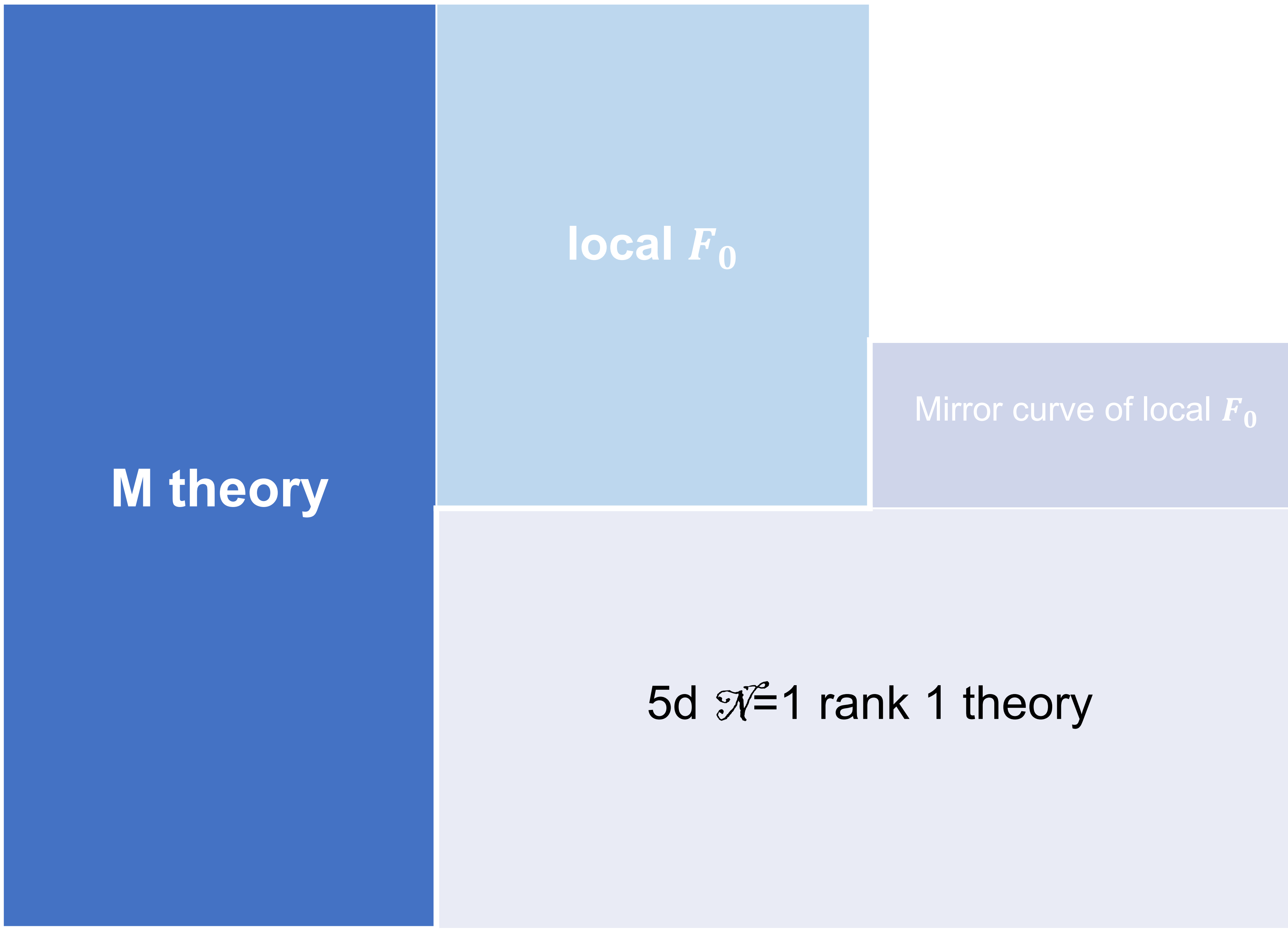
2025.11@IPMU

Trans-series representation of Hofstadter's butterfly from non-perturbative topological strings

Zhaojie Xu
(Southeast University Yau Center)
徐兆杰/徐兆傑

based on 2406.18098 w/ Jie Gu
and work in progress

Roadmap to Harper-Hofstadter model



Roadmap to Harper-Hofstadter model

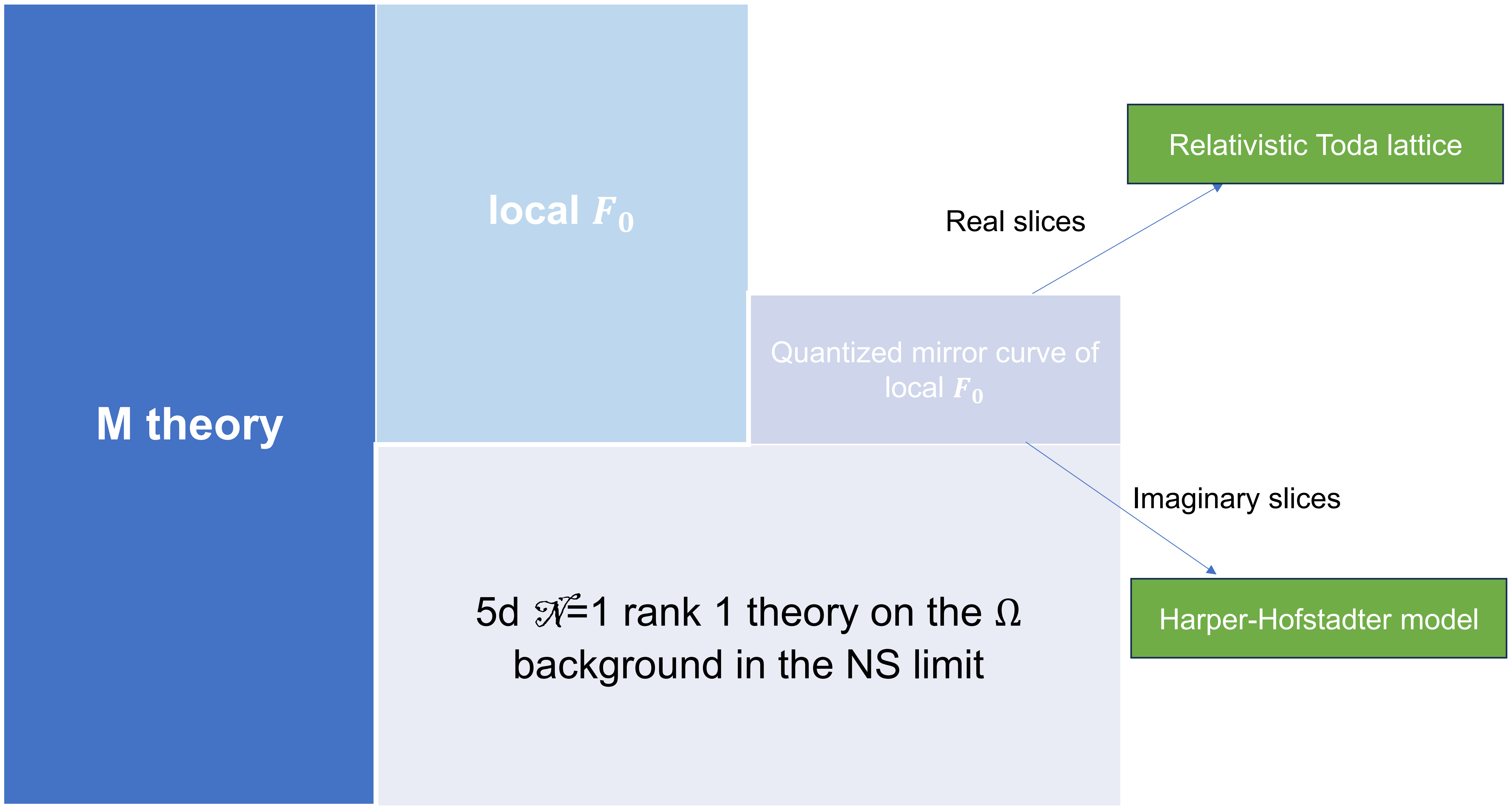
M theory

local F_0

Quantized mirror curve of local F_0

5d $\mathscr{N}$=1 rank 1 theory on the Ω background in the NS limit

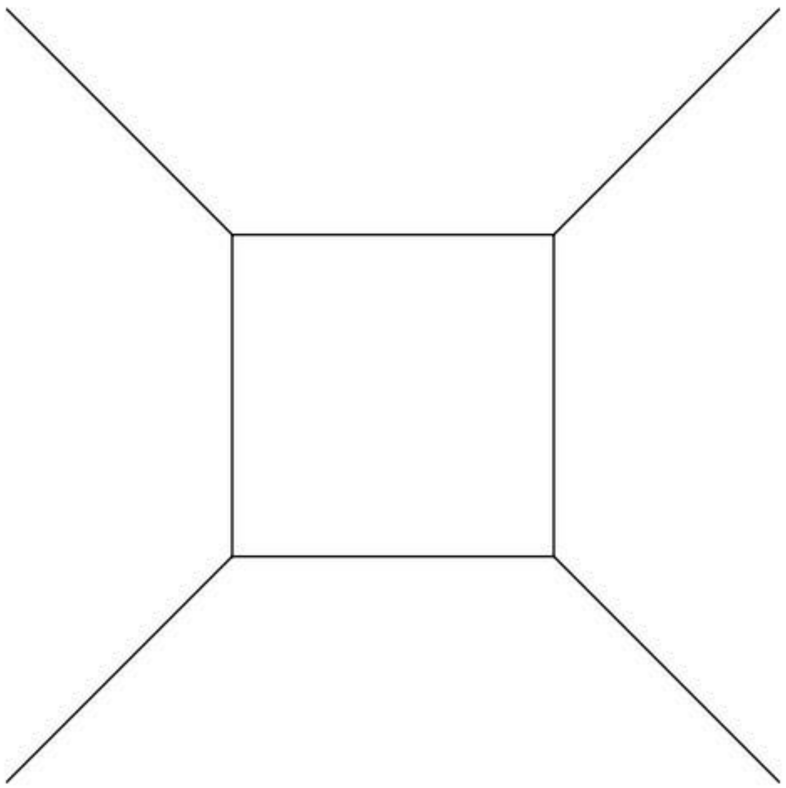
Roadmap to Harper-Hofstadter model



5d SYM from M theory on local F_0

- 5d pure SU(2) SYM can be obtained from M theory on local F_0/local $\mathbb{P}^1 \times \mathbb{P}^1$

- Under toric duality [Leung, Vafa' 97], it's also equivalent to Type IIB theory with (p,q) 5-brane web which has the same configuration as the toric diagram of local F_0



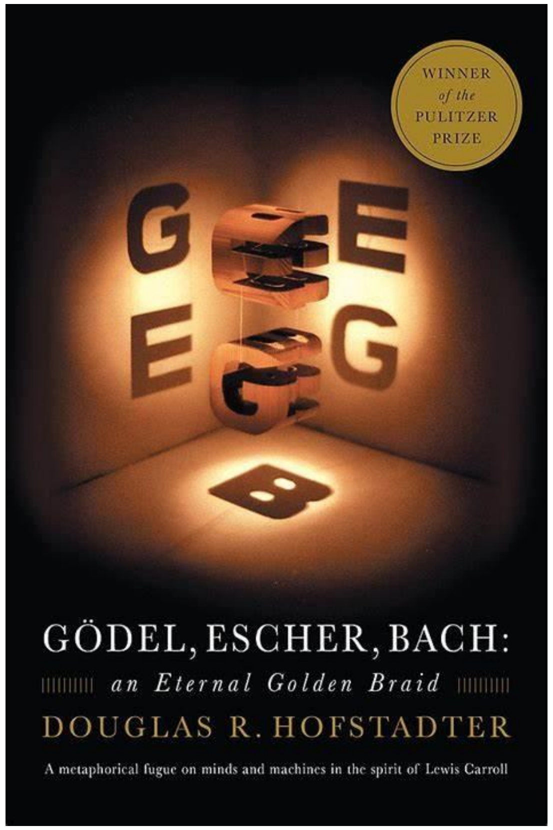
	0	1	2	3	4	5	6	7	8	9
5-branes	—	—	—	—	—	angles				

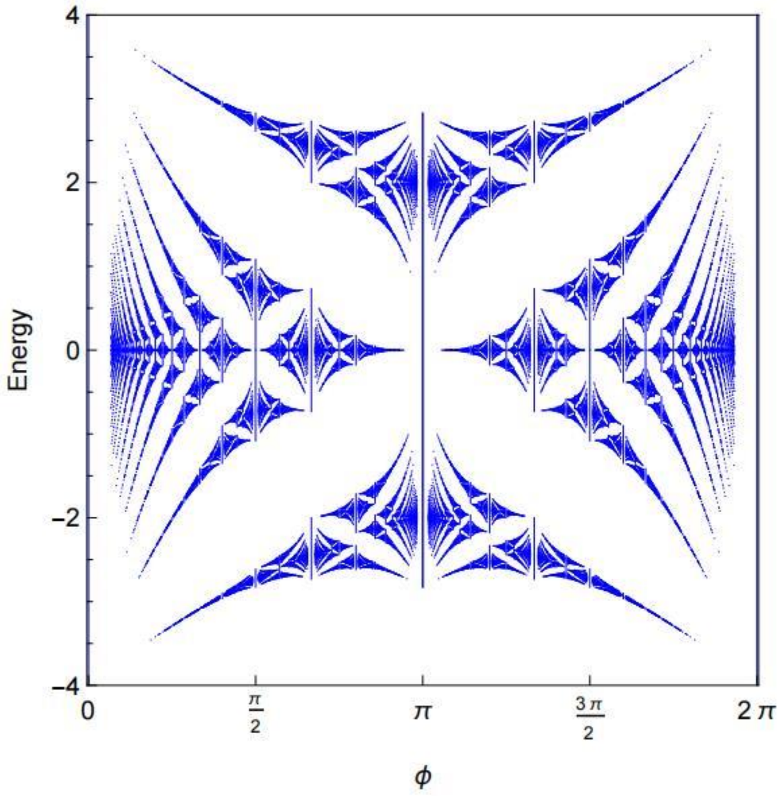
- The form of the mirror curve can be read off from the dual grid diagram

Harper-Hofstadter model

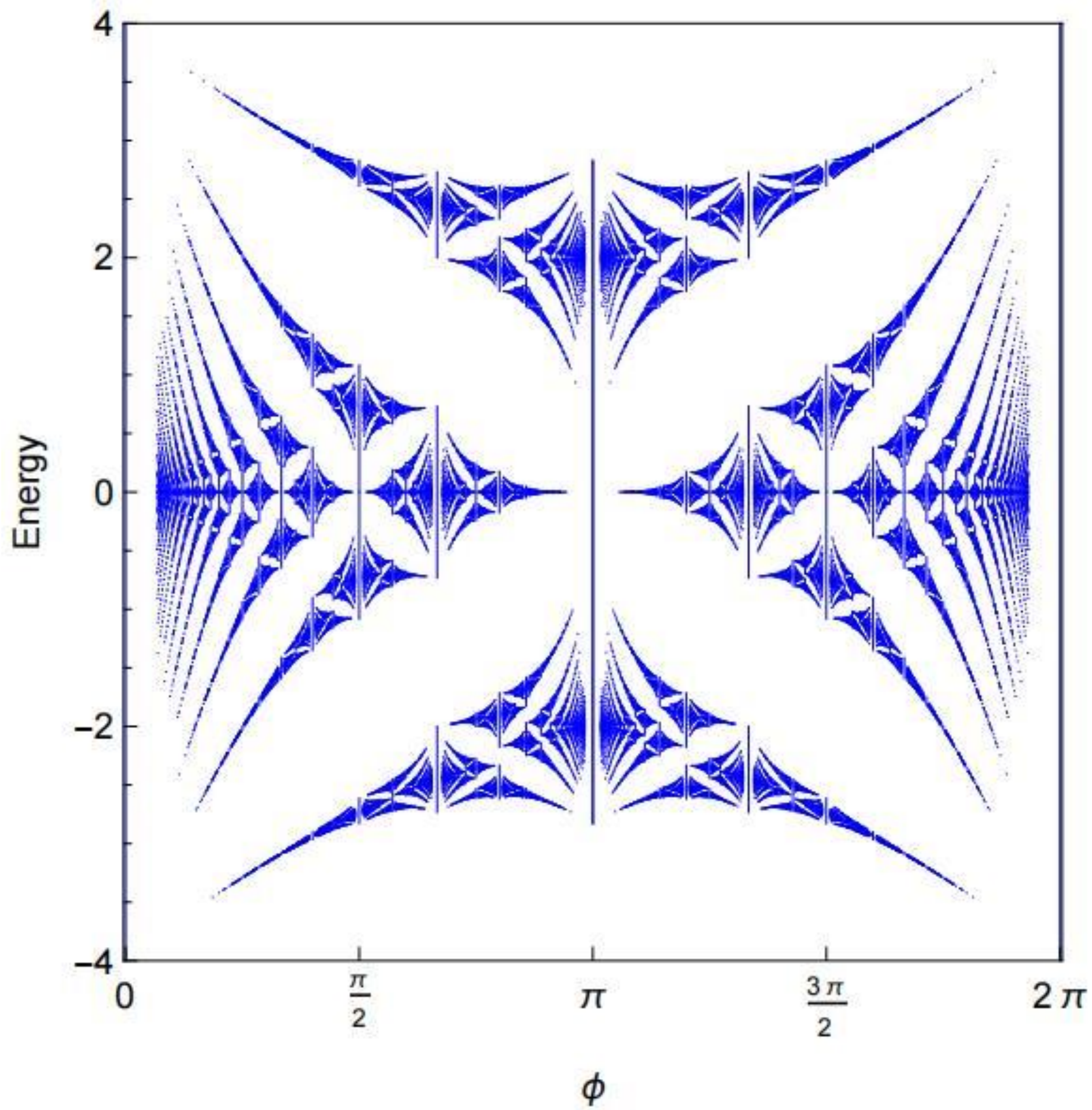
In 1976, Douglas Hofstadter (cognitive scientist, author of a famous popular science book *Gödel, Escher, Bach* and son of the Nobel laureate Robert Hofstadter) discovered a mesmerizing fractal pattern by studying the energy spectrum of the Harper model for rational flux/ (2π). The name of that energy spectrum is the famous Hofstadter's butterfly.

Credit: Basic Books



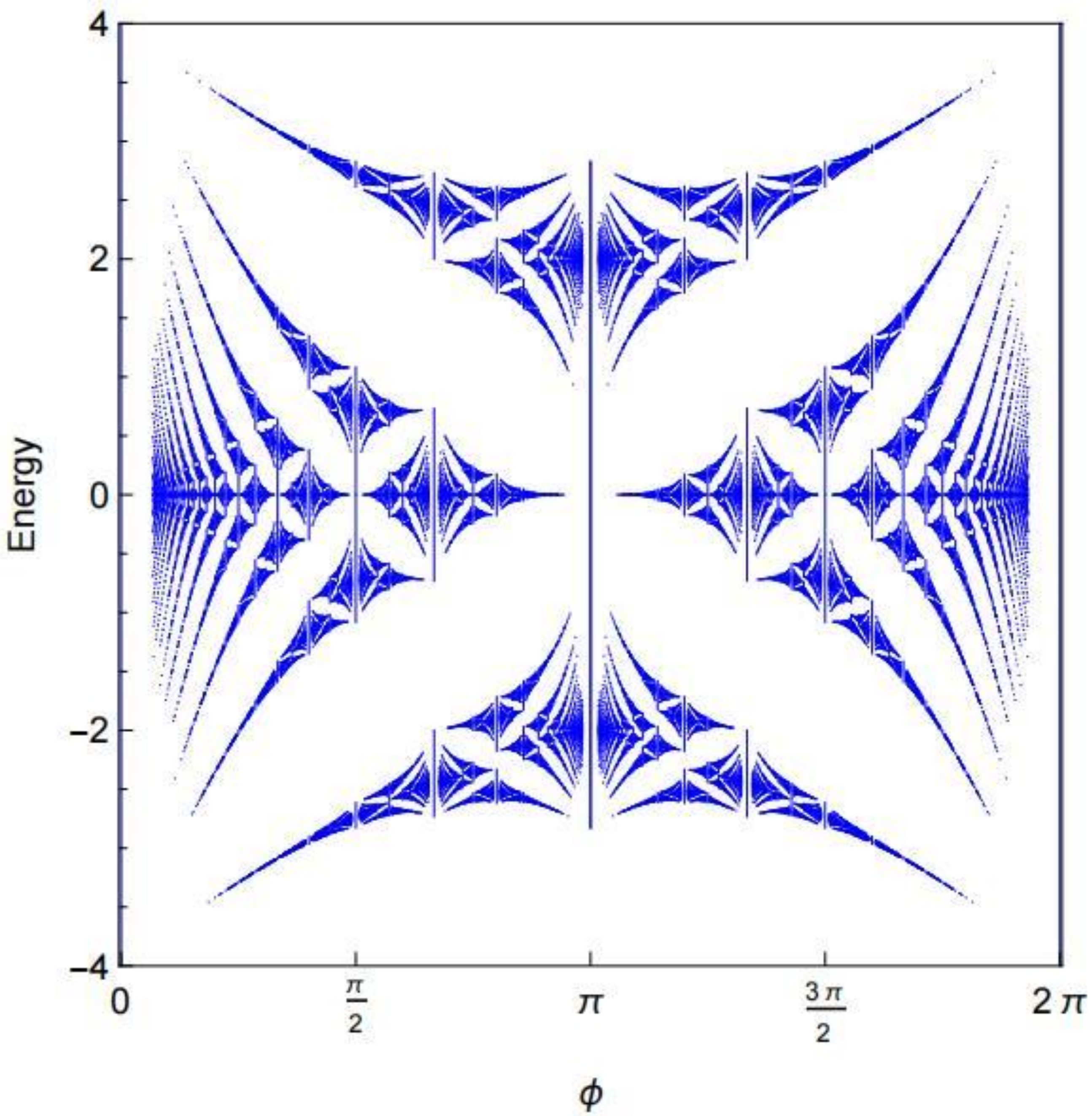


Harper-Hofstadter model



- For flux $\phi = 2\pi \frac{P}{Q}$, there are Q bands determined by the secular equation

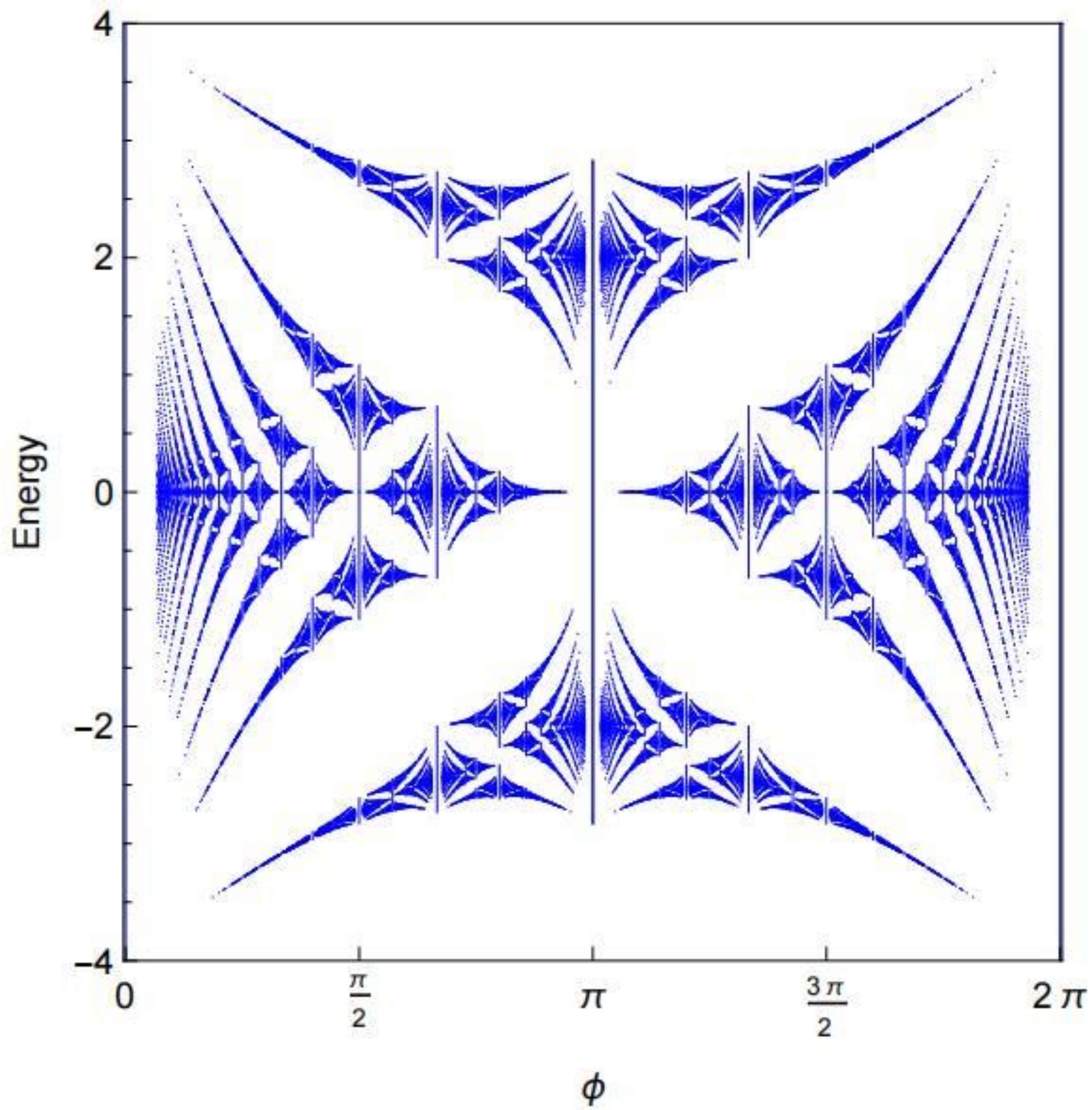
Harper-Hofstadter model



- For flux $\phi = 2\pi \frac{P}{Q}$, there are Q bands determined by the secular equation

- The same figure also shows up in quantum geometry of local F_0 [Hatsuda, Katsura, Tachikawa' 16]

Harper-Hofstadter model



- For flux $\phi = 2\pi \frac{P}{Q}$, there are Q bands determined by the secular equation
- The same figure also shows up in quantum geometry of local F_0 [Hatsuda, Katsura, Tachikawa' 16]
- Unification of high energy physics, condensed matter theory, spectral theory and *fractal art*, highly **interdisciplinary**!

Harper-Hofstadter model

- The dictionary between the topological string side and condensed matter side is summarized as follows

Topological String	Condensed Matter
Quantum mirror curve (Imaginary slices) $H = e^x + e^{-x} + e^y + e^{-y}$	Hamiltonian $H = e^{i\Pi_x} + e^{-i\Pi_x} + e^{i\Pi_y} + e^{-i\Pi_y}$
Quantum deformation parameter $\hbar$	Magnetic flux $-\phi$
Mass parameters	Hopping parameters
Imaginary part of $\frac{1}{2\pi}\frac{\partial t\left(\mathcal{E};\hbar=\frac{2\pi a}{b}\right)}{\partial \mathcal{E}}$	Density of States $\rho(\mathcal{E};\phi = \frac{2\pi a}{b})$
Branch cut structure of $t\left(\mathcal{E};\hbar = \frac{2\pi a}{b}\right)$ (large radius frame)	Band Spectrum (Hofstadter's butterfly)

Perturbative energy from perturbative Wilson loop

- Vev of ½-BPS Wilson loop wrapping S^1 and located at the origin is given by

$$W_r = \langle \mathrm{W_r} \rangle, \mathrm{W_r} = \mathrm{Tr_r} \exp \oint_{S^1} d\tau \left(A_0(\tau) - \phi(\tau) \right)$$

Perturbative energy from perturbative Wilson loop

- Vev of ½-BPS Wilson loop wrapping S^1 and located at the origin is given by

$$W_r = < \mathrm{W_r} >, \mathrm{W_r} = \mathrm{Tr_r} \exp \oint_{S^1} d\tau \, (A_0(\tau) - \phi(\tau))$$

- Take r to be the fundamental rep $\square$ and take the NS limit

$$W_{\square} = \sum_{n \geq 0} W_n(t) \hbar^{2n}$$

Perturbative energy from perturbative Wilson loop

- Vev of ½-BPS Wilson loop wrapping S^1 and located at the origin is given by

$$W_r = < \mathrm{W_r} >, \mathrm{W_r} = \mathrm{Tr_r} \exp \oint_{S^1} d\tau \, (A_0(\tau) - \phi(\tau))$$

- Take r to be the fundamental rep $\square$ and take the NS limit

$$W_{\square} = \sum_{n\geq 0} W_n(t) \hbar^{2n}$$

- Equivalent to the inverse quantum mirror map [Gaiotto, Kim' 14; Bullimore, Kim, Koroteev' 14], can be calculated by the HAE [Huang, Lee, Wang' 22; Wang' 23] efficiently

Perturbative energy from perturbative Wilson loop

- The perturbative energy of Harper-Hofstadter model corresponds exactly to the Wilson loop vev [A. Sciarappa' 16] with the identification

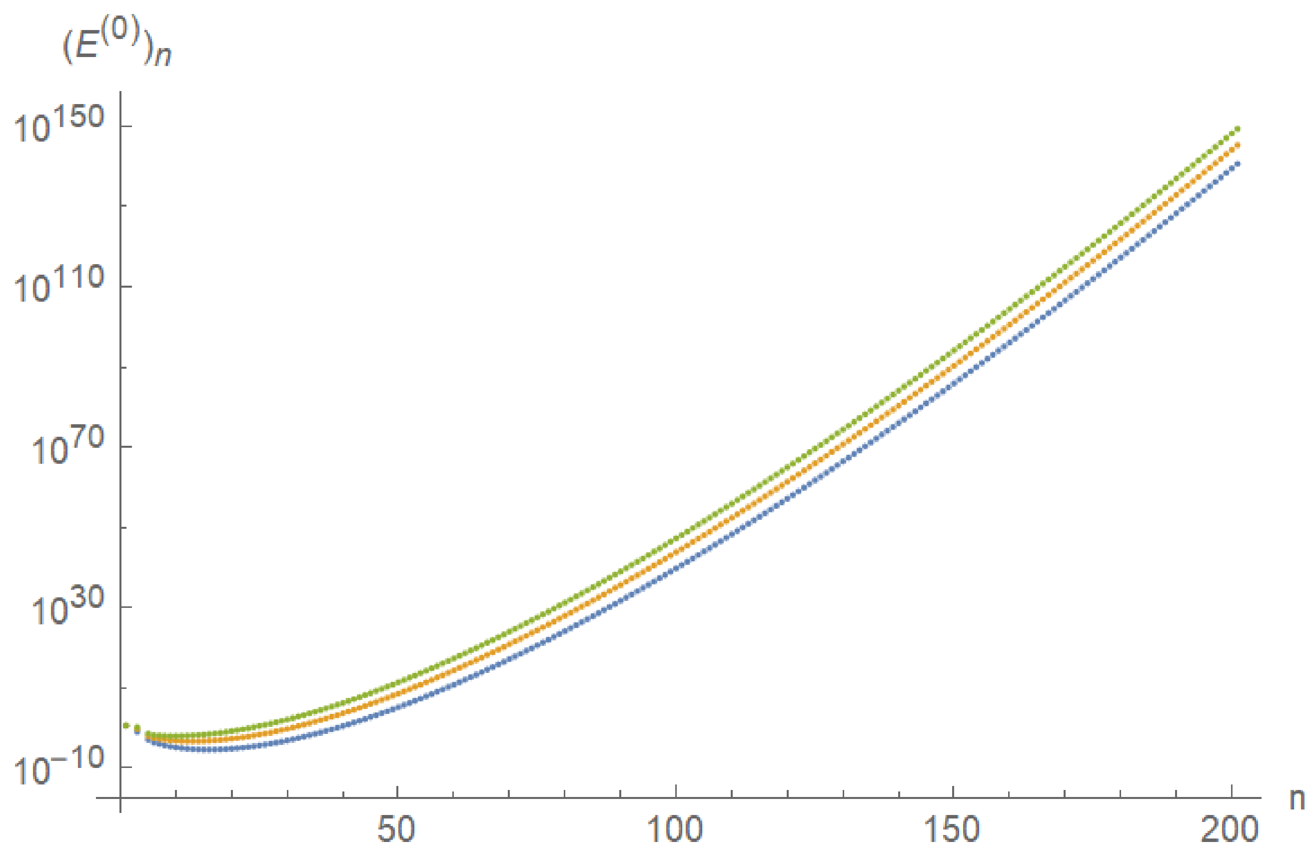
$$\hbar \to -\phi, t_c \to -\phi(N + \frac{1}{2})$$

- The Bender-Wu algorithm can help us compute the perturbative expansion in a parallel line [Sulejmanpasic, Ünsal' 16, Gu, Sulejmanpasic' 17].

$$E(N;\phi) = 4 - (1+2N)\phi + \frac{1}{8}(1+2N+2N^2)\phi^2 - \frac{1}{192}(1+3N+3N^2+2N^3)\phi^3 + \frac{(2+5N+6N^2+2N^3+N^4)\phi^4}{1536} + \frac{(67+215N+250N^2+190N^3+35N^4+14N^5)\phi^5}{245760} + \ldots + (4.970153\times10^{140} + 5.91148\times10^{141}N + \ldots)\phi^{200} + \ldots$$

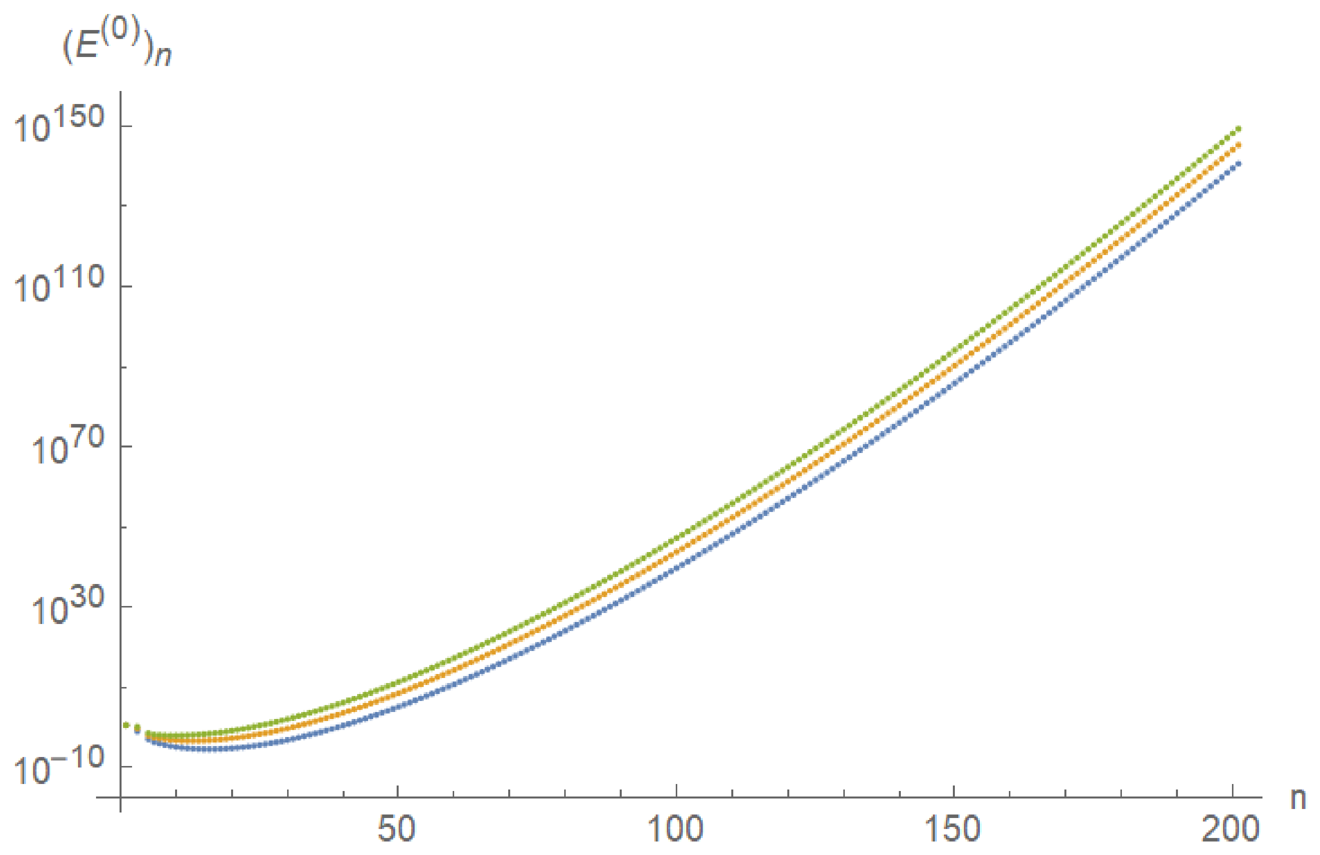
Perturbative energy from perturbative Wilson loop

- Although we can compute the perturbative expansion up to at least 200 orders efficiently, the series coefficients grow factorially, say of 1-Gevrey type



Perturbative energy from perturbative Wilson loop

- Although we can compute the perturbative expansion up to at least 200 orders efficiently, the series coefficients grow factorially, say of 1-Gevrey type



- This is when resurgence techniques come into play

Resurgence theory at work

Consider a general series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad a_n \sim n!$$

The idea of Borel summation is inserting the identity $1 = \frac{\Gamma(\mathrm{n}+1)}{n!}$ to the summand and rewrite the Gamma function with its integral representation, i.e.

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \int_0^{\infty} (z\zeta)^n e^{-\zeta} d\zeta$$

$$= \sum_{n=0}^{\infty} \frac{1}{z} \int_0^{\infty} \left(\frac{a_n}{n!} \zeta^n\right) e^{-\zeta/z} d\zeta$$

and then interchange the order of summation and integration.

Resurgence theory at work

One can define

- Borel transform

$$\mathcal{B}f(\zeta) := \sum_{n=0}^{\infty} \frac{a_n}{n!} \zeta^n$$

- Borel summation

$$\mathcal{S}f(z) = \frac{1}{z} \int_0^{\infty} d\zeta \, e^{-\zeta/z} \, \mathcal{B}f(\zeta)$$

Resurgence theory at work

One can define

- Borel transform

$$\mathcal{B}f(\zeta) := \sum_{n=0}^{\infty} \frac{a_n}{n!} \zeta^n$$

- Borel summation

$$\mathcal{S}f(z) = \frac{1}{z} \int_0^{\infty} d\zeta \, e^{-\zeta/z} \, \mathcal{B}f(\zeta)$$

- These definitions need to be modified when applied to perturbative series in physics

Resurgence theory at work

- Borel-(Pade) transform

$$\mathcal{B}E^{(0)}(\zeta) = \sum_{n=0}^{2n_{max}} \frac{a_n^{(0)}}{n!}\zeta^n \approx \frac{P_{n_{max}}(\zeta)}{Q_{n_{max}}(\zeta)}$$

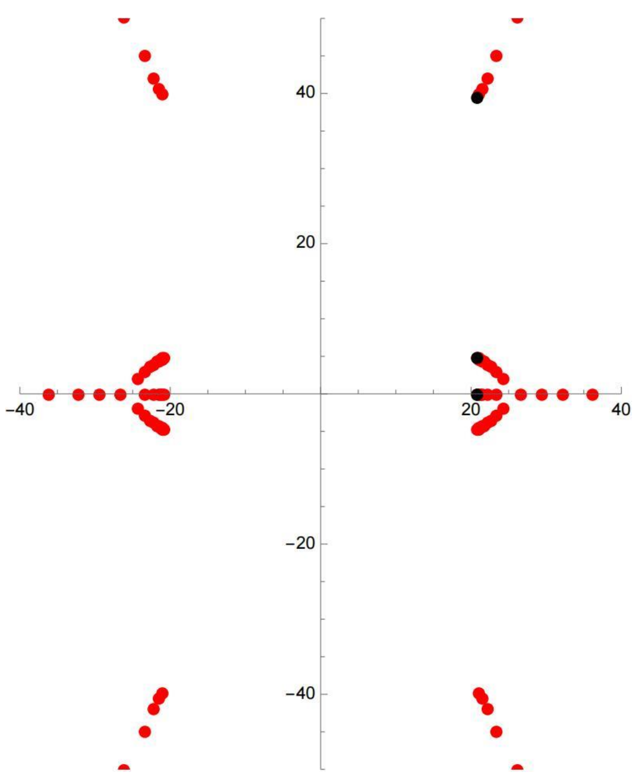
Resurgence theory at work

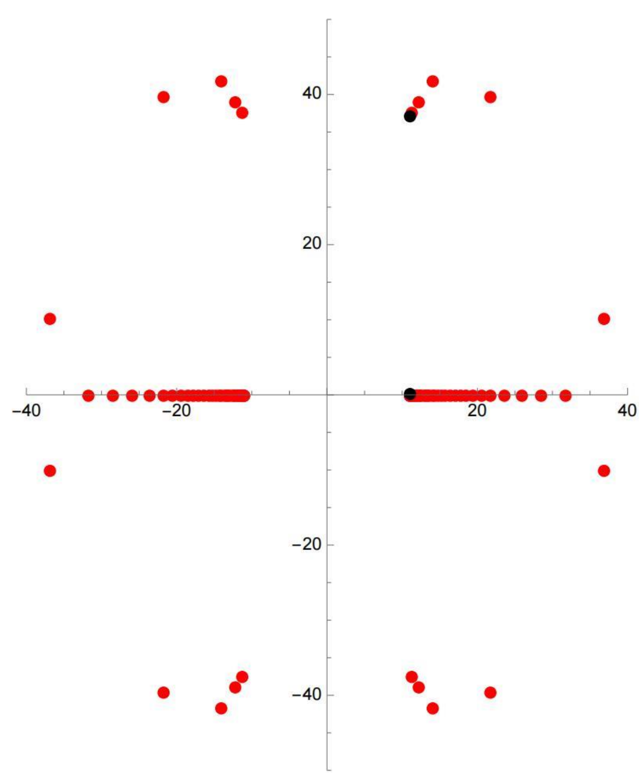
- Borel-(Pade) transform

$$\mathcal{B}E^{(0)}(\zeta) = \sum_{n=0}^{2n_{max}} \frac{a_n^{(0)}}{n!}\zeta^n \approx \frac{P_{n_{max}}(\zeta)}{Q_{n_{max}}(\zeta)}$$

- Borel plane of Wilson loop vev [Mariño, Schwick’ 24; Gu, ZX’ 24]

$z < \frac{1}{16}$





$z > \frac{1}{16}$

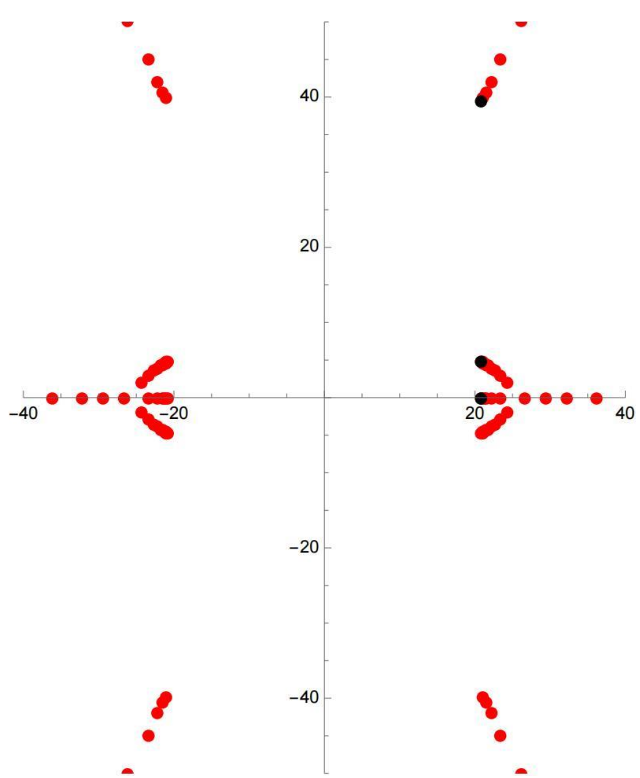
The resurgence/BPS correspondence

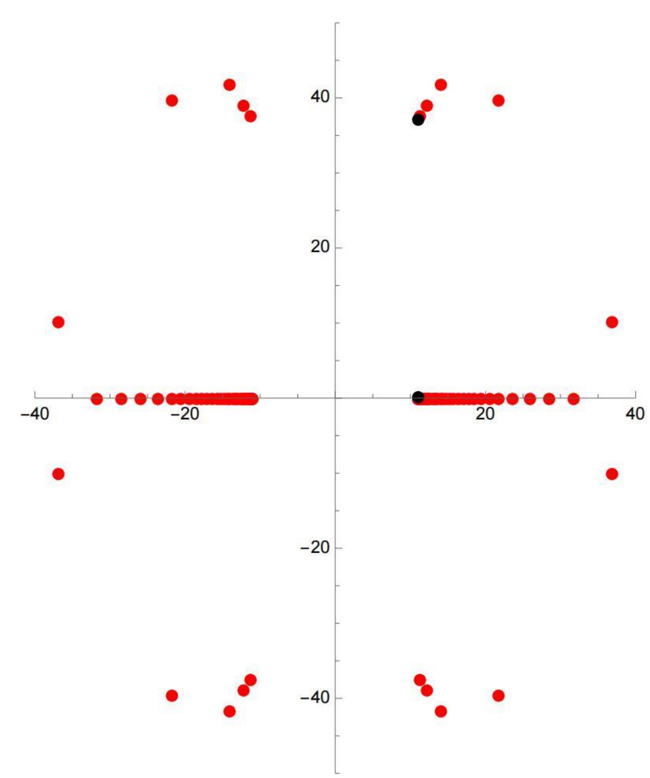
- The resurgence/BPS dictionary [Grassi, Gu, Mariño' 19]

Resurgence	BPS
$t_a^{(0)}$	Z_{γ_a}
1-cycles γ_a	EM charges γ_a
Borel singularities	BPS spectrum
Stokes constant S_γ	BPS invariant $\Omega_\gamma < \gamma_d, \gamma >$
DDP formula	KS morphism

- Borel plane of Wilson loop vev [Mariño, Schwick' 24; Gu, ZX' 24]

$z < \frac{1}{16}$





$z > \frac{1}{16}$

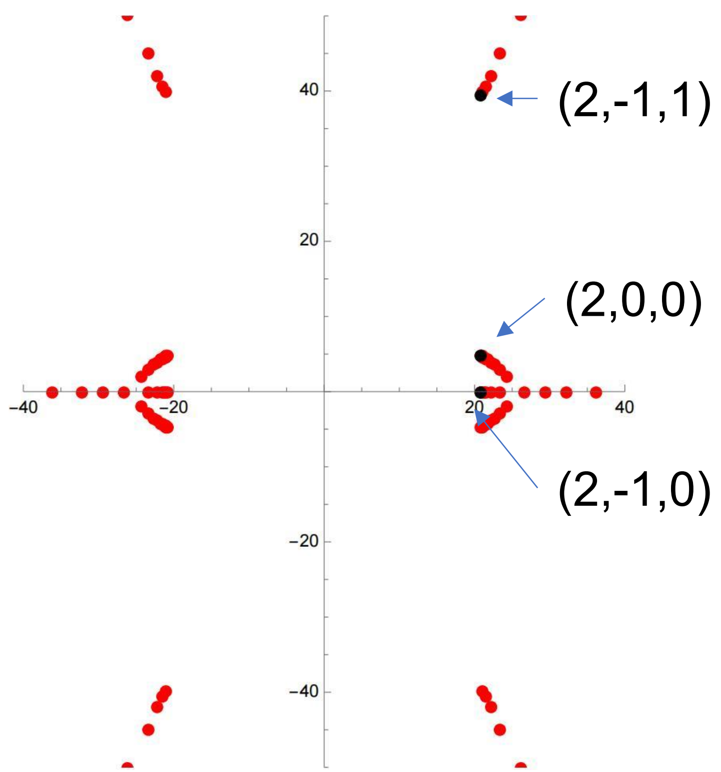
The resurgence/BPS correspondence

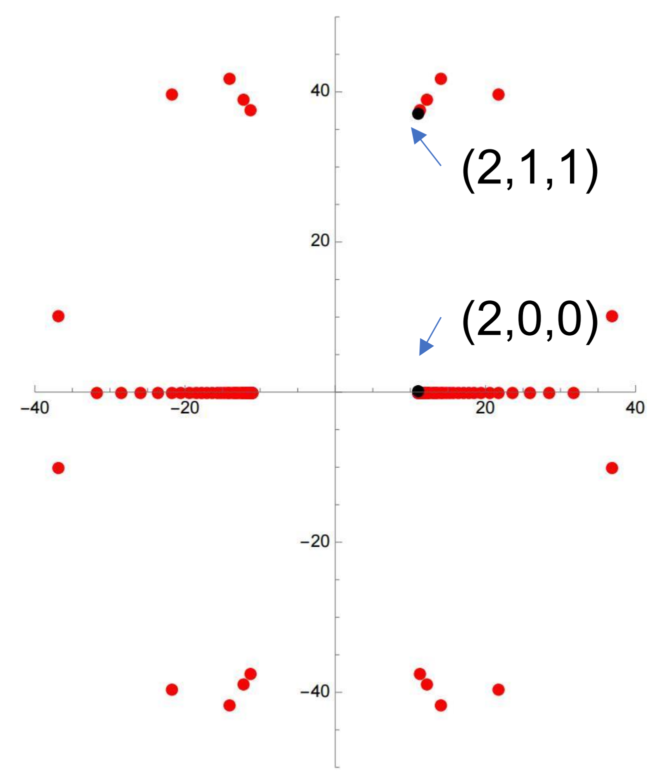
- The resurgence/BPS dictionary [Grassi, Gu, Mariño' 19]

Resurgence	BPS
$t_a^{(0)}$	Z_{γ_a}
1-cycles γ_a	EM charges γ_a
Borel singularities	BPS spectrum
Stokes constant S_γ	BPS invariant $\Omega_\gamma < \gamma_d, \gamma >$
DDP formula	KS morphism

- Borel plane of Wilson loop vev [Mariño, Schwick' 24; Gu, ZX' 24]

$z < \frac{1}{16}$



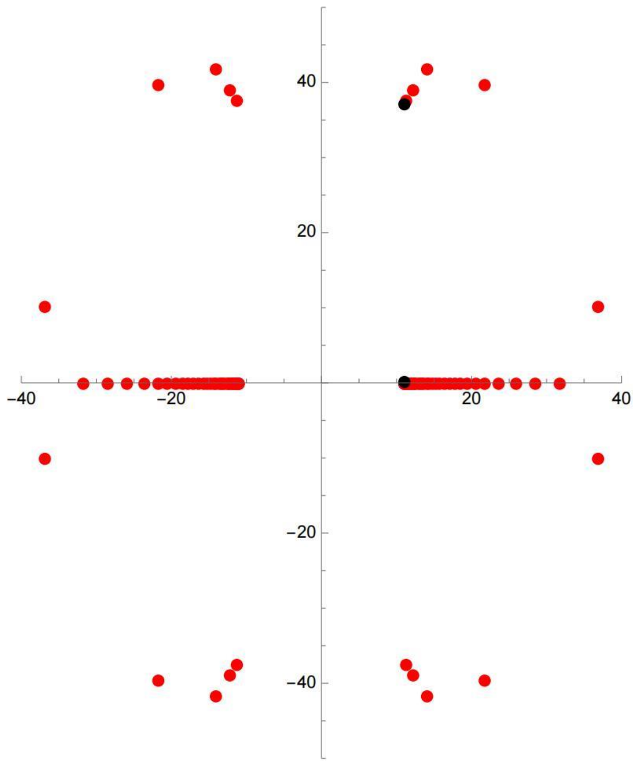


$z > \frac{1}{16}$

The resurgence/BPS correspondence

- The energy for finite magnetic flux is bounded by $-4 < E < 4$, which corresponds to the strong coupling regime $z = \frac{1}{E^2} > \frac{1}{16}$

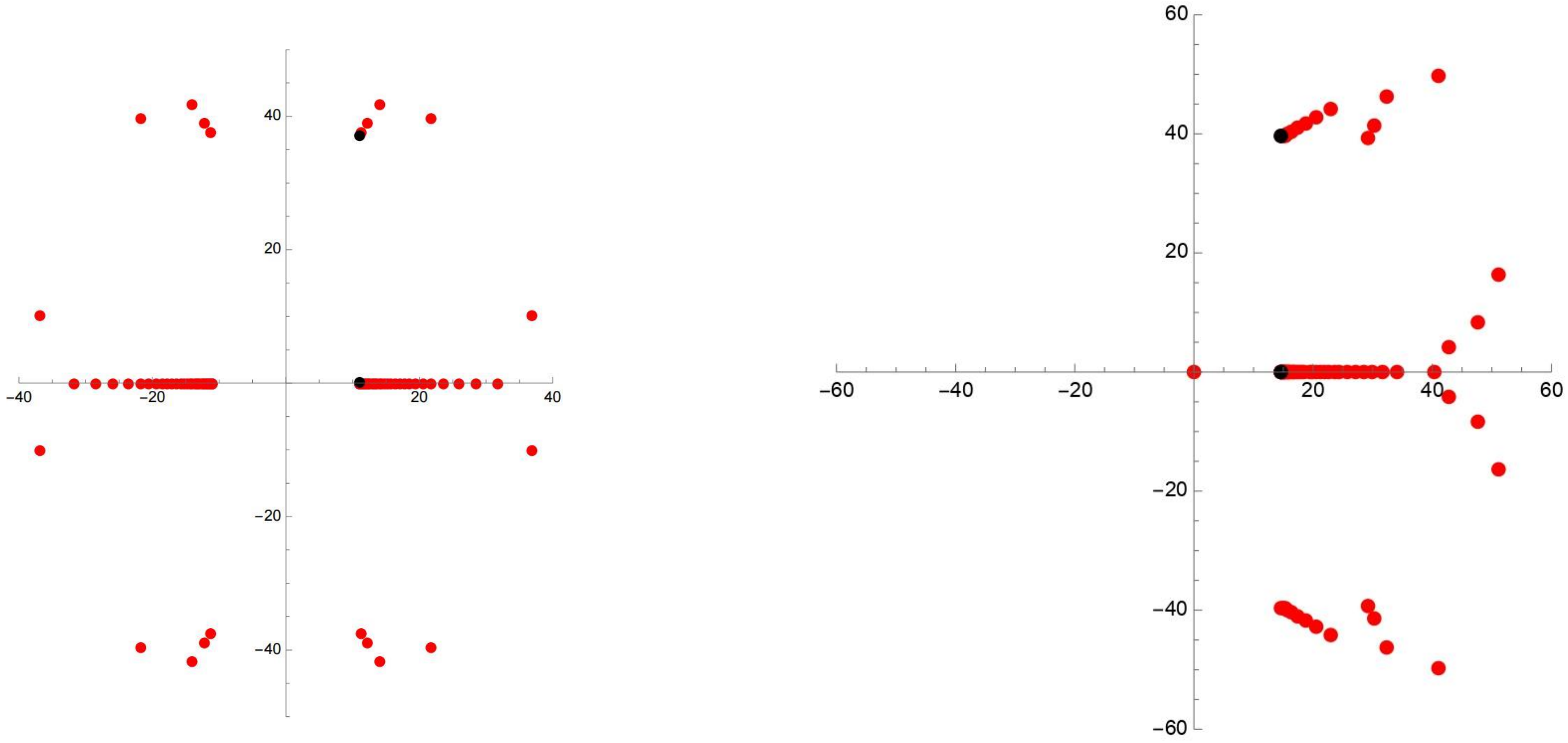
$z > \frac{1}{16}$



The resurgence/BPS correspondence

- The energy for finite magnetic flux is bounded by $-4 < E < 4$, which corresponds to the strong coupling regime $z = \frac{1}{E^2} > \frac{1}{16}$

$z > \frac{1}{16}$

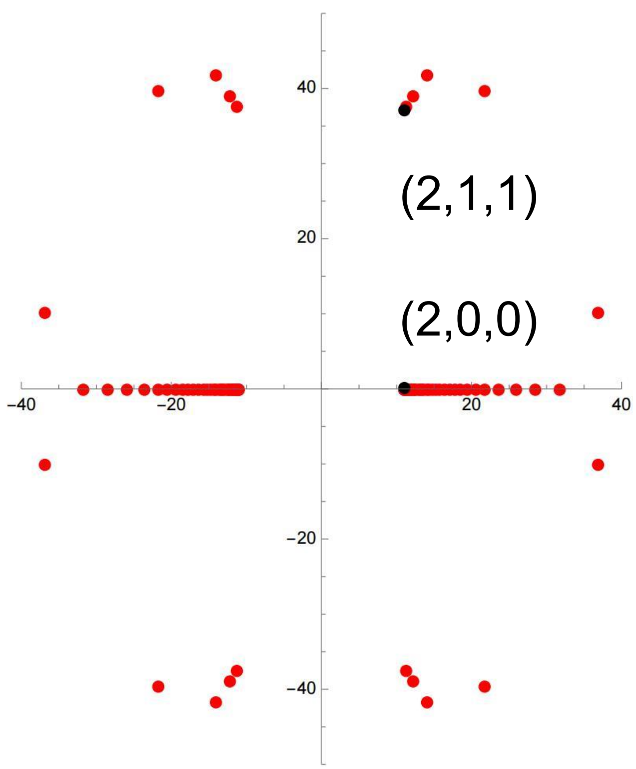


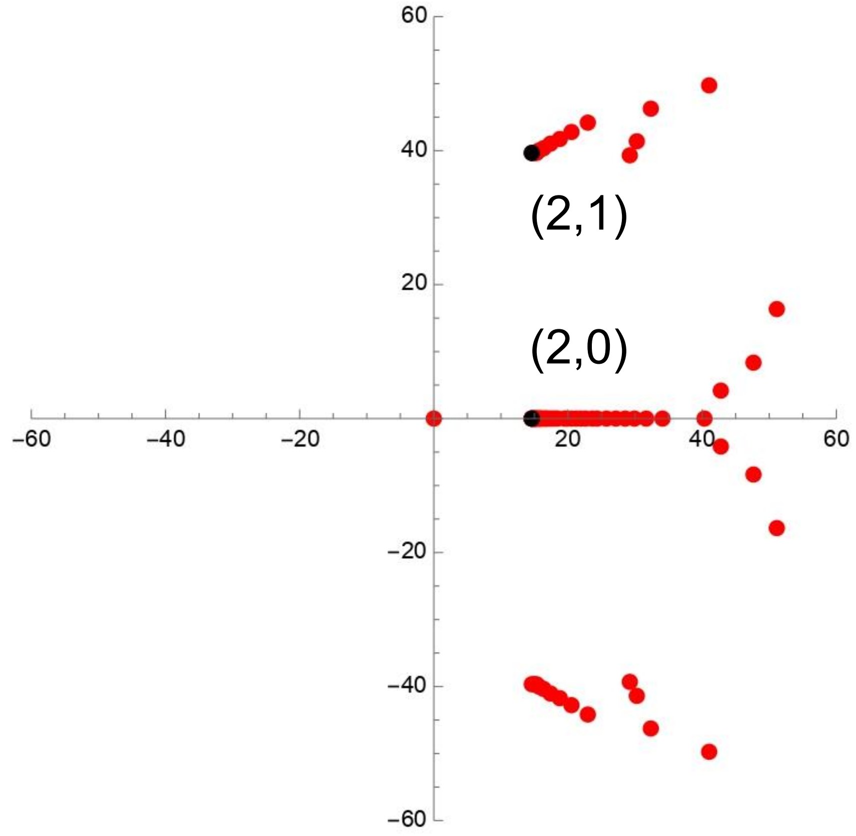
- After the identification $\hbar \to -\phi$, $t_c \to -\phi\left(N + \frac{1}{2}\right)$ and take the limit $\phi \to 0$, BPS states labeled by (p, q, r) gets mapped Borel singularities labeled by (p, r)

The resurgence/BPS correspondence

- The energy for finite magnetic flux is bounded by $-4 < E < 4$, which corresponds to the strong coupling regime $z = \frac{1}{E^2} > \frac{1}{16}$

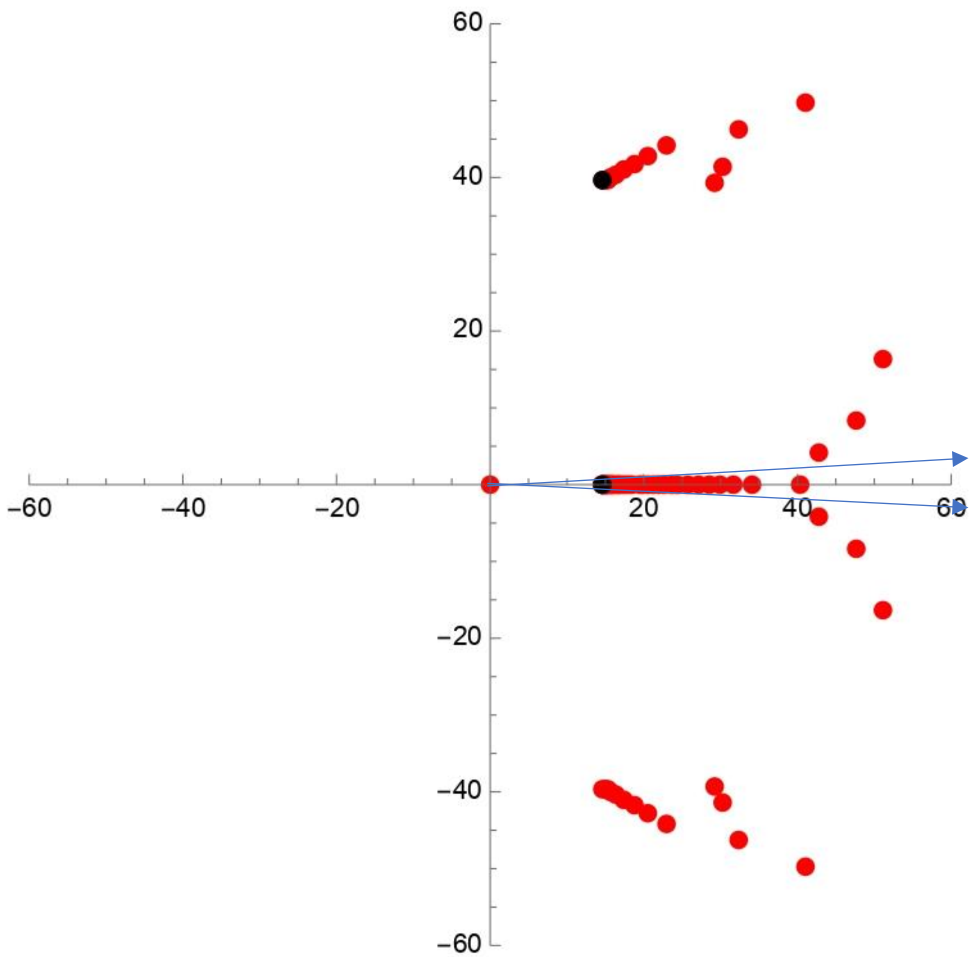
$z > \frac{1}{16}$





- After the identification $\hbar \to -\phi$, $t_c \to -\phi\left(N + \frac{1}{2}\right)$ and take the limit $\phi \to 0$, BPS states labeled by (p, q, r) gets mapped Borel singularities labeled by (p, r)

Resurgence theory at work



- Lateral Borel resummation

$$\mathcal{S}_0^{(\pm)} E^{(0)}(\phi) = \frac{1}{\phi} \int_{C_\pm} d\zeta \, e^{-\zeta/\phi} \, \mathcal{B}E^{(0)}(\zeta)$$

- The resumed energy has an imaginary ambiguity which can be cancelled by including even instanton corrections whose information is encoded in Stokes discontinuities

$$\left(\mathcal{S}_0^{(+)} - \mathcal{S}_0^{(-)}\right) E^{(0)}(\phi)$$

$$= \mathcal{S}_0^{(-)} (\mathfrak{S}_0 - id) E^{(0)}(\phi)$$

Alien calculus

- The Stokes automorphism turns series into minimal *trans-series*

$$\mathfrak{S}_0 E^{(0)} = E^{(0)} + \sum_{n\geq 1} \sum_{m=0}^{n-1} v_{n,m} \sigma^{m+1} E^{(n,m)}$$

- The Stokes automorphism can be decomposed in terms of *pointed alien derivatives*

$$\mathfrak{S}_0 = \exp\left(\sum_k \dot{\Delta}_{kA} \right)$$

- Later we will show that

$$\mathfrak{S}_0 E^{(0)} = \mathfrak{S}_0 E_{min}(\sigma = 0) = E_{min}(\sigma = 4)$$

and

$$\dot{\Delta}_{lA} E^{(0)} = \frac{4}{2\pi i} \frac{(-1)^l}{l} E^{(2l,0)}$$

$$\dot{\Delta}_{lA} E^{(n,m)} = \frac{4}{2\pi i} \frac{(-1)^l}{l} E^{(n+2l,m+1)}$$

Failure of strong resurgence program

- The bands of Hofstadter's butterfly is invisible from perturbative expansion. In addition, the minimal trans-series resurged from perturbative series does not depend on the Bloch angles θ_x, θ_y either.

Failure of strong resurgence program

- The bands of Hofstadter's butterfly is invisible from perturbative expansion. In addition, the minimal trans-series resurged from perturbative series does not depend on the Bloch angles θ_x, θ_y either.

- A preliminary analysis on the full 1-instanton and 2-instanton sector was performed in [Duan, Gu, Hatsuda, Sulejmanpasic' 18] both analytically and numerically

Failure of strong resurgence program

- The bands of Hofstadter's butterfly is invisible from perturbative expansion. In addition, the minimal trans-series resurged from perturbative series does not depend on the Bloch angles θ_x, θ_y either.

- A preliminary analysis on the full 1-instanton and 2-instanton sector was performed in [Duan, Gu, Hatsuda, Sulejmanpasic' 18] both analytically and numerically

- The technical difficulty of extracting higher instanton corrections directly was due to instanton-anti-instanton interaction terms

$$E^{(n)}(N) \sim \mathcal{P}(\phi, N)\left\{\left[\log\left(\frac{16}{\phi}\right)\right]^{n-1} - (n-1)\left[\log\left(\frac{16}{\phi}\right)\right]^{n-2}\psi^{(0)}(N+1) + \cdots\right\} e^{-n\frac{S_c}{\phi}}$$

Failure of strong resurgence program

- The bands of Hofstadter's butterfly is invisible from perturbative expansion. In addition, the minimal trans-series resurged from perturbative series does not depend on the Bloch angles θ_x, θ_y either.

- A preliminary analysis on the full 1-instanton and 2-instanton sector was performed in [Duan, Gu, Hatsuda, Sulejmanpasic' 18] both analytically and numerically

- The technical difficulty of extracting higher instanton corrections directly was due to instanton-anti-instanton interaction terms

$$E^{(n)}(N) \sim \mathcal{P}(\phi, N)\left\{\left[\log\left(\frac{16}{\phi}\right)\right]^{n-1} -(n-1)\left[\log\left(\frac{16}{\phi}\right)\right]^{n-2}\psi^{(0)}(N+1)+\cdots\right\} e^{-n\frac{S_c}{\phi}}$$

- To go further, we need insights from exact WKB analysis

Review of cosine potential model

- The quantum SW curve of 4d $\mathcal{N}$=2 pure SYM can reduce to the Hamiltonian of the cosine potential model

$$H = \boldsymbol{y}^2 + 2\cos \boldsymbol{x}$$

with

$$[\boldsymbol{x}, \boldsymbol{y}] = i\hbar$$

- Can be solved alternatively by exact WKB method

$$H\psi(x) = E\psi(x)$$

with

$$\psi_{\pm}(x) = \frac{1}{\sqrt{P_{odd}}} \exp\left(\pm \frac{i}{\hbar} \int^{x} P_{odd}(x', \hbar) dx'\right)$$

satisfying the periodic bc

$$\begin{pmatrix} \psi_+(x+2\pi) \\ \psi_-(x+2\pi) \end{pmatrix} = M \begin{pmatrix} \psi_+(x) \\ \psi_-(x) \end{pmatrix} \qquad \det(M - \mathrm{I}_{2\times 2} e^{i\theta}) = 0$$

Review of cosine potential model

- The exact quantization condition (EQC) for cos potential model is simply the Voros-Silverstone connection formula [Voros' 83, Silverstone' 85] that can be derived by Stokes graphs/spectral network techniques

$$1 + V_A^{\mp}(1 + V_B) - 2\sqrt{V_A^{\mp} V_B}\cos\theta = 0$$

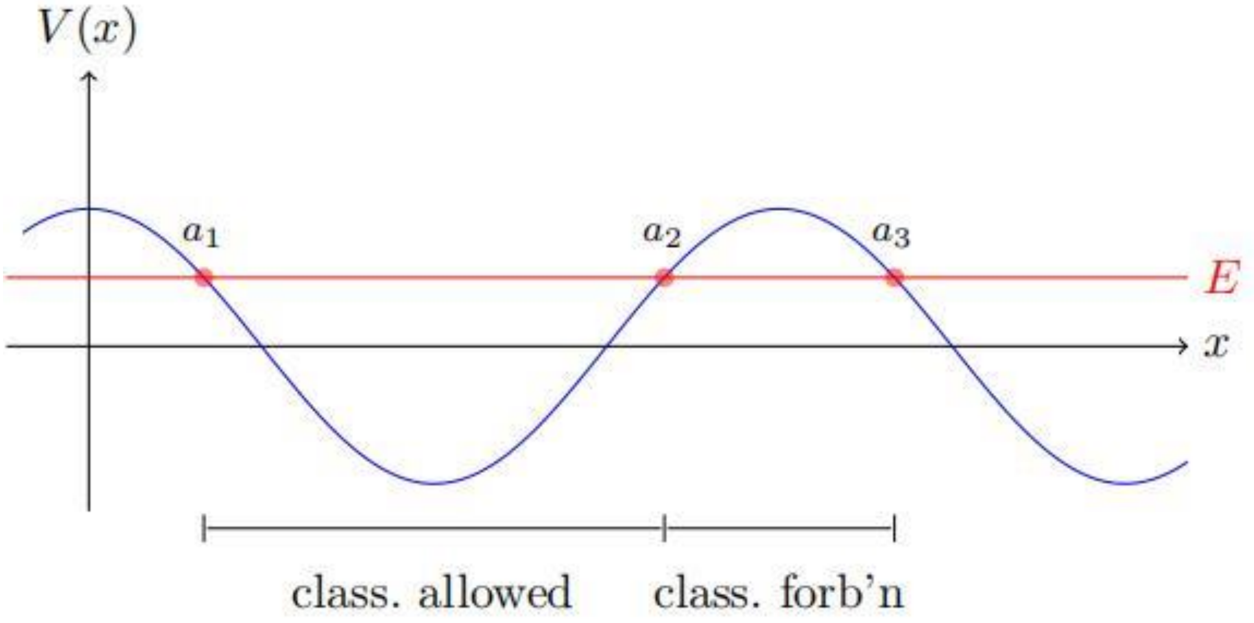
where

$$V_A = e^{2\pi i \frac{t(E,\hbar)}{\hbar}}, \qquad V_B = e^{-\frac{t_D(E,\hbar)}{\hbar}}$$

and

$$t = \frac{1}{\pi}\int_{a_1}^{a_2} P_{odd}(x,\hbar)dx$$

$$t_D = -2i\int_{a_2}^{a_3} P_{odd}(x,\hbar)dx$$



Review of cosine potential model

- $t = \left(N + \frac{1}{2}\right)\hbar$ solves the perturbative quantization condition
$$1 + V_A^{\mp} = 0$$
- In order to solve the exact quantization condition
$$1 + V_A^{\mp}(1 + V_B) - 2\sqrt{V_A^{\mp} V_B}\cos\theta = 0,$$
we have to promote the period to instanton *trans-series*
$$\hat{t} = t + \Delta t$$
- Δt satisfies the implicit equation
$$\Delta t = \mp\frac{\hbar}{\pi}\log\left(\pm \mathrm{i}\cos\theta\,\lambda(t + \Delta t) + \sqrt{1 + \sin^2\theta\,\lambda^2(t + \Delta t)}\right)$$

with $\lambda = V_B^{1/2} = e^{-\frac{t_D(E(t),\hbar)}{2\hbar}}$

Van Spaendonck-Vonk universal trans-series structure

- From the implicit EQC, the full trans-series can be determined by the Lagrange inversion thm [van Spanendonck, Vonk' 23]

$$\hat{E}(N,\hbar) = E^{(0)}(N,\hbar) + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} u_{n,m}(\theta,\epsilon)E^{(n,m)}(N,\hbar)$$

with

$$E^{(n,m)}(N,\hbar) = \left(\frac{\partial}{\partial N}\right)^m \left(\frac{\partial E^{(0)}}{\partial N} e^{-n\frac{t_D}{2\hbar}}\right)$$

$$u_{n,m}(\theta,\epsilon) = \frac{1}{n!}B_{n,m+1}(1!\,r_1, 2!\,r_2, \dots, (n-m)!\,r_{n-m})$$

where r_i is the series coefficient of Δt in terms of λ

Van Spaendonck-Vonk universal trans-series structure

- From the implicit EQC, the full trans-series can be determined by the Lagrange inversion thm [van Spanendonck, Vonk' 23]

$$\hat{E}(N,\hbar) = E^{(0)}(N,\hbar) + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} u_{n,m}(\theta,\epsilon)E^{(n,m)}(N,\hbar)$$

with

$$E^{(n,m)}(N,\hbar) = \left(\frac{\partial}{\partial N}\right)^m \left(\frac{\partial E^{(0)}}{\partial N} e^{-n\frac{t_D}{2\hbar}}\right)$$

$$u_{n,m}(\theta,\epsilon) = \frac{1}{n!}B_{n,m+1}(1!\,r_1, 2!\,r_2, \ldots, (n-m)!\,r_{n-m})$$

where r_i is the series coefficient of Δt in terms of λ

- Turns out to be universal for a wide class of 1d QM models (been tested for cubic potential, double well model and cosine potential model)

Van Spaendonck-Vonk universal trans-series structure

- The minimal trans-series can be determined by the DDP formula [Delabaere, Dillinger, Pham' 97]

$$\hat{E}_{min}(\hbar,\sigma;N) = E^{(0)}(N,\hbar) + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} v_{n,m}(\theta,\sigma)E^{(2n,m)}(\hbar;N)$$

with

$$E^{(n,m)}(N,\hbar) = \left(\frac{\partial}{\partial N}\right)^m \left(\frac{\partial E^{(0)}}{\partial N} e^{-n\frac{t_D}{2\hbar}}\right)$$

$$v_{n,m}(\theta,\sigma) = \frac{\sigma^{m+1}}{n!} B_{n,m+1}(1!\, s_1, 2!\, s_2, \dots, (n-m)!\, s_{n-m})$$

where

$$s_j = \frac{1}{2\pi \mathrm{i}}\frac{(-1)^j}{j}$$

- The energy at van Hove singularity for $\phi = 2\pi/Q$ match with Borel resumed minimal trans-series

$$E^{VHS}(\phi = 2\pi/Q) = \mathcal{S}_0^{(\pm)} E_{min}(\hbar = -\phi, \sigma = \mp 2)$$

Van Spaendonck-Vonk universal trans-series structure

- The full trans-series can be factorized into minimal trans-series and medium trans-series [van Spanendonck, Vonk’ 23]

$$Full\ transseries \simeq minimal\ transseries \otimes medium\ transseires$$

- for energy trans-series, this tensor factorization is manifested as

$$\hat{E}\ (N,\hbar) = E_{\min}\ (N,\hbar,\sigma) + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} w_{n,m}(\theta)E_{\min}^{(n,m)}(N,\hbar,\sigma)$$

where $w_{n,m}(\theta)$ are the trans-series coefficients for medium trans-series

Reverse engineering the exact WKB QC for Harper-Hofstadter model

- The WKB QC for 5d SW curve/Harper-Hofstadter model is difficult to derive. For example, it's hard to generalize the success of the ODE/IM correspondence [Dorey, Tateo' 98 …, Ito, Mariño, Shu' 18] to systems governed by difference equations. Nevertheless, we can still numerically extract the trans-series coefficients from the Hofstadter's butterfly for certain values of the flux

- For $E^{(0)}(\phi, N)$, the BW algorithm is sufficient to extrapolate to $O(\phi^{200})$. For the computation of t_D, we have to rely on refined holomorphic anomaly equation [BCOV 93, Huang, Klemm' 10, Krefl, Walcher' 10]

Holomorphic Anomaly Equations

- The holomorphic anomaly equations are useful tools of calculating quantum free energy $\mathrm{F}_{NS}(t_c;\hbar) := \sum_n F_n(t_c)\hbar^{2n}$ and the dual period recursively.
- Refined HAE in NS limit for genus-1 curves simplifies to

$$\frac{\partial F_n}{\partial S} = -\frac{1}{2}\sum_{r=1}^{n-1} D_{t_c}F_r\, D_{t_c}F_{n-r}$$

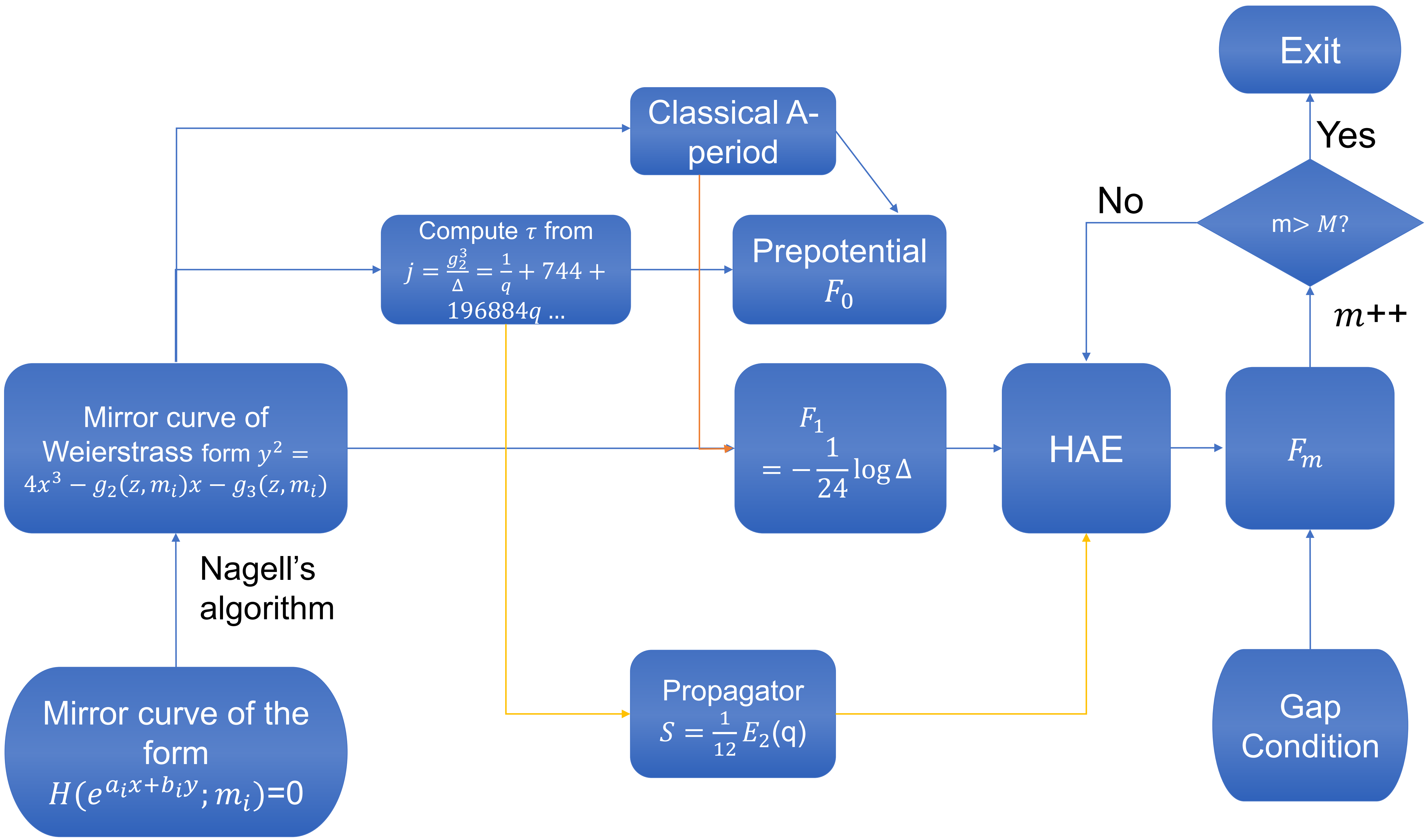
- The algorithm should be implemented by the first level of NS free energy, which for local F_0 is simply

$$F_1 = -\frac{1}{24}\log\Delta$$

- The integration constants are fixed by

$$F_n^{sing}(t_c) = \frac{(1-2^{1-2n})B_{2n}}{(2n)(2n-1)(2n-2)t_c^{2n-2}}, \qquad n \geq 2$$

Algorithm of calculating F_m up to order M



Full trans-series

- With the series expansion of $E^{(0)}$ and t_D ready, we are ready to extract the trans-series coefficients by comparing the spectrum at flux $\phi = \frac{2\pi\, P}{Q}$, with $\gcd(P,Q)=1$

- For $\phi = \frac{2\pi}{Q}$, we find the first few coefficients up to 6-instanton orders [Gu, ZX' 24]

m	0	1	2
$w_{1,m}$	$\frac{\Theta}{\pi}$		
$w_{2,m}$	0	$\frac{\Theta^2}{2\pi^2}$	
$w_{3,m}$	$-\frac{\Theta}{\pi}+\frac{\Theta^3}{6\pi}$	0	$\frac{\Theta^3}{6\pi^3}$

with $\Theta = (-1)^{N+1}(\cos\ \theta_x + \cos\theta_y)$

Full trans-series

- With the series expansion of $E^{(0)}$ and t_D ready, we are ready to extract the trans-series coefficients by comparing the spectrum at flux $\phi = \frac{2\pi P}{Q}$, with $\mathrm{gcd}(P,Q)=1$

- For $\phi = \frac{2\pi}{Q}$, we find the first few coefficients up to 6-instanton orders [Gu, ZX' 24]

m	0	1	2
$u_{1,m}$	$\frac{\Theta}{\pi}$		
$u_{2,m}$	$\frac{\mathrm{i}\epsilon}{\pi}$	$\frac{\Theta^2}{2\pi^2}$	
$u_{3,m}$	$-\frac{\Theta}{\pi}+\frac{\Theta^3}{6\pi}$	$\frac{\mathrm{i}\epsilon\Theta}{\pi^2}$	$\frac{\Theta^3}{6\pi^3}$

with $\Theta = (-1)^{N+1}(\cos\ \theta_x + \cos\theta_y)$

Full trans-series

- We conjectured a formula for all coefficients

$$u_{n,m}\left(\theta_{x,y},\epsilon\right)=\frac{1}{n!}B_{n,m+1}(1!\,r_1,2!\,r_2,\dots,(n-m)!\,r_{n-m})$$

$$\sum_{j\geq 1} r_j\lambda^j=\frac{i}{\epsilon\pi}\log\left(\sqrt{1+(2-\Theta^2)\lambda^2+\lambda^4}-i\epsilon\Theta\lambda\right)$$

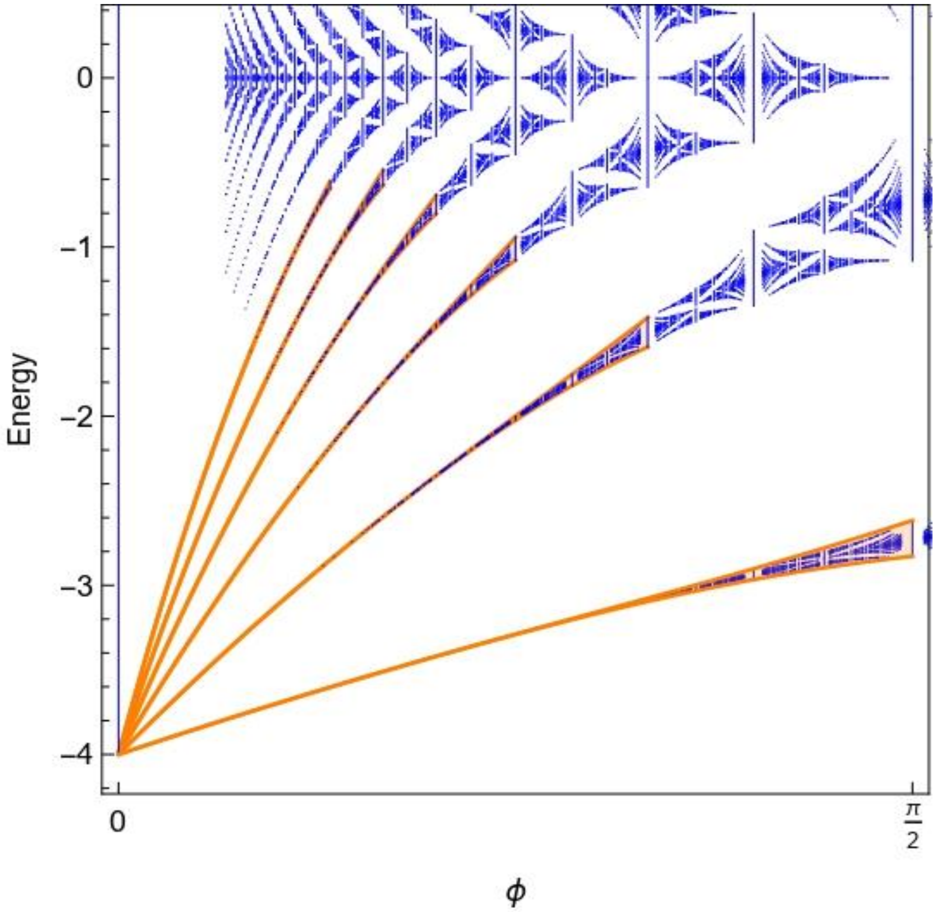
Full trans-series

- We conjectured a formula for all coefficients

$$w_{n,m}(\theta_{x,y}) = \frac{1}{n!} B_{n,m+1}(1!\, t_1, 2!\, t_2, \dots, (n-m)!\, t_{n-m})$$

$$t_i = \frac{1}{\pi}[\lambda^i] \arcsin \frac{\Theta}{\lambda + \lambda^{-1}}$$

- veins of the butterfly and resummed instanton trans-series



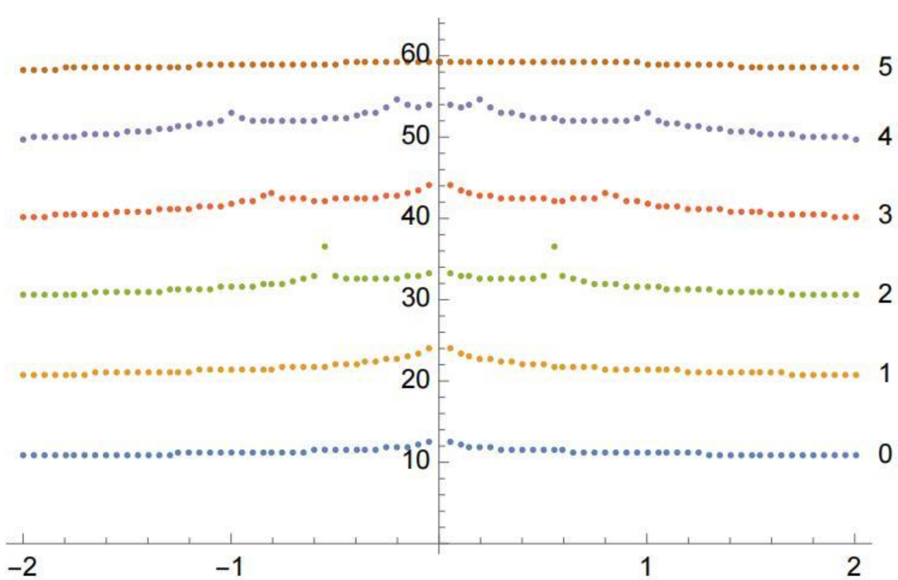
Full trans-series

- We conjectured a formula for all coefficients

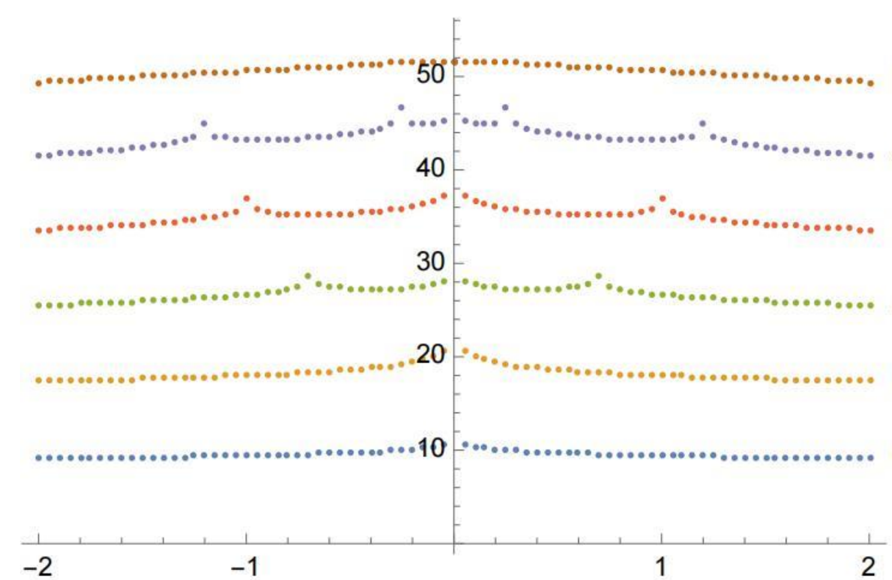
$$w_{n,m}(\theta_{x,y}) = \frac{1}{n!} B_{n,m+1}(1!\,t_1, 2!\,t_2, \dots, (n-m)!\,t_{n-m})$$

$$t_i = \frac{1}{\pi}[\lambda^i] \arcsin\frac{\Theta}{\lambda + \lambda^{-1}}$$

- # of matching digits between exact spectrum and resummed instanton trans-series



$\phi = \frac{2\pi}{23}, \qquad N = 0$



$\phi = \frac{2\pi}{23}, \qquad N = 1$

EQC

- The implied EQC from the generating series

$$D_{\pm}: 1 + V_A^{\mp}(1 + V_B)^2 - 2\sqrt{V_A^{\mp} V_B}\,\Theta = 0$$

and the two lateral conditions are related by the Stokes automorphism

$$\mathfrak{S}_0 D_+ = D_-$$

which is a consequence of

$$\mathfrak{S}_0\, V_A = V_A(1 + V_B)^4$$

EQC

- The implied EQC from the generating series

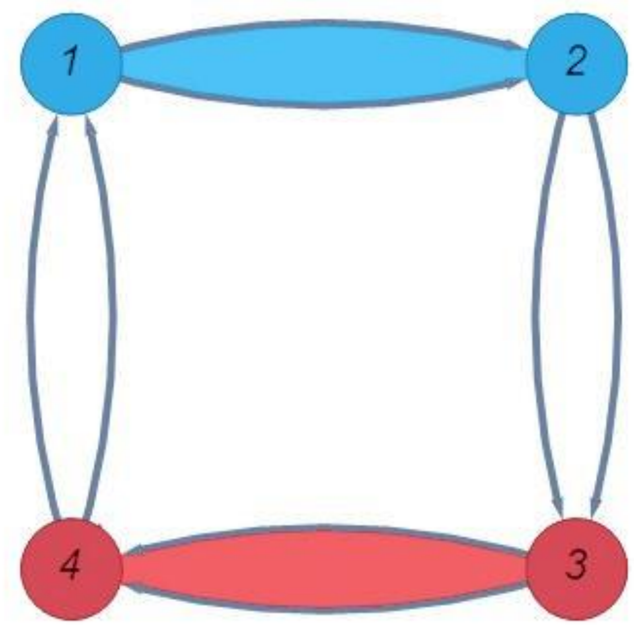
$$D_{\pm}: 1 + V_A^{\mp}(1+V_B)^2 - 2\sqrt{V_A^{\mp} V_B}\,\Theta = 0$$

and the two lateral conditions are related by the Stokes automorphism

$$\mathfrak{S}_0 D_+ = D_-$$

which is a consequence of

$$\mathfrak{S}_0\, V_A = V_A(1+V_B)^4$$



- Comparison with 4d case

$$D_{\pm}: 1 + V_A^{\mp}(1+V_B) - 2\sqrt{V_A^{\mp} V_B}\cos\theta = 0$$

$$\mathfrak{S}_0\, V_A = V_A(1+V_B)^2$$

can be understood by the fact that the BPS quiver of 5d SU(2) SYM is a **double copy** of that of 4d SU(2) SYM [Del Monte, Longhi' 21]

EQC

- The implied EQC from the generating series

$$D_{\pm}: 1 + V_A^{\mp}(1 + V_B)^2 - 2\sqrt{V_A^{\mp} V_B}\Theta = 0$$

  and the two lateral conditions are related by the Stokes automorphism

$$\mathfrak{S}_0 D_+ = D_-$$

  which is a consequence of

$$\mathfrak{S}_0\, V_A = V_A(1 + V_B)^4$$

- Consistent with the resurgence/BPS dictionary

$$\mathfrak{S}_0\, V_A = V_A(1 + V_B)^{<\gamma_A, \gamma_B>\Omega_B}$$

  with

$$\gamma_A = (0,1,0), \gamma_B = (2,0,0)$$

EQC

- The implied EQC from the generating series

$$D_{\pm}:\ 1+V_A^{\mp}(1+V_B)^2-2\sqrt{V_A^{\mp}V_B\Theta}=0$$

and the two lateral conditions are related by the Stokes automorphism

$$\mathfrak{S}_0 D_+ = D_-$$

which is a consequence of

$$\mathfrak{S}_0\, V_A = V_A(1+V_B)^4$$

- The medium QC looks more symmetric for 5d

$$D_{med}:\ (1+V_A)(1+V_B)-2\sqrt{V_A V_B}\Theta=0$$

which can be obtained by acting on the full QC by “half” Stokes automorphism

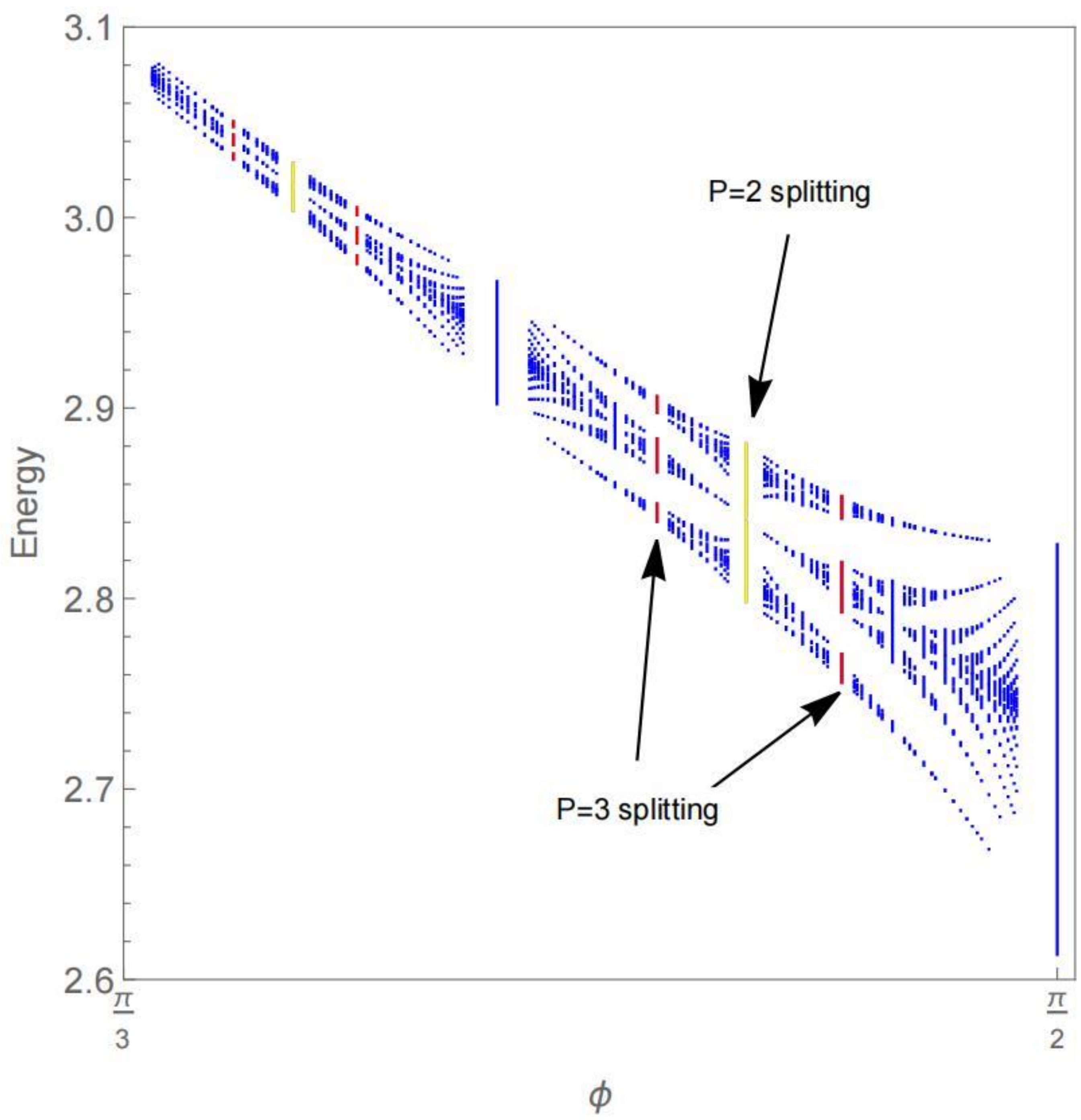
$$\mathfrak{S}_0^{\pm 1/2} D_{\pm} = D_{med}$$

Splitting band phenomenon

- For P>1, the trans-series coefficients are much more complicated

- Within each primary Landau level, there are P subbands

- For $\phi = 2\pi\frac{P}{Q}$, we found self-similarity relation with $\tilde{\phi} = 2\pi\frac{Q}{P}\ mod\ 2\pi$ @ 1-instanton order [Gu, ZX' 24]



Continued fractions and Wilkinson-Rammal formula

- One can avoid dealing with splitting bands by identifying the proper expansion base point for $\alpha = \frac{\phi}{2\pi}$ rational

Continued fractions and Wilkinson-Rammal formula

- One can avoid dealing with splitting bands by identifying the proper expansion base point for $\alpha = \frac{\phi}{2\pi}$ rational

- For any $\alpha \in \mathbb{Q}$, we can express it in terms of the continued fraction

$$\alpha = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cdots \cfrac{1}{n_l}}}$$

and denote it by $[n_0, n_1, n_2, \ldots, n_l]$. Here we require $n_i \in \mathbb{Z}$ and $|n_i| \geq 2$.

Continued fractions and Wilkinson-Rammal formula

- One can avoid dealing with splitting bands by identifying the proper expansion base point for $\alpha = \frac{\phi}{2\pi}$ rational

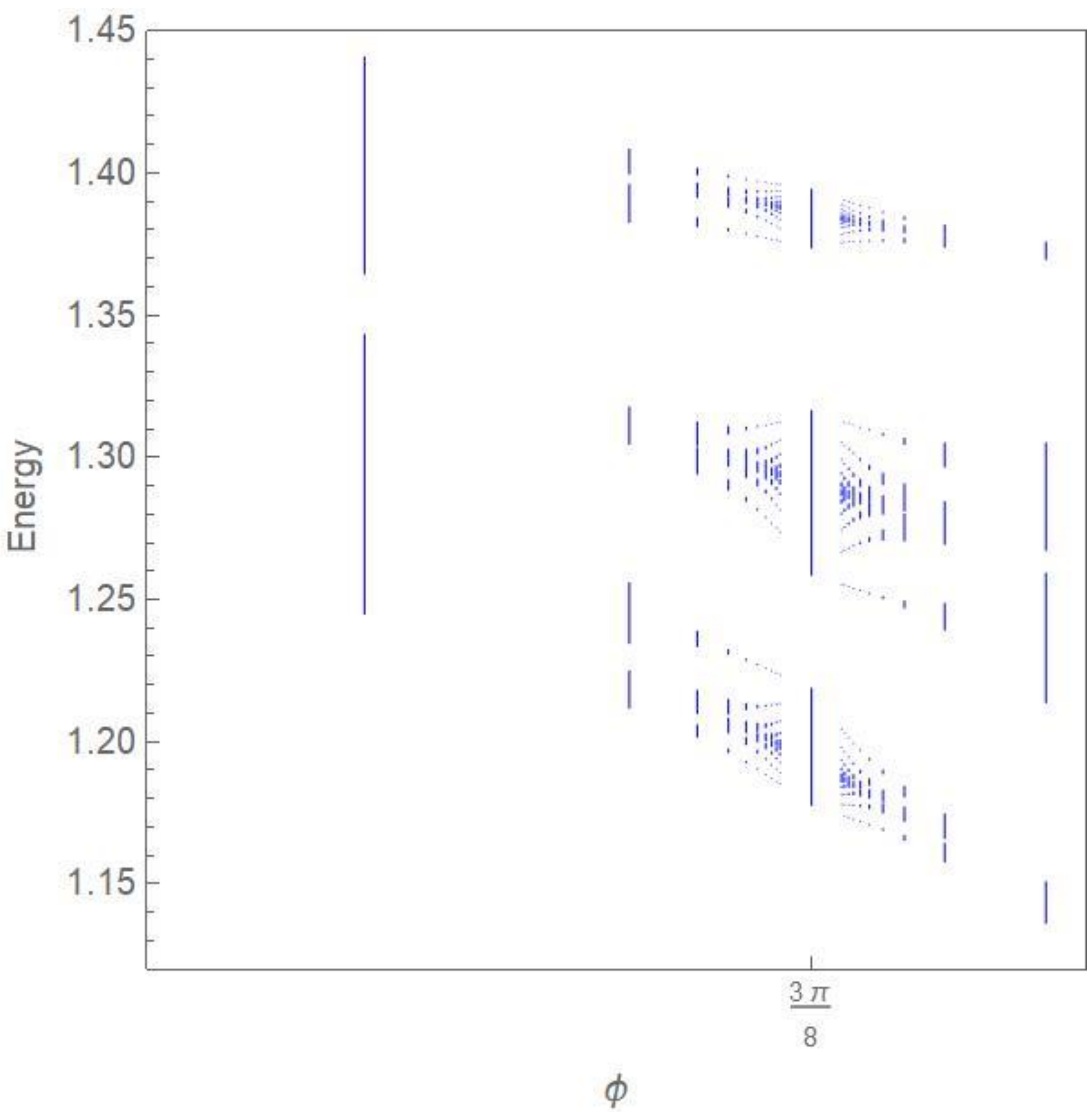
- For any $\alpha \in \mathbb{Q}$, we can express it in terms of the continued fraction

$$\alpha = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cdots \cfrac{1}{n_l}}}$$

and denote it by $[n_0, n_1, n_2, \ldots, n_l]$. Here we require $n_i \in \mathbb{Z}$ and $|n_i| \geq 2$.

- For the bands at $[n_0, n_1, n_2, \ldots, n_l]$, their proper expansion base point should be $[n_0, n_1, n_2, \ldots, n_{l-1}]$.

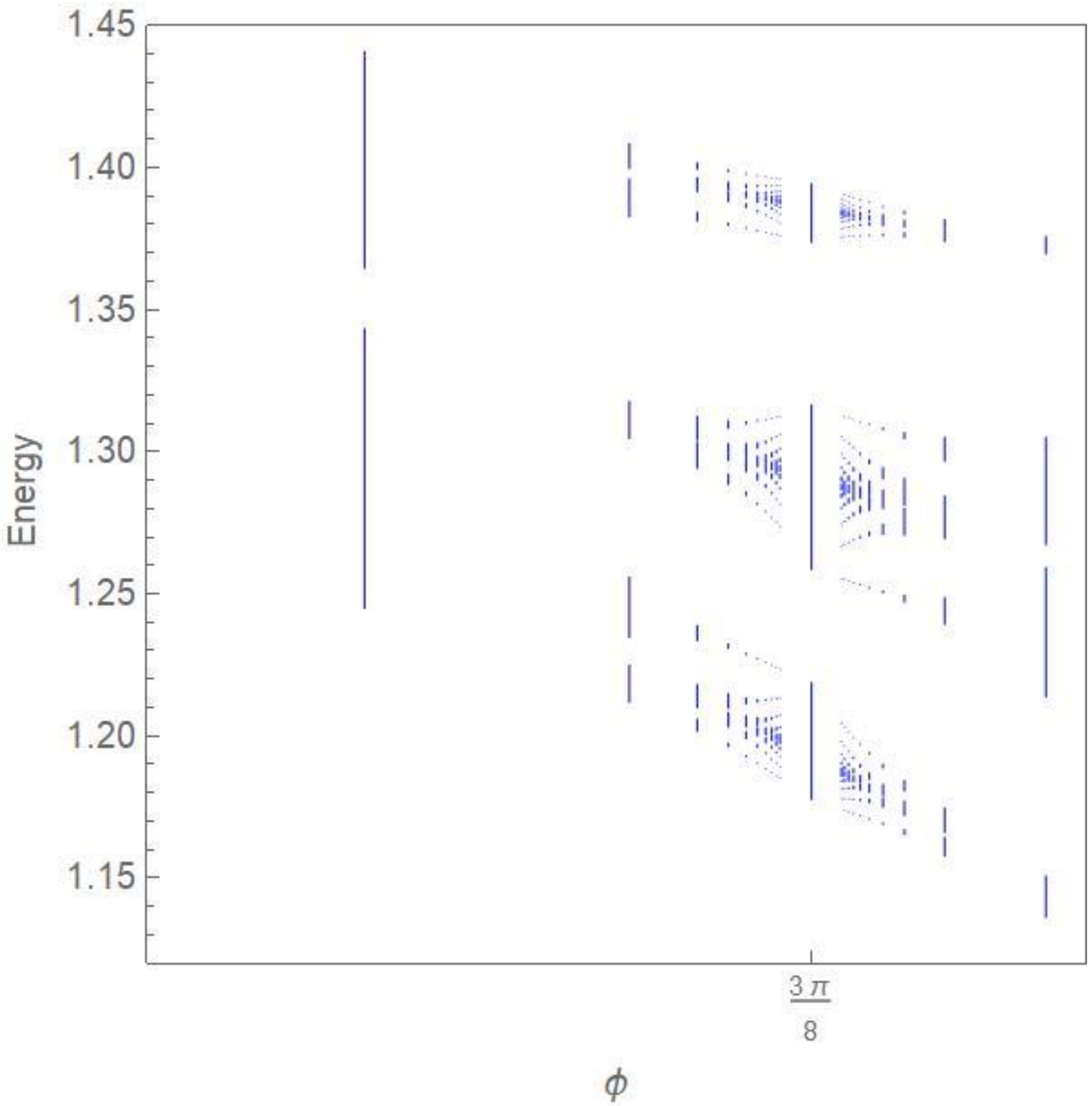
Continued fractions and Wilkinson-Rammal formula



Bands for $\alpha = [0,5,3,n_3]$ near $\alpha_0 = [0,5,3]$

- The perturbative expansion near general α was worked out by [Wilkinson' 84, Rammal, Bellisard' 90]

Continued fractions and Wilkinson-Rammal formula

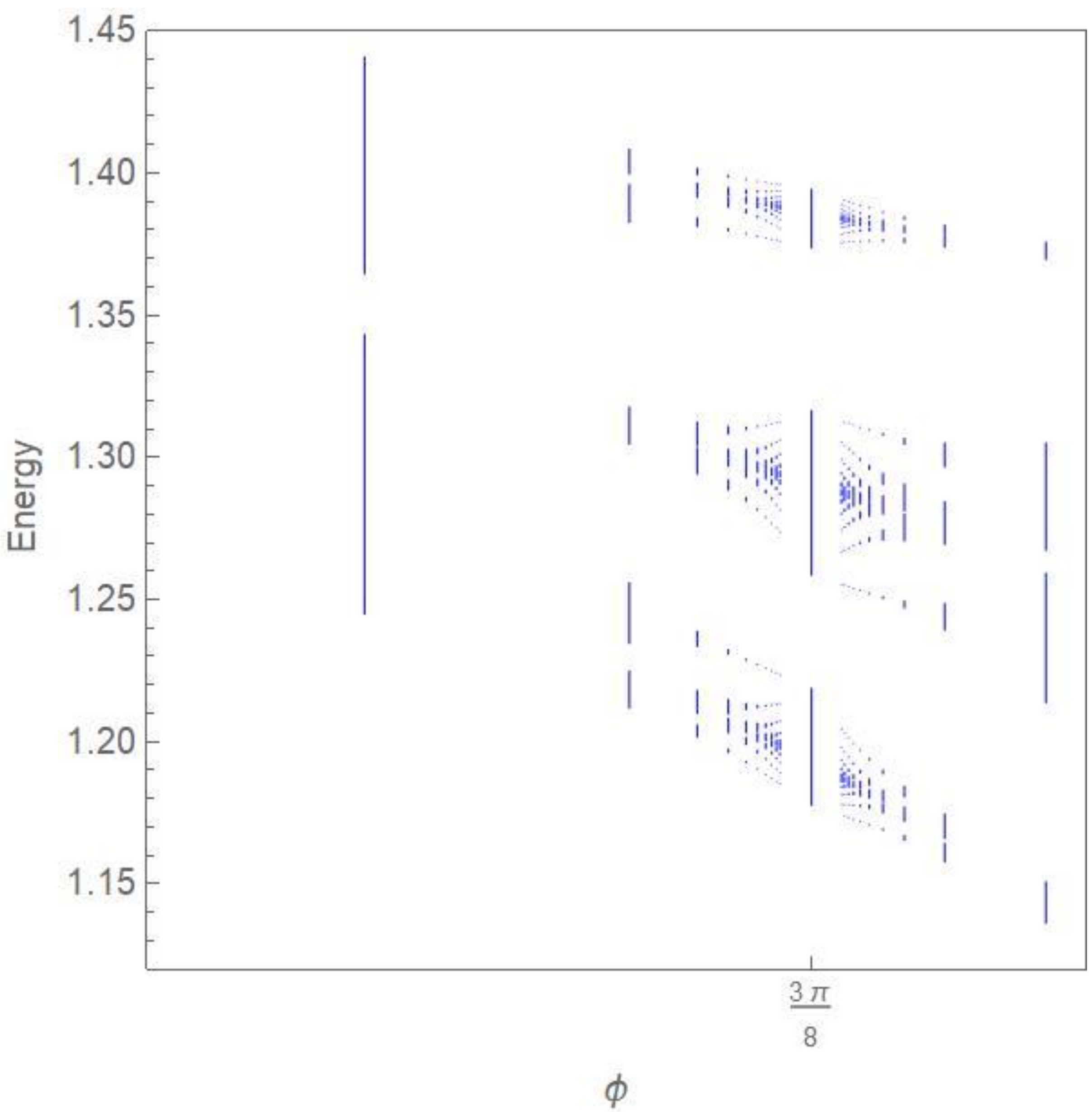


Bands for $\alpha = [0,5,3,n_3]$ near $\alpha_0 = [0,5,3]$

- The perturbative expansion near general α was worked out by [Wilkinson' 84, Rammal, Bellisard' 90], but only to the first order

- By numerical fitting using Richardson extrapolation, we are able to calculate the perturbative expansion near $\alpha_0 = \frac{1}{2} = [0,2]$ and $\alpha_0 = \frac{1}{3} = [0,3]$ up to the second order

Continued fractions and Wilkinson-Rammal formula



Bands for $\alpha = [0,5,3,n_3]$ near $\alpha_0 = [0,5,3]$

- The perturbative expansion near general α was worked out by [Wilkinson' 84, Rammal, Bellisard' 90], but only to the first order
- By numerical fitting using Richardson extrapolation, we are able to calculate the perturbative expansion near $\alpha_0 = \frac{1}{2} = [0,2]$ and $\alpha_0 = \frac{1}{3} = [0,3]$ up to the second order
- Need to find a way to calculate those expansions systematically

  (Generalized Bender-Wu?)

Generalization to local $\mathbb{P}^2$

- Local $\mathbb{P}^2$ geometry is the vacuum manifold of the GLSM with the scalar potential

$$U \supset \frac{e^2}{2}(|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 - 3|\phi_0|^2 - r)^2$$

Generalization to local $\mathbb{P}^2$

- Local $\mathbb{P}^2$ geometry is the vacuum manifold of the GLSM with the scalar potential
$$U \supset \frac{e^2}{2}\left(|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 - 3|\phi_0|^2 - r\right)^2$$
- Under local mirror symmetry [CKYZ 99, Hori, Vafa' 00], the twisted superpotential subject to the constraint on twisted chiral superfields would give us the mirror curve
$$H = e^x + e^y + e^{-x-y} \quad (m = 1)$$

Generalization to local $\mathbb{P}^2$

- Local $\mathbb{P}^2$ geometry is the vacuum manifold of the GLSM with the scalar potential

$$U \supset \frac{e^2}{2}(|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 - 3|\phi_0|^2 - r)^2$$

- Under local mirror symmetry [CKYZ 99, Hori, Vafa' 00], the twisted superpotential subject to the constraint on twisted chiral superfields would give us the mirror curve

$$H = e^x + e^y + e^{-x-y} \quad (m = 1)$$

- We are interested in the spectral problem of taking imaginary slices of the curve and study the quantized curve, i.e.

$$H = e^{i\boldsymbol{x}} + e^{i\boldsymbol{y}} + e^{-i\boldsymbol{x}-i\boldsymbol{y}}$$

with

$$[\boldsymbol{x}, \boldsymbol{y}] = i\phi$$

Generalization to local $\mathbb{P}^2$

- The resulting difference equation

$$e^{ix}\psi(x) + \psi(x+\phi) + q^{\frac{1}{2}}e^{-ix}\psi(x-\phi) = E\psi(x)$$

can be simplified into the form

$$\psi(x+\phi) + 2\cos x\,\psi\left(x - \frac{\phi}{2}\right) = E\psi(x)$$

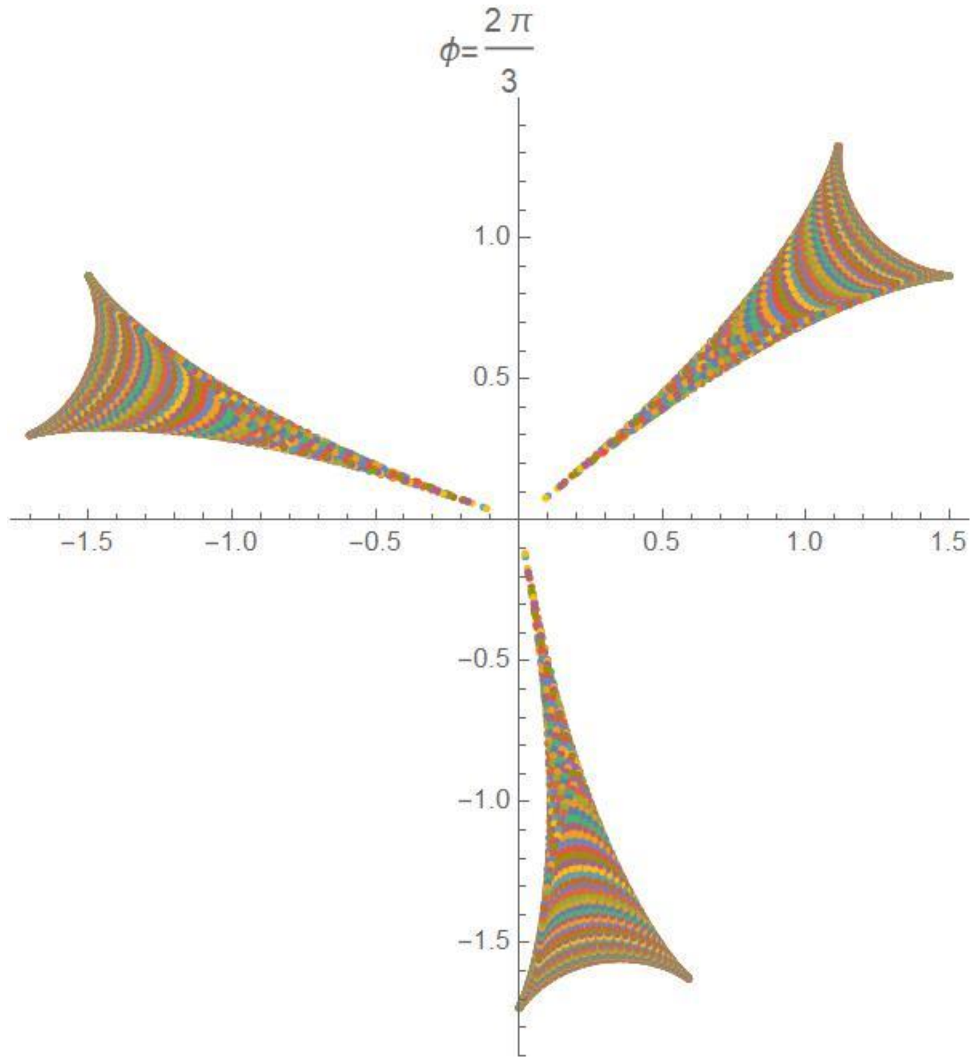
- For $\phi = 2\pi\frac{P}{Q}, P, Q \in \mathbb{Z}, (P,Q) = 1$,the secular equation

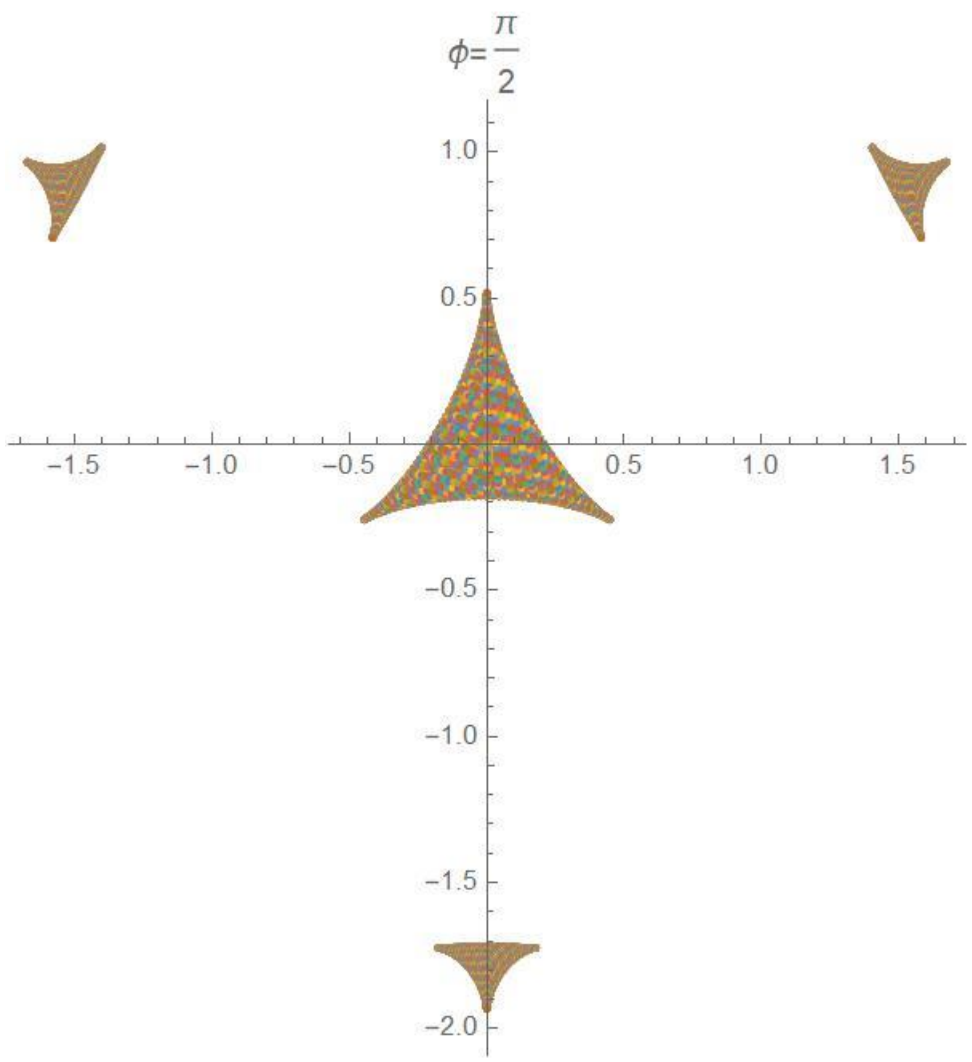
$$\det\begin{pmatrix} e^{i\theta_x}T_1 - E & e^{i\theta_y} & 0 & \dots & q^{1/2}e^{-i\theta_x - i\theta_y}T_{-1} \\ q^{1/2}e^{-i\theta_x - i\theta_y}T_{-2} & e^{i\theta_x}T_2 - E & e^{i\theta_y} & \dots & 0 \\ 0 & q^{1/2}e^{-i\theta_x - i\theta_y}T_{-3} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & e^{i\theta_y} \\ e^{i\theta_y} & 0 & 0 & q^{1/2}e^{-i\theta_x - i\theta_y}T_{-Q} & e^{i\theta_x}T_Q - E \end{pmatrix} = 0$$

with $T_j := e^{\mathrm{i}j\phi}$

Spectrum of local $\mathbb{P}^2$ model

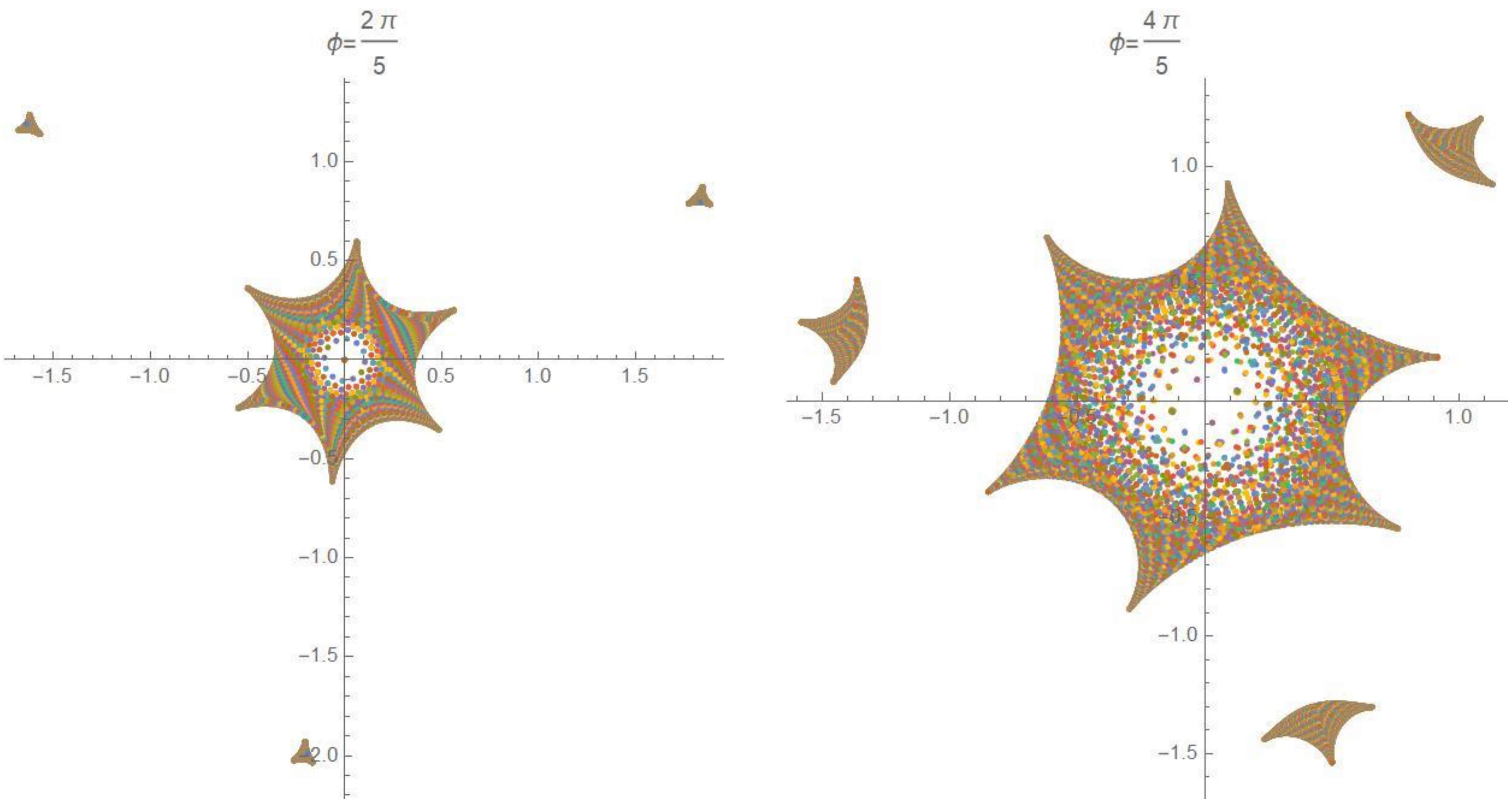
- The spectrum determined from the secular equation is complex and exhibits $\mathbb{Z}_3$ symmetry, energy bands become “energy blocks”





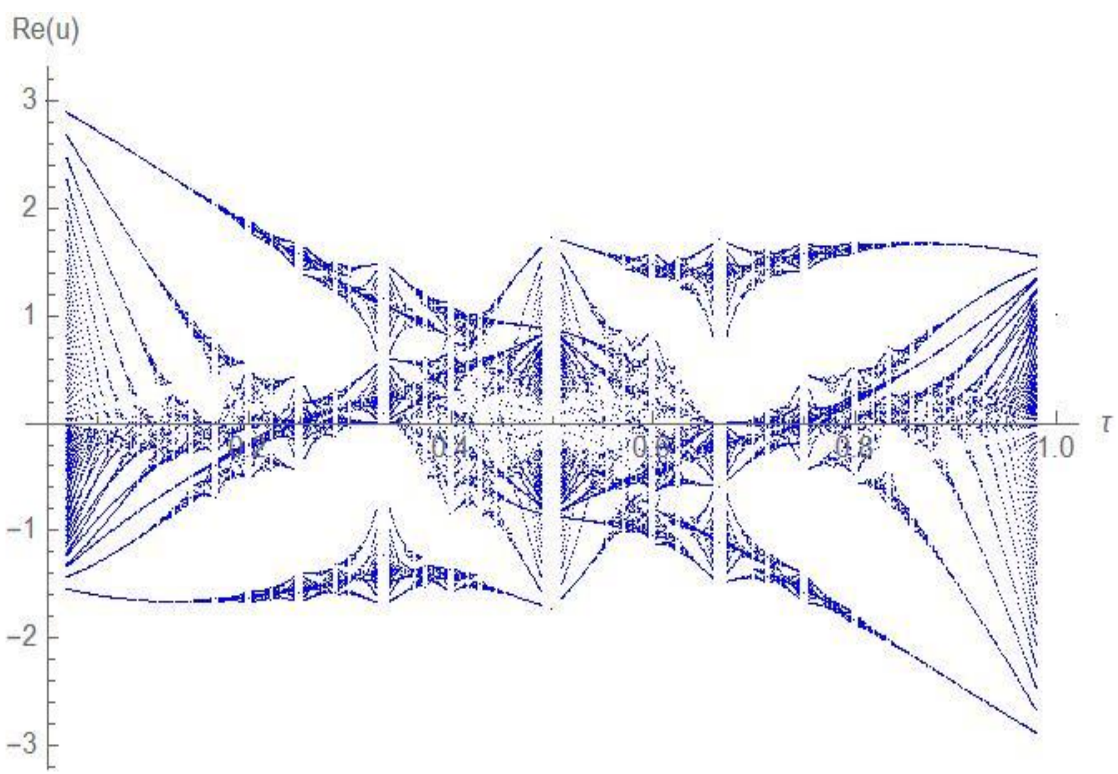
Spectrum of local $\mathbb{P}^2$ model

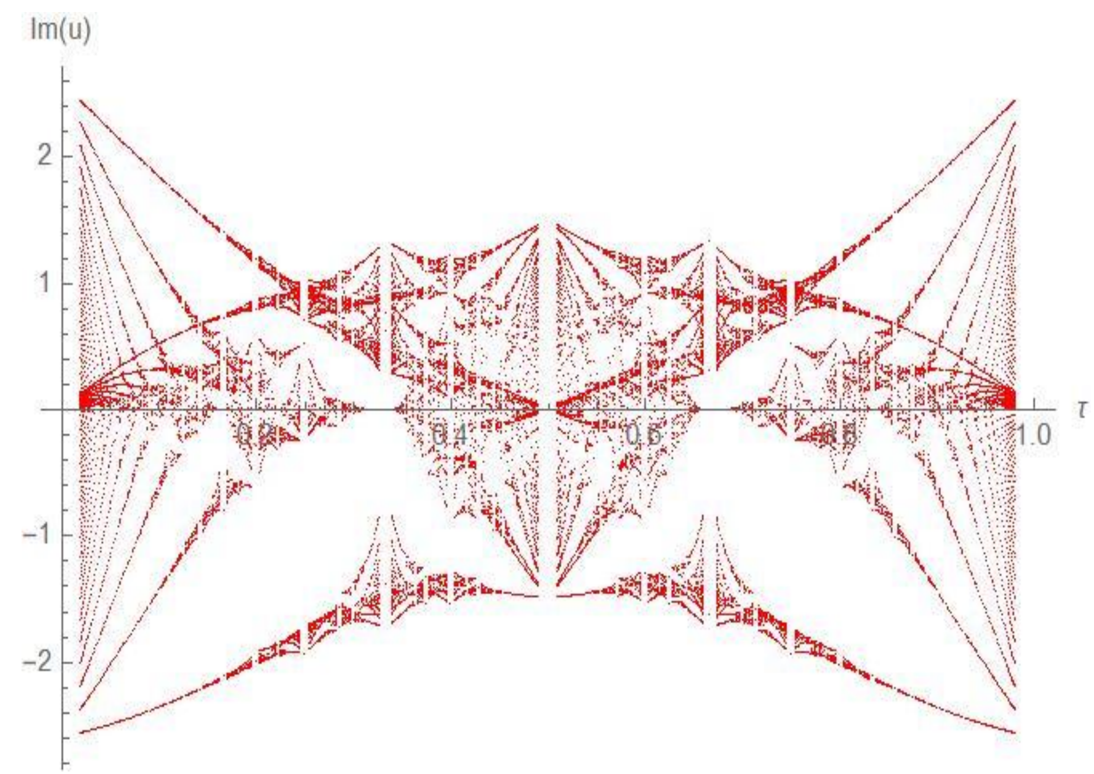
- The spectrum determined from the secular equation is complex and exhibits $\mathbb{Z}_3$ symmetry, energy bands become “energy blocks”



Spectrum of local $\mathbb{P}^2$ model

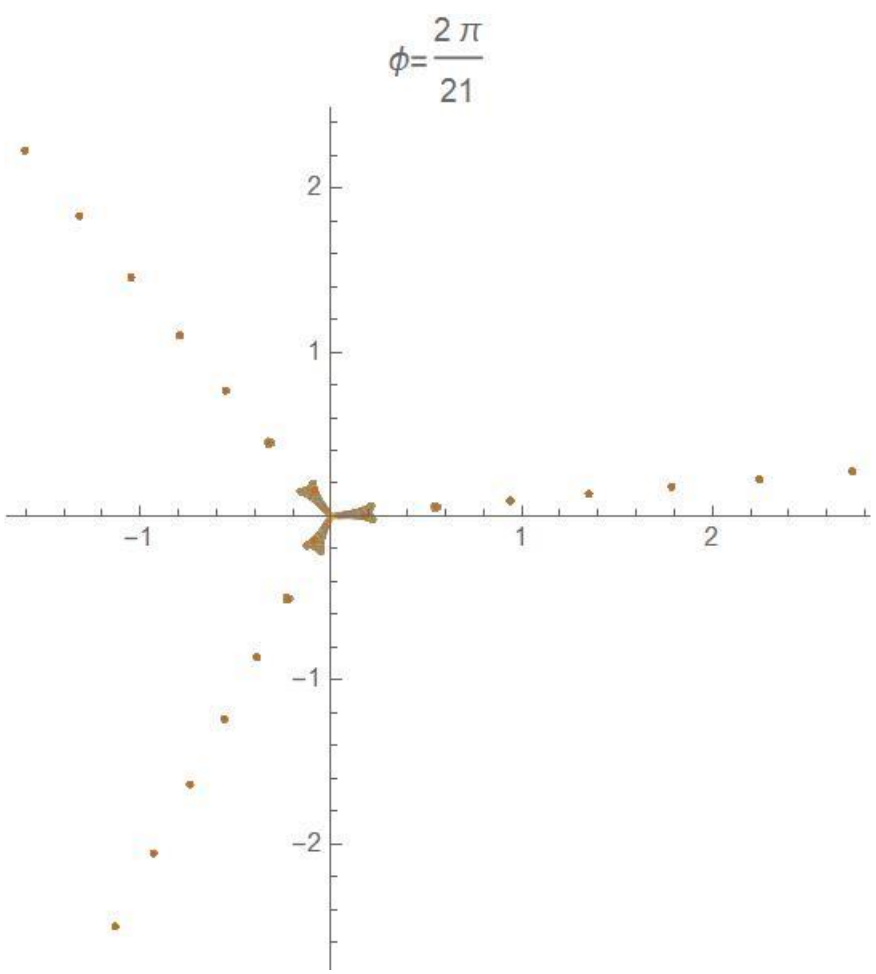
- The spectrum has the symmetry $\mathrm{Re}(u) \to -\mathrm{Re}(u)$, $\mathrm{Im}(u) \to \mathrm{Im}(u)$ under $\phi \to 2\pi - \phi$

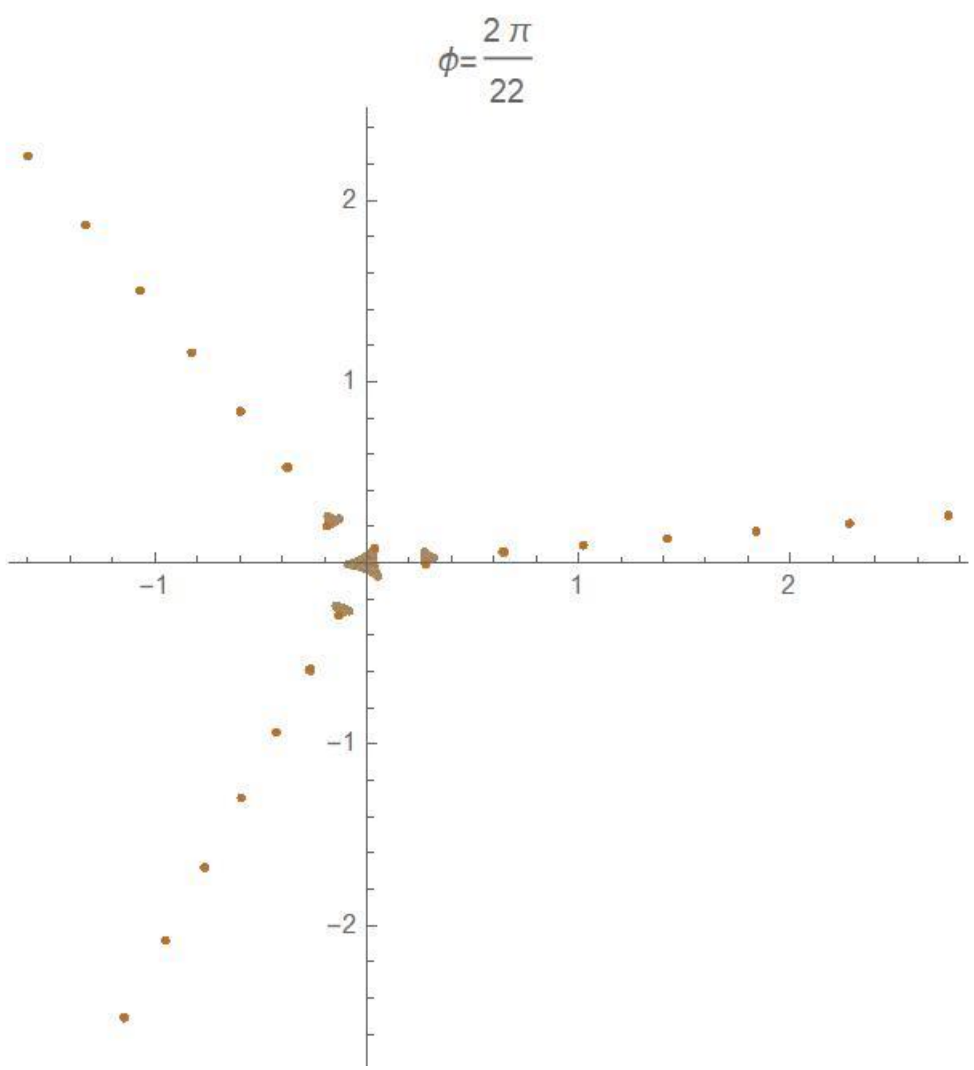




Spectrum of local $\mathbb{P}^2$ model

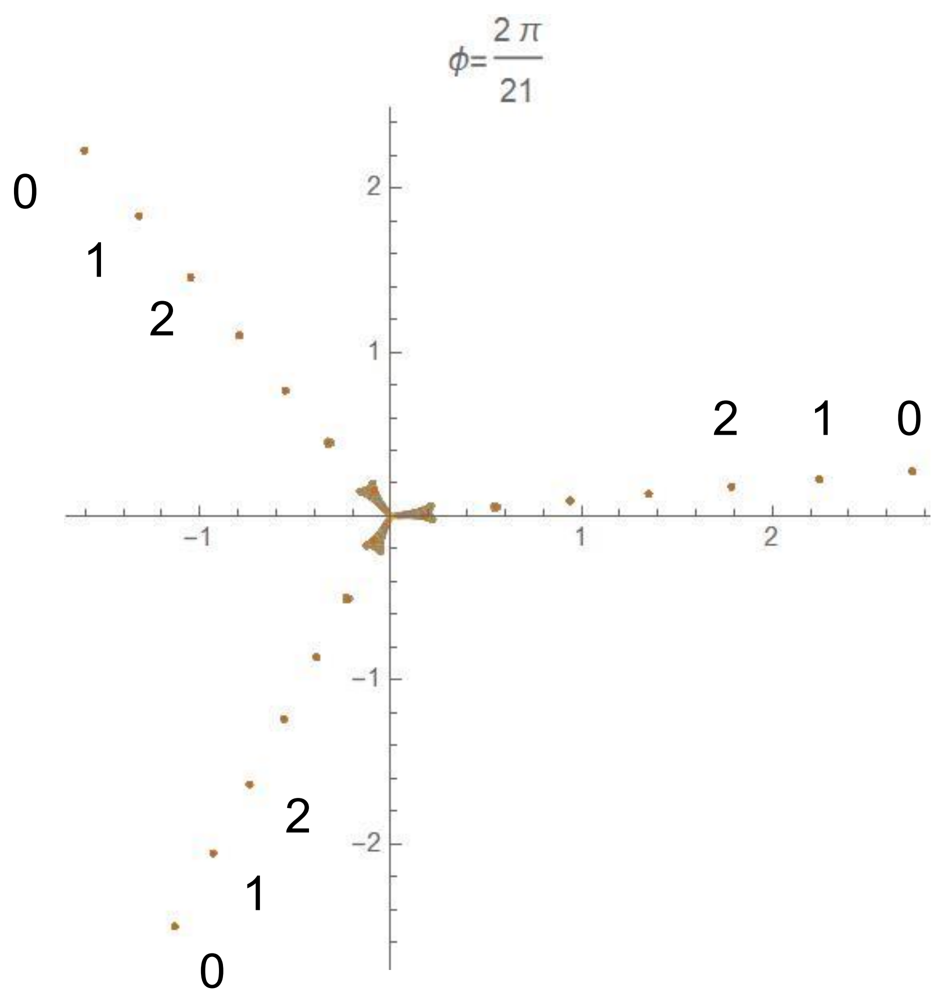
- Near the semiclassical regime, Landau levels start to emerge, and each level comes as a triplet

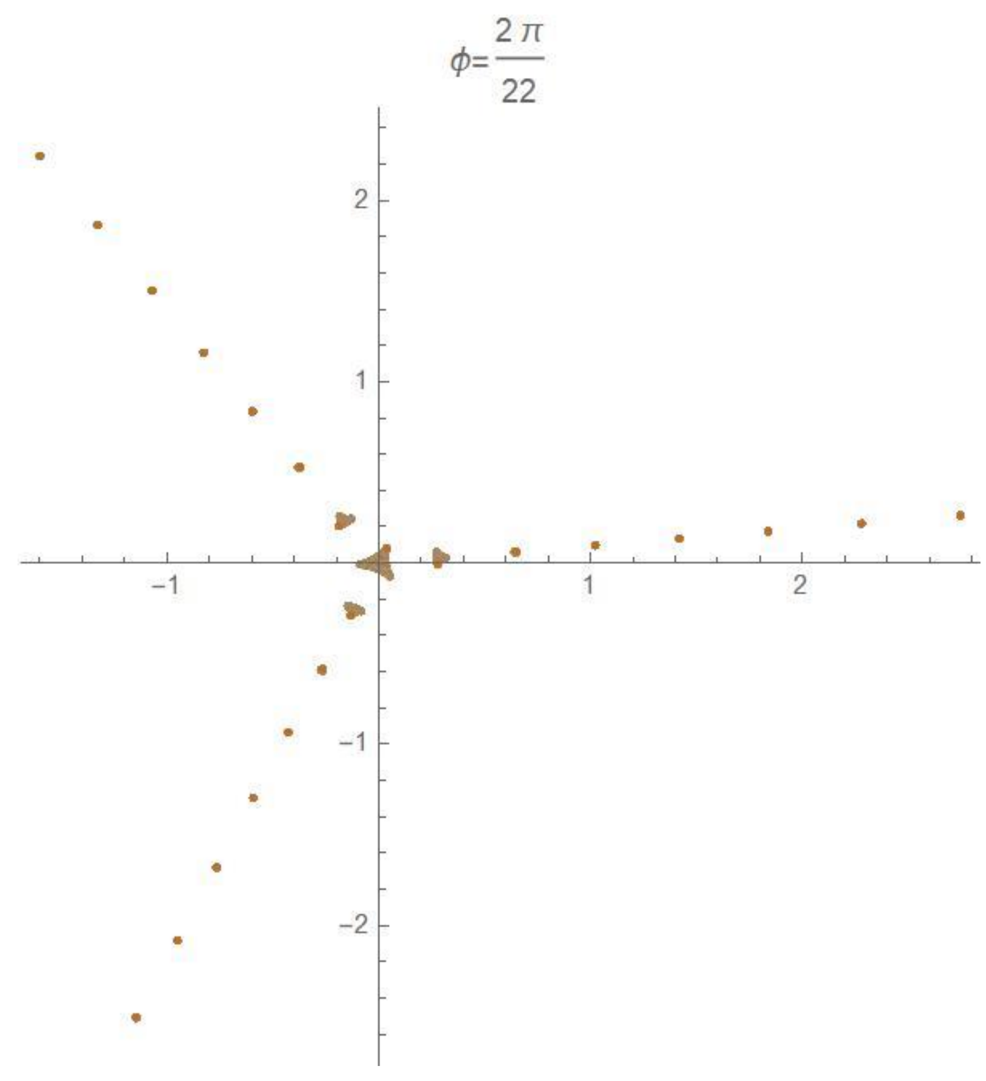




Spectrum of local $\mathbb{P}^2$ model

- Near the semiclassical regime, Landau levels start to emerge, and each level comes as a triplet





Spectrum of local $\mathbb{P}^2$ model

- The perturbative series is given by

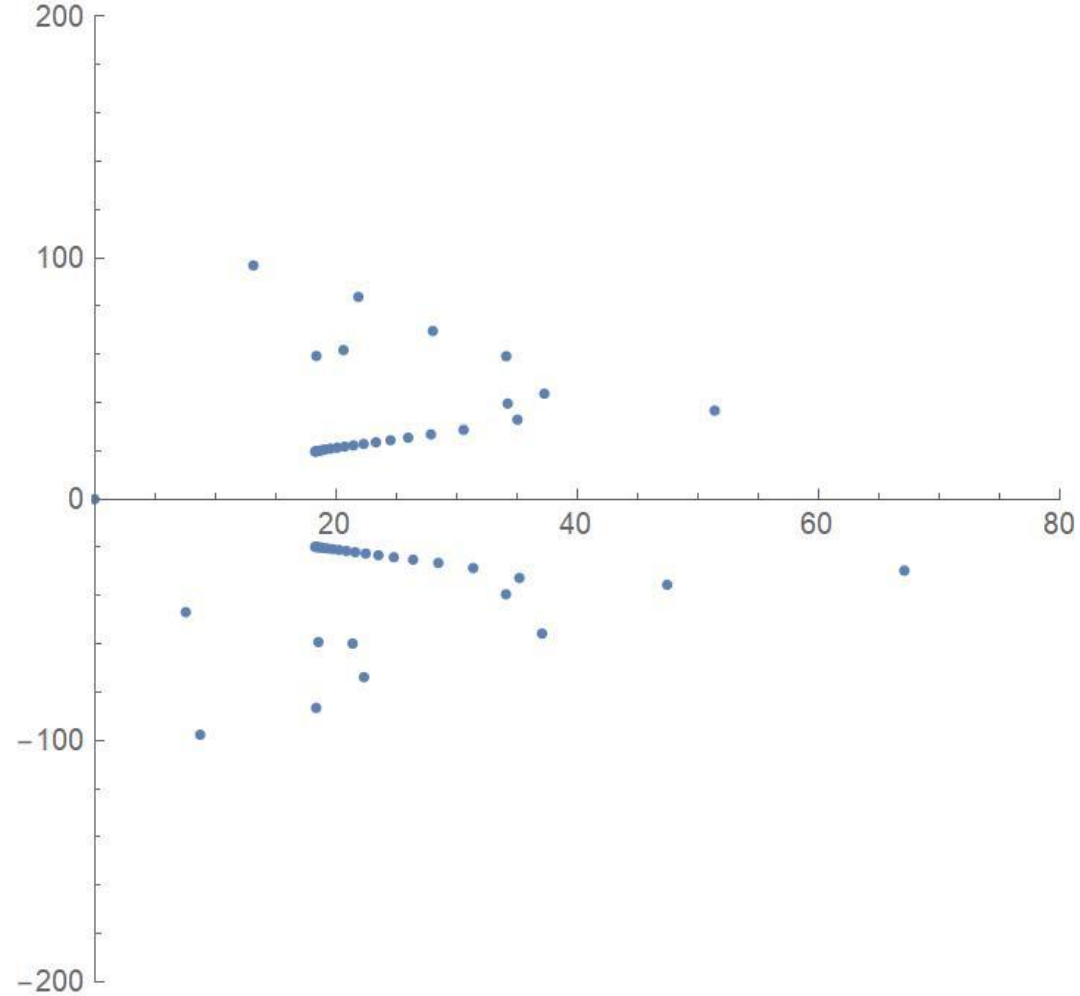
$$E^{(0)}(\phi; N) = e^{\frac{i\phi}{3}}\left(3 - \sqrt{3}\left(N + \frac{1}{2}\right)\phi + \cdots\right)$$

Spectrum of local $\mathbb{P}^2$ model

- The perturbative series is given by

$$E^{(0)}(\phi; N) = e^{\frac{i\phi}{3}}\left(3 - \sqrt{3}\left(N + \frac{1}{2}\right)\phi + \cdots\right)$$

- This series is Borel summable therefore we expect the full trans-series is given by the median trans-series



Spectrum of local $\mathbb{P}^2$ model

- Use the van Spaendonck-Vonk trans-series structure as an ansatz for numerical fitting, we find for $\phi = 2\pi/Q$

$$E^{(1)}(\phi;N) = \frac{\Theta}{\pi}\frac{\partial E^{(0)}}{\partial N} e^{-\frac{\sqrt{3}t_c^D}{\phi}}$$

$$E^{(2)}(\phi;N) = \frac{\Theta^2}{2\pi^2}\frac{\partial}{\partial N}\left(\frac{\partial E^{(0)}}{\partial N} e^{-\frac{2\sqrt{3}t_c^D}{\phi}}\right)$$

$$E^{(3)}(\phi;N) = \frac{\Theta^3}{6\pi^3}\frac{\partial^2}{\partial N^2}\left(\frac{\partial E^{(0)}}{\partial N} e^{-\frac{3\sqrt{3}t_c^D}{\phi}}\right) + \left(\frac{\Theta^3}{6\pi} - \frac{1}{2\pi}\right)\frac{\partial E^{(0)}}{\partial N} e^{-\frac{3\sqrt{3}t_c^D}{\phi}}$$

where

$$\Theta = (-1)^{N+1}\,\frac{\mathrm{e}^{i\theta_x} + \mathrm{e}^{i\theta_y} + (-1)^Q \mathrm{e}^{-i\theta_x - i\theta_y}}{2}$$

$$t_c^D(\phi;N) = 2\sqrt{3}\,\mathrm{Im}\left(\mathrm{Li}_2 \mathrm{e}^{\frac{i\pi}{3}}\right) + \left(N + \frac{1}{2}\right)\phi\log\phi + \cdots$$

work in progress for the WKB quantization condition

Open Questions

- Can we find useful tools to obtain the EWKBQC systematically for general SW curves for 5d gauge theories?
- Obtain the exact Wilkinson-Rammal formula from the corresponding EQC
- Further test the universality of van Spaendonck-Vonk trans-series structure on other quantum mechanical systems. How does the situation change for the cases where n-parameter trans-series show up?
- Is there other examples of Gevrey-1 series encountered in QM and QFT whose full trans-series structure cannot be determined by resurgence theory alone? If so, what’s the extra input needed that goes beyond the minimal trans-series?


Thank you for your attention