambiguity. But plateau indicates that $\partial_\lambda M_T \approx 0$. This demonstrates approximate QFT representation.

Open string amplitude

Crossing symmetric dispersion relation

S-matrix bootstrap

Closed string amplitude

Conclusion

CSDR Set up

The series representation is obtained using crossing symmetric dispersion relation. Auberson, Khuri, 1973; Sinha, Zahed, 2012.04877; Raman, Sinha, 2107.06559; Song, 2305.03669

Mandelstam variables are shifted $(s + t + u = \mu)$

$$s_1 = s - \frac{\mu}{3}, \qquad s_2 = t - \frac{\mu}{3}, \qquad s_3 = u - \frac{\mu}{3}.$$

Here s_3 is held fixed so that we can ignore u-channel cut. Singularities are present for $s_1, s_2 > 0$.

s_1 and s_2 are mapped to the complex plane:

$$s_k = \frac{4aze^{i\pi k}}{\left(z - e^{i\pi k}\right)^2}, \qquad k = 1, 2, \qquad a = \frac{s_1 s_2}{s_1 + s_2} = \text{constant}.$$

This satisfies $(s_1(z) - a)(s_2(z) - a) = a^2$. On this hyper-surface amplitude is analytic except at the physical cuts.

Parametrization

Singularities in s_1 are mapped to arcs on unit circle in the complex plane

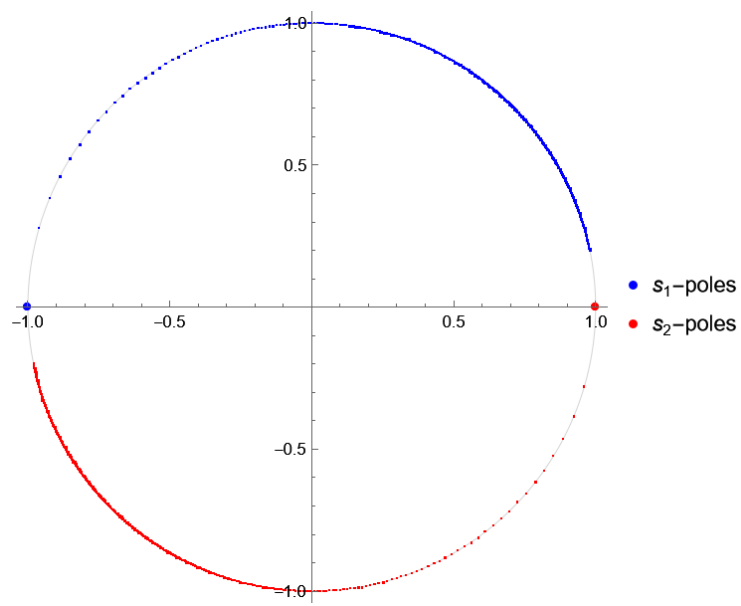
$$z = \frac{(s_1 - 2a) + 2i\sqrt{a\,(s_1 - a)}}{s_1}, \qquad a \leq s_1 < \infty.$$

This implies $z = e^{i\phi}$ with
$\phi = \cos^{-1}\left(\frac{s_1 - 2a}{s_1}\right)$.

At $z \to e^{i\pi k}$, s_j is fixed and $s_i \to \infty$.

Fall-off condition:

$$\widetilde{\mathcal{M}}\,(z, a) = \mathcal{O}\left(\frac{1}{(z - e^{i\pi k})^2}\right), \quad z \to e^{i\pi k}$$



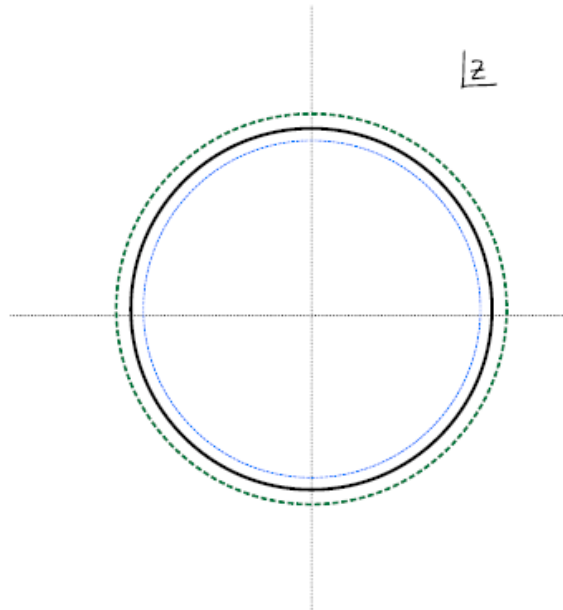
This implies $\mathcal{M}\,(s_1, s_2)$ can grow slower than $\mathcal{O}\,(s_1)$ in the large s_1 and fixed s_2 limit.

$\widetilde{\mathcal{M}}(z,a)$ has the following properties:

- $\widetilde{\mathcal{M}}(z,a) = \widetilde{\mathcal{M}}^*(1/z^*,a)$ Real analyticity
- $\widetilde{\mathcal{M}}(z,a)$ is function of z^2 Crossing symmetry

Evaluate the following integral

$$\frac{1}{2\pi i}\oint \frac{\mathrm{d}z'}{z'-z}\frac{z'^2-1}{z'^2}\widetilde{\mathcal{M}}(z',a)\,.$$



Discontinuity of the amplitude across unit disc is

$$\widetilde{\mathcal{A}}(z,a) = \lim_{\epsilon\to 0}\frac{1}{2i}\left[\widetilde{\mathcal{M}}\left((1+\epsilon)\,e^{i\phi},a\right) - \widetilde{\mathcal{M}}\left((1-\epsilon)\,e^{i\phi},a\right)\right].$$

In the complex plane dispersion relation is

$$\widetilde{\mathcal{M}}(z,a) = \widetilde{\mathcal{M}}(0,a) + \frac{1}{\pi}\frac{z^2}{1-z^2}\int_{\text{Disc }|z'|=1}\frac{\mathrm{d}z'}{z'}\frac{z'^2-1}{z'^2-z^2}\widetilde{\mathcal{A}}(z',a)$$

$$\mathcal{M}(s_1, s_2) = \mathcal{M}(0,0) + \frac{1}{\pi}\int_{s_0}^{\infty} \frac{\mathrm{d}\sigma}{\sigma}\mathcal{A}^{(s_1)}\left(\sigma, \frac{a\sigma}{\sigma - a}\right)\left[\frac{s_1}{\sigma - s_1} + \frac{s_2}{\sigma - s_2}\right]$$

Since $a = \frac{s_1 s_2}{s_1 + s_2}$, there are spurious singularities when the denominator vanishes.

The goal is to remove negative powers of $x = s_1 + s_2$ from CSDR.

- Perform Taylor series expansion of $\mathcal{A}$ around $a = 0$.
- Kernel: $H = \frac{s_1}{\sigma - s_1} + \frac{s_2}{\sigma - s_2} = \frac{\sigma^2 - y}{\sigma^2 - x\sigma + y} - 1$. Since $y = ax$,

$$a^n \frac{1}{\sigma^2 - x\sigma + y} = \sum_{k=0}^{\infty} \frac{y^n x^{k-n} \sigma^k}{(\sigma^2 + y)^{k+1}}$$

- Spurious singularities come from $k < n$. We remove those terms.
- a in the argument of $\mathcal{A}$ is replaced by $\frac{\sigma y}{\sigma^2 + y}$.
- With the above substitution $\mathcal{A}\left(\sigma, \frac{a\sigma}{\sigma - a}\right)$ becomes $\mathcal{A}\left(\sigma, \frac{y}{\sigma}\right)$.
- This procedure is equivalent to imposing null constraints. This implies $\mathcal{M} = \sum_{p,q \geq 0} W_{pq} x^p y^q$.

Local 2-channel symmetric dispersion relation

$$\mathcal{M}(s_1, s_2)$$
$$= \mathcal{M}(0,0) + \frac{1}{\pi}\int_{s_0}^{\infty} \mathrm{d}\sigma \left\{ \left[\frac{1}{\sigma - s_1} + \frac{1}{\sigma - s_2} - \frac{1}{\sigma} \right] \mathcal{A}^{(s_1)}\left(\sigma, \frac{s_1 s_2}{\sigma}\right) - \frac{1}{\sigma}\mathcal{A}^{(s_1)}(\sigma, 0) \right\}$$

APS, Aninda Sinha; 2401.05733

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We can introduce a free parameter in the above expression:

$$\mathcal{M}(s_1, s_2) = \mathcal{M}(-\lambda, -\lambda) - \frac{1}{\pi}\int_{s_0}^{\infty} d\sigma \left\{ \frac{1}{\sigma + \lambda}\mathcal{A}^{(s_1)}(\sigma, -\lambda) + \left(\frac{1}{s_1 - \sigma} + \frac{1}{s_2 - \sigma} + \frac{1}{\sigma + \lambda} \right) \mathcal{A}^{(s_1)}\left(\sigma, \frac{(s_1 + \lambda)(s_2 + \lambda)}{\sigma + \lambda} - \lambda\right) \right\}$$

Bhat, Chowdhury, APS, Sinha; 2409.18259

Thia is obtained by shifting $x \to x + 2\lambda$ and $y \to y + \lambda x + \lambda^2$.

Parametric dispersion relation

$$\mathcal{M} = -\frac{1}{\pi}\int_{s_0}^{\infty} \mathrm{d}\sigma \left(\frac{1}{s_1-\sigma}+\frac{1}{s_2-\sigma}+\frac{1}{\sigma+\lambda}\right)\mathcal{A}^{(s_1)}\left(\sigma, \frac{(s_1+\lambda)(s_2+\lambda)}{\sigma+\lambda}-\lambda\right)$$

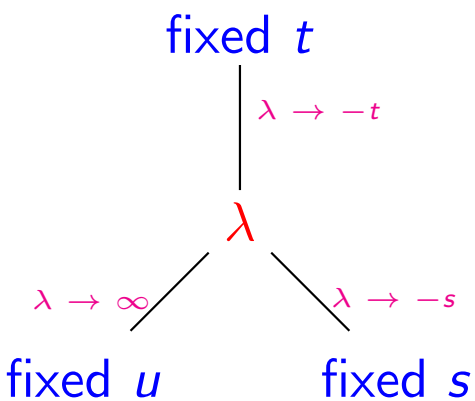
$$\downarrow \quad \lambda = -s_2$$

$$\frac{1}{\pi}\int_{s_0}^{\infty} \mathrm{d}\sigma \frac{1}{\sigma - s_1}\mathcal{A}^{(s_1)}(\sigma, s_2)$$

we retrieve fixed t dispersion relation.

Bhat, Chowdhury, APS, Sinha, (PRD) 2409.18259

Dispersion relations, which have different domains of analyticity, are interpolated among each other using λ.



Subtracted dispersion relations

$$\oint \mathrm{d}\sigma \left[\frac{1}{s_1-\sigma}+\frac{1}{s_2-\sigma}+\frac{1}{\sigma+\lambda}\right]\mathcal{M}\left(\sigma, s_2'(\sigma)\right)=0$$

if $\mathcal{M}(s_1, s_2) \sim \mathcal{O}(1)$ at large s_1

Crossing symmetry implies integrations along s_1-channel and s_2-channel cut are equal. This requirement leads to

$$s_2'(\sigma)=\frac{(s_1+\lambda)(s_2+\lambda)}{\sigma+\lambda}-\lambda$$

Bhat, APS, Sinha; 2506.03862

UV-IR duality: $s_2'(-\lambda)=\infty$ and $s_2'(\infty)=-\lambda$

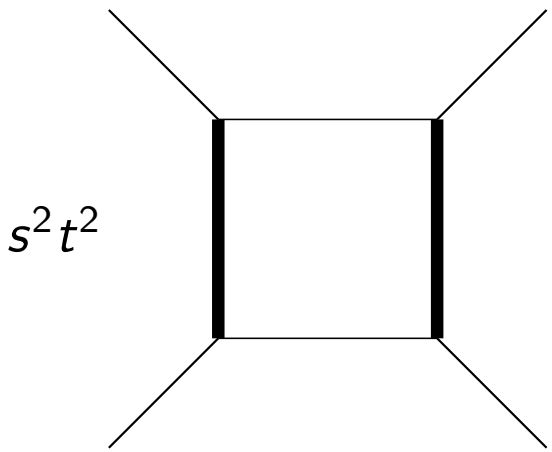
If $\mathcal{M}(s_1, s_2)$ grows faster than $\mathcal{O}\left(s_1^0\right)$ at large s_1, then we can add subtractions to the dispersion relation in the form Taylor expansions around $s_1=s_2=-\lambda$. This is equivalent to adding counterterms such that the arc at infinity does not contribute in the contour integration.

$s_2'(\sigma)$ is a monotonically decreasing function for $\sigma>0$. For amplitudes having Regge behaviour (e.g. $\mathcal{M} \sim s_1^{2+s_2}$), even the un-subtracted dispersion relation works if λ is chosen sufficiently large.

Examples

• This dispersion relation works for amplitudes containing massless poles. Examples: $\frac{1}{st}$, $\frac{1}{s}+\frac{1}{t}$. Here λ acts as IR regulator.

• Dilaton scattering in $\mathcal{N}=4$ SYM on the Coulomb branch



$$\mathcal{I}_{\square}(s,t) = \frac{1}{t\sqrt{s(s-4)}}\left(\frac{1}{\epsilon}-\log(-t)\right) \log\left(1-\frac{s-\sqrt{s(s-4)}}{2}\right) + (s\leftrightarrow t)+\mathcal{O}(\epsilon)$$

Open string amplitude

Crossing symmetric dispersion relation

S-matrix bootstrap

Closed string amplitude

Conclusion

Bootstrap

Partial wave expansion of $2 \to 2$ massless scalar scattering

$$\mathcal{M}(s_1, s_2) = \sum_{\ell} f_\ell(s_1)\, \mathcal{G}_\ell^{\left(\frac{D-3}{2}\right)}\left(1 + \frac{2s_2}{s_1}\right)$$

Assumptions at tree-level:

- Poles of the amplitude are at positive integer values.

$$\text{Disc}_s\left[f_\ell(s)\right] = -\pi \sum_{n=1}^{\infty} c_\ell^{(n)} \delta(s - n)$$

- Unitarity implies $c_\ell^{(n)} > 0$.
- Spins of the exchanges are bounded from above.
- We have to fix the form of the massless pole.

A parametric representation of amplitudes

$$\mathcal{M}(s_1, s_2) = \text{Massless pole} + W_{00} + \sum_{n=1}^{\infty} \sum_{\ell} c_\ell^{(n)} \left\{ \left(\frac{2}{n} - \frac{1}{\lambda + n} \right) \mathcal{G}_\ell^{\left(\frac{D-3}{2}\right)} \left(\frac{n - \lambda}{n + \lambda} \right) \right.$$
$$\left. + \left[\frac{1}{s_1 - n} + \frac{1}{s_2 - n} + \frac{1}{\lambda + n} \right] \mathcal{G}_\ell^{\left(\frac{D-3}{2}\right)} \left[1 + \frac{2}{n} \left(\frac{(s_1 + \lambda)(s_2 + \lambda)}{\lambda + n} - \lambda \right) \right] \right\}$$

W_{00} is a Wilson coefficient.

This one-parameter family of representations holds for amplitudes

1. which have any number of only scalar exchanges,
2. if exchanges of spinning states are allowed then there should be infinite of higher spin states.

This representation is valid for open string amplitude, Veneziano amplitude (Euler-Beta function) and their hypergeometric deformations.

cf. Cheung, Remmen, 2302.12263; Härring, Zhiboedov, 2311.13631; Mansfield, Spradlin, 2409.09561

Objective of the bootstrap method is to solve for $\left\{ c_\ell^{(n)} \right\}$ subjected to set of constraints, $\frac{\partial}{\partial \lambda} \mathcal{M}(s_1, s_2) = 0$.

Constraints for bootstrap

Open string amplitude, $\frac{\Gamma(-s_1)\Gamma(-s_2)}{\Gamma(1-s_1-s_2)} - \frac{1}{s_1 s_2}$ satisfies

$$\underbrace{W_{00}(0,0)}_{-\pi^2/6} + \frac{1}{\pi}\int_{s_0}^{\infty} \mathrm{d}\sigma \left[\frac{1}{\sigma - s_1} + \frac{1}{\sigma - s_2} - \frac{1}{\sigma}\right] \mathcal{A}^{(s_1)}\left(\sigma, \frac{s_1 s_2}{\sigma}\right) - \int_{s_0}^{\infty} \frac{\mathrm{d}\sigma}{\sigma} \mathcal{A}^{(s_1)}(\sigma, 0).$$

This follows from fixed t dispersion relation.

We can impose this condition as a constraint on the amplitude

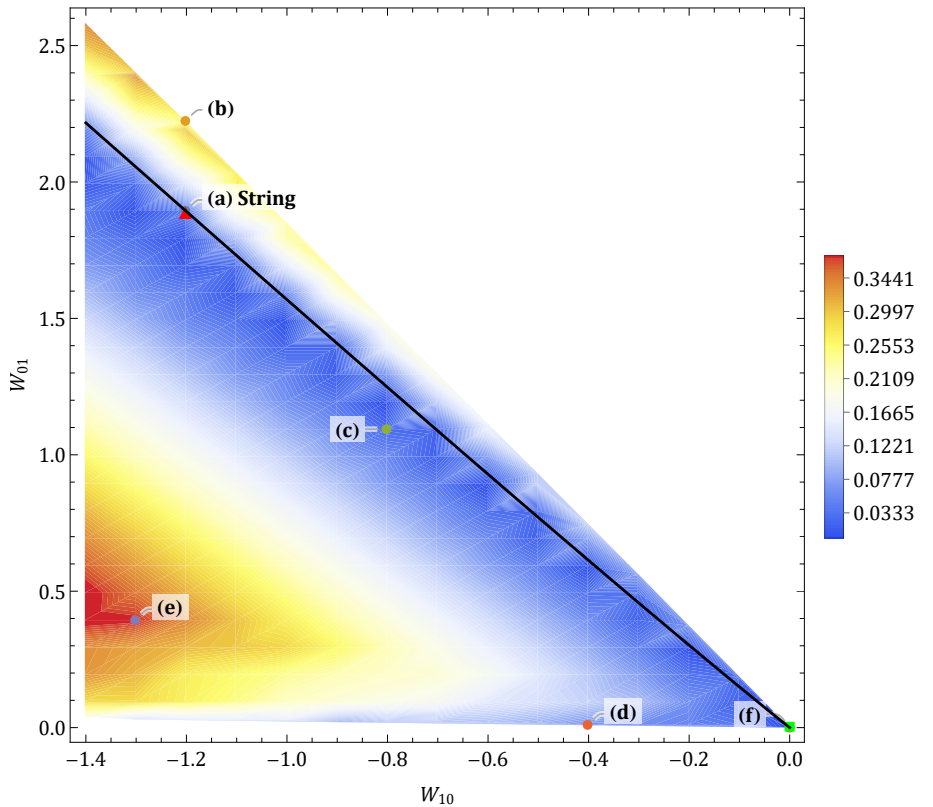
$$W_{00} = -\sum_{n=1}^{\infty} \sum_{\ell} \frac{(D-3)_n}{n\, n!} c_{\ell}^{(n)}.$$

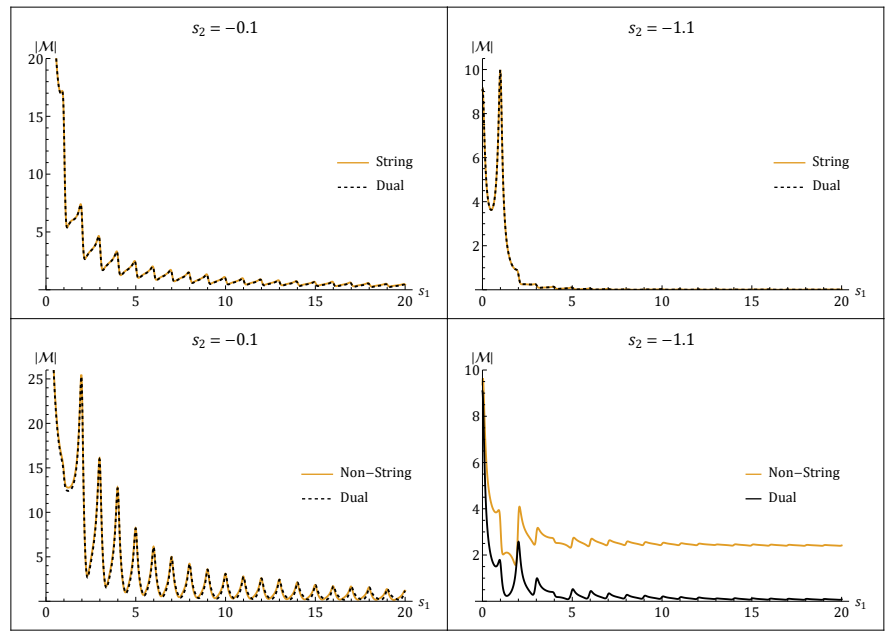
This ensures that the amplitude satisfies dual resonance (single channel expansion) in the forward limit.

Numerical analysis includes following steps:

- Maximize W_{00} with respect to $c_{\ell}^{(n)}$.
- Specify few Wilson coefficients. Here we fix W_{10} and W_{01}.
- Impose $\frac{\partial^k}{\partial \lambda^k} \mathcal{M}(s_1, s_2) = 0$ for higher values of k.

Dual resonance in the forward scattering





Allowed region of S-matrix in the space of Wilson coefficients. Contours denote deviations from the corresponding dual amplitudes. Any generic amplitude shows duality in the forward limit. Black curve represents hypergeometric deformed open string amplitude.

When single channel expansion holds beyond the forward limit, we obtain the following constraint for the partial waves

$$\sum_{n=1}^{\infty}\sum_{\ell} c_{\ell}^{(n)}\left\{\frac{(D-3)_n}{n\,n!}+\left(-\frac{2}{n}+\frac{1}{\lambda+n}\right)\mathcal{G}_{\ell}^{\left(\frac{D-3}{2}\right)}\left(\frac{n-\lambda}{n+\lambda}\right)\right\}=0.$$

Analytical solutions for amplitudes which satisfy the above equation

$$\frac{1}{s_1 s_2}-\frac{\Gamma(1-s_1)\Gamma(1-s_2)}{(r+1)\Gamma(-s_1-s_2+2)}\,{}_3F_2\left(r+1,1-s_1,1-s_2;r+2,-s_1-s_2+2;1\right).$$

Mansfield, Spradlin, 2409.09561

- $r=0$ is the open string amplitude.
- For $r\gg 1$, amplitude reduces to $\frac{1}{s_1 s_2}$, which is $(W_{10},W_{01})=(0,0)$.
- $(W_{10},W_{01})\to(-\infty,\infty)$ as $r\to -1$.

Numerical results

n	Exact	Bootstrap
1	1	1.000
2	0.0714	0.0717
3	0.0119	0.0121
4	0.00289	0.00297
5	0.000867	0.000898
6	0.000300	0.000311
7	0.000114	0.000119
8	0.0000469	0.0000486
9	0.0000204	0.0000211
10	9.26×10^{-6}	9.59×10^{-6}
11	4.37×10^{-6}	4.52×10^{-6}
12	2.12×10^{-6}	2.19×10^{-6}
13	1.06×10^{-6}	1.09×10^{-6}
14	5.41×10^{-7}	5.57×10^{-7}
15	2.81×10^{-7}	2.90×10^{-7}
16	1.49×10^{-7}	1.53×10^{-7}
17	7.99×10^{-8}	8.21×10^{-8}
18	4.35×10^{-8}	4.47×10^{-8}
19	2.39×10^{-8}	2.46×10^{-8}
20	1.33×10^{-8}	1.37×10^{-8}

(a) Leading

n	Exact	Bootstrap
3	0	0
4	0.00108	0.000367
5	0.000636	0
6	0.000325	0.000123
7	0.000163	0.000118
8	0.0000831	0.0000752
9	0.0000430	0.0000427
10	0.0000227	0.0000235
11	0.0000122	0.0000130
12	6.64×10^{-6}	7.20×10^{-6}
13	3.67×10^{-6}	4.01×10^{-6}
14	2.05×10^{-6}	2.25×10^{-6}
15	1.16×10^{-6}	1.27×10^{-6}
16	6.65×10^{-7}	7.23×10^{-7}
17	3.84×10^{-7}	4.14×10^{-7}
18	2.23×10^{-7}	2.39×10^{-7}
19	1.31×10^{-7}	1.39×10^{-7}
20	7.73×10^{-8}	8.05×10^{-8}

(b) Subleading

n	Exact	Bootstrap
5	0.0000842	0
6	0.000133	0
7	0.000104	0
8	0.0000698	0.0000491
9	0.0000444	0
10	0.0000277	0
11	0.0000170	4.93×10^{-6}
12	0.0000105	5.68×10^{-6}
13	6.44×10^{-6}	4.22×10^{-6}
14	3.96×10^{-6}	2.76×10^{-6}
15	2.44×10^{-6}	1.83×10^{-6}
16	1.51×10^{-6}	1.27×10^{-6}
17	9.39×10^{-7}	8.74×10^{-7}
18	5.84×10^{-7}	5.79×10^{-7}
19	3.65×10^{-7}	3.73×10^{-7}
20	2.29×10^{-7}	2.52×10^{-7}

(c) Subsubleading

Figure: $c_\ell^{(n)}$ on the Regge trajectories for open string amplitude in $D = 10$

Entanglement minimization

We define linear entropy by $\mathcal{E}[\Omega] = 1 - \mathrm{tr}_A\left[\rho_A^2\right]$.

ρ_{AB} is the density matrix which is constructed from the state

$$|\Omega\rangle = \sum_{a,b} \Omega_{ab} |p_1, a\rangle_A \otimes |p_2, b\rangle_B .$$

If $|\Omega\rangle$ is a product state, then difference in linear entropy between initial and final states in a scattering is proportional to S-matrix in the forward limit,

$$\Delta\mathcal{E}\left[\Omega^{\text{prod}}\right] = 4\mathcal{N}\,\mathrm{Im}\mathcal{M}^{\alpha\beta}_{\alpha\beta}\left(p_1, p_2 \to p_1, p_2\right).$$

Bose, Haldar, Sinha, Sinha, Tiwari, 2006.12213; Aoude, Chung, Huang, Machado, Tam, 2007.09486; Bose, Sinha, Tiwari, 2011.07944; Aoude, Elor, Remmen, Sumensari, 2402.16956

Four-gluon scattering in open string theory is

$$\mathcal{M} = \underbrace{\left(F_{\mu\nu}F^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - 4F_{\mu\nu}F^{\nu\alpha}F_{\alpha\beta}F^{\beta\mu}\right)}_{\sim s^2 \varepsilon_1\cdot\varepsilon_1^*\ \varepsilon_2\cdot\varepsilon_2^* \text{ in the forward limit}} \frac{\Gamma(-s)\Gamma(-t)}{\Gamma(1-s-t)}.$$

Maximization W_{00} corresponds to minimizing linear entropy $\int \mathrm{d}s \frac{\Delta\mathcal{E}}{s^2}$.

Closed string amplitude

Parametric representation from full three-channel crossing symmetric dispersion relation

$$\begin{aligned}
&\frac{\Gamma(-s_1)\,\Gamma(-s_2)\,\Gamma(-s_3)}{\Gamma(1+s_1)\,\Gamma(1+s_2)\,\Gamma(1+s_3)} \\
= \quad & -\frac{1}{s_1 s_2 s_3} + \sum_{n=1}^{\infty} \frac{1}{(n!)^2} \left[\frac{1}{s_1-n} + \frac{1}{s_2-n} + \frac{1}{s_3-n} + \frac{1}{\lambda+n}\right] \\
& \left(1 - \frac{n}{2} + \frac{n-2\lambda}{2}\sqrt{1 - \frac{4\,(s_1+\lambda)\,(s_2+\lambda)\,(s_3+\lambda)}{(n+\lambda)\,(n-2\lambda)^2}}\right)^2_{n-1}
\end{aligned}$$

Rosengren, 2409.06658; Bhat, Chowdhury, APS, Sinha, 2409.18259

For a generic crossing symmetric amplitude we have a parametric representation

$$\begin{aligned}
\mathcal{M}(s_1, s_2, s_3) \quad = \quad & -\frac{1}{s_1 s_2 s_3} + \sum_{n=1}^{\infty} \sum_{\ell \in \text{spectrum}} \left[\frac{1}{s_1-n} + \frac{1}{s_2-n} + \frac{1}{s_3-n} + \frac{1}{\lambda+n}\right] \\
& c_\ell^{(n)}\, \mathcal{G}_\ell^{\left(\frac{D-3}{2}\right)} \left(1 - \frac{2}{n}\left\{\frac{n-2\lambda}{2}\left(1 \pm \sqrt{1 - \frac{4\,(s_1+\lambda)\,(s_2+\lambda)\,(s_3+\lambda)}{(\lambda+n)\,(n-2\lambda)^2}}\right) + \lambda\right\}\right).
\end{aligned}$$

Double copy

At a massive pole residues exhibit double copy relation

$$\left(1-\frac{n}{2}+\frac{n-2\lambda}{2}\sqrt{1-\frac{4\left(s_1+\lambda\right)\left(s_2+\lambda\right)\left(s_3+\lambda\right)}{\left(n+\lambda\right)\left(n-2\lambda\right)^2}}\right)^2_{n-1} \xrightarrow{s_1=n} \left(1+s_2\right)^2_{n-1}$$

From the series representation

$$\mathcal{M}_{\text{cl}}\left(s_1,s_2,s_3\right) = \mathcal{M}_{\text{op}}\left(s_1,s_2\right)\frac{\sin\left(\pi s_1\right)\sin\left(\pi s_2\right)}{\pi\sin\left(\pi\left(s_1+s_2\right)\right)}\mathcal{M}_{\text{op}}\left(s_1,s_2\right)$$

$$\downarrow \quad s_1,\ s_2 \sim 0$$

$$= -\frac{1}{s_1 s_2 s_3}+\frac{2}{\left(s_1+s_2\right)}\left[\sum_n \frac{\left(2\lambda+n\right)}{\lambda n^2 n!}\left(-\frac{n\lambda}{n+\lambda}\right)_n+\zeta\left(2\right)\right]+\mathcal{O}\left(1\right)$$

Sum over infinite terms required to cancel the crossing symmetry violating term.

Double copy relations can be used to constrain Wilson coefficients.

Work in progress. . .

Open string amplitude

Crossing symmetric dispersion relation

S-matrix bootstrap

Closed string amplitude

Conclusion

Zeta functions

$$\zeta(2) = -\sum_{n=1}^{\infty} \frac{(2\lambda + n)\Gamma\left(\frac{n^2}{n+\lambda}\right)}{\lambda n^3 \Gamma(n)\Gamma\left(-\frac{n\lambda}{n+\lambda}\right)}$$

With $\lambda = 1.5$ if we sum over only 10 terms the error is 0.01%, whereas the same for the $\lambda = 0$ produces 6% error.

$$\zeta(3) = -\sum_{n=1}^{\infty} \frac{\Gamma\left(\frac{n^2}{n+\lambda}\right)\left(\lambda n(2\lambda + n)\left(\psi^{(0)}\left(\frac{n^2}{n+\lambda}\right) - \psi^{(0)}\left(1 - \frac{n\lambda}{n+\lambda}\right)\right) + (\lambda + n)^2\right)}{\lambda n^4(\lambda + n)\Gamma(n)\Gamma\left(-\frac{n\lambda}{n+\lambda}\right)}$$

```
N[Zeta[3], 20]
```

```
1.2020569031595942854
```

```
N[Sum[-(Gamma[n^2/(n+α)] ((n+α)^2 + n α (n+2 α) (EulerGamma - HarmonicNumber[-(n α)/(n+α)] + PolyGamma[0, n^2/(n+α)])))/(n^4 α (n+α) Gamma[n] Gamma[-(n α)/(n+α)]) /. α → 535/12, {n, 1, 25}], 20]
```

```
1.2020569031587786376
```

```
N[Sum[1/n^3, {n, 1, 100}], 20]
```

```
1.2020074006596776104
```

Generating function:

$$\mathcal{Z}(x) \equiv -\sum_{n=1}^{\infty} \frac{x^2}{n!}\left(\frac{x-2n}{n^2-nx}+\frac{1}{\lambda+n}\right)\left(1-\lambda+\lambda\frac{x+\lambda}{n+\lambda}\right)_{n-1}$$

Coefficient of x^k, $k \geq 2$ gives $\zeta(k)$.

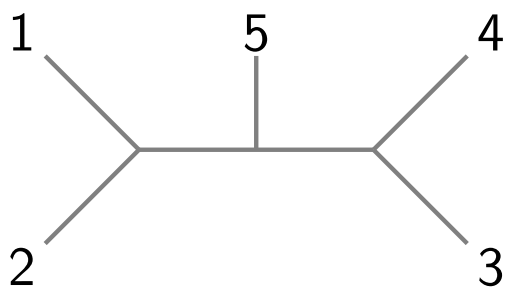
General expression:

$$\zeta(j) = \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}(-1)^{k+n+1}\frac{1}{n^2 n!}\left(\frac{\lambda}{n+\lambda}\right)^{j-2}\left(1-\frac{n\lambda}{n+\lambda}\right)^{k-j+2}$$
$$\left(\frac{n(n+2\lambda)}{n+\lambda}{}^{k}C_{j-2}+\frac{n+\lambda-n\lambda}{\lambda}P_{j-3}^{(\alpha_k,\beta_k)}(\xi_n)\right)S_{n-1}^{(k)}$$

$\xi_n = 2\frac{n\lambda}{n+\lambda} - 1$ and $P^{(\alpha,\beta)}(x)$ is the Jacobi Polynomial. Here $\alpha_k = 3-j+k$, $\beta_k = -k$. $S_n^{(k)}$ are the Stirling numbers of the first kind

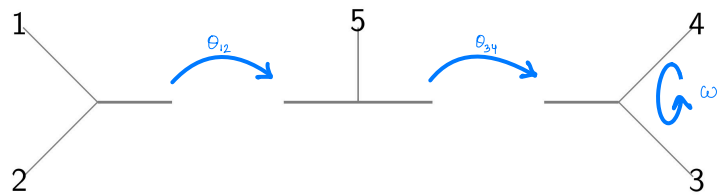
Beyond 4-point amplitudes: 5-point kinematics

Ongoing work with **Soumen Saren** and **Aninda Sinha**



A 5-point amplitude can be described by 5 variables. We choose $\{s_{12}, s_{34}, \theta_{12}, \theta_{34}, \omega\}$. Angular variables are the parameters of little group for massive particles.

Every 3-point vertex can be associated a COM frame



Amplitude can be expanded in terms of spherical harmonics

$$\mathcal{M} = \sum_{j,l=0}^{\infty} \sum_{n=-\min\{|j|,|l|\}}^{\min\{|j|,|l|\}} f_{jln}(s_{12}, s_{34})\, e^{i\omega n} \left(1-x^2\right)^{\frac{|n|}{2}} \mathcal{G}^{m+|n|}_{j-|n|}(x) \left(1-y^2\right)^{\frac{|n|}{2}} \mathcal{G}^{m+|n|}_{l-|n|}(y)$$

Here $x = \cos\theta_{12}$, $y = -\cos\theta_{34}$, $m = \frac{d-3}{2}$.

Crossing symmetry

Planar-ordered cubic scalar amplitude

$$\mathcal{M}_5^{\phi^3} = \frac{1}{(s_{12}-m^2)(s_{34}-m^2)} + \frac{1}{(s_{23}-m^2)(s_{45}-m^2)} + \frac{1}{(s_{34}-m^2)(s_{51}-m^2)} + \frac{1}{(s_{45}-m^2)(s_{12}-m^2)} + \frac{1}{(s_{51}-m^2)(s_{23}-m^2)}.$$

The amplitude is invariant under the simultaneous exchanges

$s_{12} \leftrightarrow s_{34}$ $\quad$ $s_{51} \leftrightarrow s_{45}$,

$s_{12} \leftrightarrow s_{23}$ $\quad$ $s_{34} \leftrightarrow s_{51}$,

$s_{12} \leftrightarrow s_{45}$ $\quad$ $s_{23} \leftrightarrow s_{34}$,

$s_{12} \leftrightarrow s_{51}$ $\quad$ $s_{23} \leftrightarrow s_{45}$,

$s_{23} \leftrightarrow s_{51}$ $\quad$ $s_{34} \leftrightarrow s_{45}$.

Crossing symmetry

Planar-ordered cubic scalar amplitude

$$\mathcal{M}_5^{\phi^3} = \frac{1}{(s_{12}-m^2)(s_{34}-m^2)} + \frac{1}{(s_{23}-m^2)(s_{45}-m^2)} + \frac{1}{(s_{34}-m^2)(s_{51}-m^2)} + \frac{1}{(s_{45}-m^2)(s_{12}-m^2)} + \frac{1}{(s_{51}-m^2)(s_{23}-m^2)}.$$

The amplitude is invariant under the simultaneous exchanges

$$\begin{aligned} &s_{12} \leftrightarrow s_{34} && s_{51} \leftrightarrow s_{45}, \\ &s_{12} \leftrightarrow s_{23} && s_{34} \leftrightarrow s_{51}, \\ &s_{12} \leftrightarrow s_{45} && s_{23} \leftrightarrow s_{34}, \\ &s_{12} \leftrightarrow s_{51} && s_{23} \leftrightarrow s_{45}, \\ &s_{23} \leftrightarrow s_{51} && s_{34} \leftrightarrow s_{45}. \end{aligned}$$

Symmetry is satisfied by tachyonic amplitude

$$\mathcal{M}_5^{\text{string}} = \frac{\Gamma(-1-s_{12})\,\Gamma(-1-s_{23})\,\Gamma(-1-s_{34})\,\Gamma(-1-s_{45})\,\Gamma(-3-s_{23}-s_{34}-s_{45})\,\Gamma(-1-s_{51})}{\Gamma(-2-s_{12}-s_{23})\,\Gamma(-2-s_{23}-s_{34})\,\Gamma(-2-s_{34}-s_{45})\,\Gamma(-2-s_{45}-s_{51})} \times {}_6F_5\left(\begin{matrix} -1-s_{23},\ -1-s_{45},\ -1+s_{12}-s_{34}-s_{45}, \\ -4-s_{23}-s_{34}-s_{45},\ -1-\frac{s_{23}}{2}-\frac{s_{34}}{2}-\frac{s_{45}}{2},\ -1-s_{23}-s_{34}+s_{51} \\ -2-s_{12}-s_{23},\ -2-s_{23}-s_{34},\ -2-s_{34}-s_{45},\ -2-s_{45}-s_{51} \end{matrix}\ ;-1\right).$$

Arkani-Hamed, Figueiredo, Remmen, 2412.20639

Summary & future works

- We have obtained a parametric crossing symmetric dispersive representation. By tuning the free parameter we can recover the known dispersion relations.
- This provides a suitable framework for bootstrap analysis for amplitudes with massive higher spin exchanges.
- Numerical techniques used in the analysis involve Semi Definite Programming for Bootstrap (SDPB) and Machine Learning.
- We obtained a class of solutions which include string amplitude. More constraints are required to uniquely fix string amplitudes. We need to extend the analysis beyond four-point tree-level amplitudes.
- We can constrain the non-analytic structures at higher loops using the present set up.
- What is the domain of analyticity of the dispersion relation?
- Study of Regge analysis using crossing symmetric dispersion relation.
- Can we obtain crossing-symmetric dispersion relation for higher point amplitudes?

Thank you