Witten Anomaly and Fermionic Instanton Zero Mode

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Motivation and Conclusion

AN SU(2) ANOMALY

Edward WITTEN [1]
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A new restriction on fermion quantum numbers in gauge theories is derived. For instance, it is shown that an SU(2) gauge theory with an odd number of left-handed fermion doublets (and no other representations) is mathematically inconsistent.

(In an ordinary instanton field, the number of fermion zero modes is $2 \operatorname{tr} T_3^2$, so the inconsistent theories are precisely those with an odd number of fermion zero modes in an instanton field.)

Witten Anomaly

- Anomaly under large gauge transformation
- Characterized by $\pi_4(SU(2)) = \mathbb{Z}_2$

v.s.

Fermionic Instanton Zero modes

- Inconsistent when the fermion zero mode count is odd
- Instantons are related to $\pi_3(SU(2)) = \mathbb{Z}$

Question 1 : How exactly they are related?

For other gauge groups G, one should check $\pi_4(G)$:

$$\pi_4(SU(N)) = 0 \quad (N \geq 3)$$

$$\pi_4(Sp(N)) = \mathbb{Z}_2$$

$$\pi_4(SO(N)) = 0 \quad (N \geq 7)$$

Therefore only Sp-groups might suffer Witten anomaly.

Question 2: For $SU(N)$ theory, it is easy to have odd number fermionic zero modes. Then why $SU(N)$ theory is safe?

To solve **Question 1**: We use the *Pontryagin-Thom* construction of $\pi_4(SU(2))$ to build a mapping torus

$$\hat{\mathbb{Y}}_5 = S^1_y \times \underbrace{S^3 \times S^1_\beta}_{4D \text{ spacetime}}$$

where $y = 1$ is glued with $y = 0$ via the large gauge transformation. The mapping torus compute the 5D mod-2 index, which characterizes the Witten anomaly.

If we switch the role between S^1_y and S^1_β

$$\hat{\mathbb{Y}}_5 = \underbrace{S^1_y \times S^3}_{4D \text{ spacetime}} \times S^1_\beta$$

the mapping torus count the fermionic instanton zero modes instead.

Regarding on **Question 2**:

The vanishing of pertrubative anomaly restricts the chiral matter contents, and automaically impose even number of fermionic instanton zero modes.

We then raise **Question 3**:

What happens if the perturbative anomaly is canceled via *anomaly inflow*?

To answer **Question 3**, we examine two cases :

- Witten-Yonekura setup of anomaly inflow
- (fractional) D3-branes probing singularities

Both are safe from odd number of fermionic zero modes.

Before moving on...

In the modern language, global anomaly in D-dimension is

$$\exp\left(-2\pi i\eta(X_{D+1})\right) \in \mathrm{Hom}(\Omega^{\mathrm{Spin}}_{D+1}(BG), U(1))$$

And in general, the non-vanishing of $\pi_D(G)$ is *neither necessary nor sufficient.* [Davighi, Lohitsiri,2021] [Lee, Tachikawa,2021]

The reason $\pi_4(SU(2))$ applies is, the 5D mapping torus constructed by gluing via large gauge transformation is a *generator* of the spin bordism group

$$\Omega^{\mathrm{Spin}}_5(BSU(2), U(1)) = \mathbb{Z}_2\,.$$

So what we have achieved through Pontryagin-Thom, from the bordism perspective, is *constructing a generator where the relation between Witten anomaly with fermionic instanton zero mode problem becomes manifest.*

An overview of Witten Anomaly

Gauge anomaly

- Consider a 4D gauge theory with gauge group G and gauge field $\mathcal{A}_\mu^a$, and massless (chiral) fermionic matters ψ_i charged under G.

$$Z[\mathcal{A}_\mu] = \int \prod_i \left[\mathcal{D}\psi_i \mathcal{D}\bar{\psi}_i\right] e^{-S_E[\mathcal{A}_\mu, \psi_i, \bar{\psi}_i]} = e^{-W[\mathcal{A}_\mu]}$$

- Under the gauge transformation

$$\mathcal{A}_\mu \to g(x) \circ \mathcal{A}_\mu$$

If the partition function is NOT invariant

$$Z[\mathcal{A}_\mu] \neq Z[g(x) \circ \mathcal{A}_\mu]$$

then there exists gauge anomaly.

Perturbative gauge anomaly

The perturbative anomaly contributed by a 4D chiral fermion ψ transformed under the representation $\mathbf{R}$ of G is

- Step 0. Begin with a 6-form anomaly polynomial $\mathcal{I}_6^{(0)}$

$$\begin{aligned}\mathcal{I}_6^{(0)} &= \pm \frac{1}{3!(2\pi)^3}\mathrm{Tr}\left[\mathcal{F}^{\mathbf{R}} \wedge \mathcal{F}^{\mathbf{R}} \wedge \mathcal{F}^{\mathbf{R}}\right] \\ &\sim \mathcal{F}^a_{\mu_1\mu_2}\mathcal{F}^b_{\mu_3\mu_4}\mathcal{F}^c_{\mu_5\mu_6}\mathrm{Tr}_{\mathbf{R}}[T_a T_{(b}T_{c)}]dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_6}\end{aligned}$$

- Step 1. $\mathcal{I}_6^{(0)}$ is closed due to $d\mathcal{F} = \partial_{[\mu}\mathcal{F}_{\nu\rho]} = 0$, it implies

$$d\mathcal{I}_6^{(0)} = 0 \to \mathcal{I}_6^{(0)} = d\mathcal{I}_5^{(1)} \quad \text{(locally)}$$

where $\mathcal{I}_5^{(1)}$ is the 5D Chern-Simons density

$$\mathcal{I}_5^{(1)} = \frac{1}{3!(2\pi)^3}\mathrm{Tr}\left[\mathcal{A}^{\mathbf{R}} \wedge \mathcal{F}^{\mathbf{R}} \wedge \mathcal{F}^{\mathbf{R}} + \cdots\right]$$

- Step 2. Notice that $\mathcal{I}_6^{(0)}$ is gauge invariant by construction. One has

$$d\left(\delta_\epsilon \mathcal{I}_5^{(1)}\right) = \delta_\epsilon\left(d\mathcal{I}_5^{(1)}\right) = \delta_\epsilon \mathcal{I}_6^{(0)} = 0\,,$$

where ϵ is the gauge transformation parameter

$$\delta_\epsilon \mathcal{A}_\mu^a = \partial_\mu \epsilon^a + \mathbb{i} f^a_{\ bc} \mathcal{A}_\mu^b \epsilon^c\,.$$

Therefore we can further write

$$\delta_\epsilon \mathcal{I}_5^{(1)} = d\left(\mathcal{I}_4^{(2)}[\epsilon]\right) \quad \text{(locally)}$$

Then $\mathcal{I}_4^{(2)}[\epsilon]$ is the anomaly satisfying

$$\delta_\epsilon W[\mathcal{A}_\mu^a] = 2\pi i \int \mathcal{I}_4^{(2)}[\epsilon]$$

The total anomaly polynomial is then

$$\mathcal{I}^{(0)}_{6;\text{total}} = \sum_i \mathcal{I}_6^{(0)}[\Psi_i] \sim \sum_i \pm \mathrm{Tr}_{\mathbf{R}_i}[T_a T_{(b} T_{c)}]\,.$$

- If the representation is real ($SS^* = 1$) or pseudo-real ($SS^* = -1$), which means the gauge generators satisfy

$$(T_a)^* = (T_a)^T = -ST_aS^{-1}, \quad SS^* = \pm 1$$

Then it does not contribute to the anomaly since

$$\mathrm{Tr}[T_aT_{(b}T_{c)}] = \mathrm{Tr}[T^T_{(c}T^T_{b)}T^T_a] = -\mathrm{Tr}[T_{(c}T_{b)}T_a] = -\mathrm{Tr}[T_aT_{(b}T_{c)}]$$

which implies $\mathrm{Tr}[T_aT_{(b}T_{c)}] = 0$.

- For a single generation in Standard Model

$SU(2)$ representation	**1**	**2**
Particles	e_R, u_R, d_R	$(e, \nu_e)_L, (u, d)_L$

The fundamental representation of $SU(2)$ is pesudo-real

$$(\sigma_a)^* = -(\sigma_2)(\sigma_a)(\sigma_2)^{-1}, \quad \sigma_2(\sigma_2)^* = -1.$$

The Standard Model is free of perturbative pure $SU(2)$ anomaly.

Witten anomaly

- Witten pointed out, 4D $SU(2)$ theory with odd number of fundamental chiral fermions suffer a "non-perturbative" anomaly
- In the Euclidean signature, the spacetime is $\mathbb{R}^4 \bigcup \infty = S^4$, and the gauge transformation $g(x)$ is the map

$$g(x) : S^4 \to SU(2) = S^3$$

- This map is classified by the 4-th homotopy group $\pi_4(S^3)$

$$\pi_4(SU(2)) = \pi_4(S^3) = \mathbb{Z}_2 = \{e, a\}$$

- Witten argued that, the 4D $SU(2)$ theory with N fundamental chiral fermion transform as

$$Z[\mathcal{A}_\mu^g] = (-1)^N Z[\mathcal{A}_\mu]$$

under *large gauge transformation* $g(x)$ with $[g(x)] = a$.

- Let us begin with the partition function of a Dirac fermion Ψ

$$Z[\mathcal{A}_\mu^a] = \int \left[\mathcal{D}\bar{\Psi}\mathcal{D}\Psi\right] e^{-\int d^4x \bar{\Psi} \mathbb{i}\gamma^\mu \mathscr{D}_\mu \Psi} = \mathrm{Det}\left(\mathbb{i}\gamma^\mu \mathscr{D}_\mu\right)$$

  where the determinant is formally

$$\mathrm{Det}\left(\mathbb{i}\gamma^\mu \mathscr{D}_\mu\right) = \prod_\lambda \lambda \quad \text{with} \quad \mathbb{i}\gamma^\mu \mathscr{D}_\mu \Psi_\lambda = \lambda \Psi_\lambda.$$

  Due to $\{\gamma^5, \mathbb{i}\gamma^\mu \mathscr{D}_\mu\} = 0$, the eigenvalues are pairwise $(\lambda, -\lambda)$.
- For a chiral fermion, the eigenvalue problem is ill-defined

$$\begin{pmatrix} 0 & \bar{\chi}_R \end{pmatrix} \underbrace{\begin{pmatrix} 0 & \mathbb{i}\bar{\sigma}^\mu \bar{\mathcal{D}}_\mu \\ \mathbb{i}\sigma^\mu \mathcal{D}_\mu & 0 \end{pmatrix}}_{\mathbb{i}\gamma^\mu \mathscr{D}_\mu} \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}$$

  since the Dirac operator map between chiral/anti-chiral components.

- Nevertheless, $(\mathbb{i}\gamma^\mu \mathscr{D}_\mu)^2$ has well-defined eigenvalues

$$\left(\begin{array}{cc} (\mathbb{i}\bar{\sigma}^\mu \bar{\mathcal{D}}_\mu)(\mathbb{i}\sigma^\mu \mathcal{D}_\mu) & 0 \\ 0 & (\mathbb{i}\sigma^\mu \mathcal{D}_\mu)(\mathbb{i}\bar{\sigma}^\mu \bar{\mathcal{D}}_\mu) \end{array} \right) \left(\begin{array}{c} \psi_L \\ 0 \end{array} \right) = \lambda^2 \left(\begin{array}{c} \psi_L \\ 0 \end{array} \right)$$

- We can pick one of the two, either $+\sqrt{\lambda^2}$ or $-\sqrt{\lambda^2}$, to define the effective action. For example, let us take

$$Z[\mathcal{A}^a_\mu] = \text{Pfaff}\,(\mathbb{i}\gamma^\mu \mathscr{D}_\mu) = \prod_{\lambda>0} \lambda$$

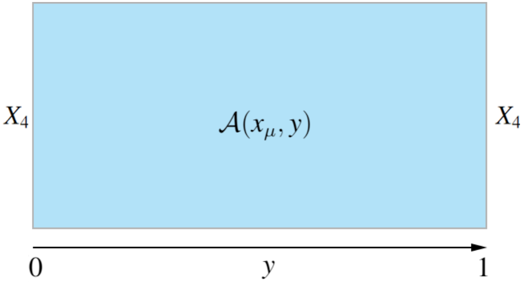
where we assume that no zero mode exists.
- The Witten anomaly arises because the sign choice is ambiguous.

- Imagine a fictitious 5-dimensional manifold $\mathbb{Y}_5$ with the topology $[0,1] \times X_4$, where X_4 is the 4D spacetime.
- Let the gauge fields $\mathcal{A}(x_\mu, y)$ change continuously with $y \in [0,1]$ such that

$$\mathcal{A}(x_\mu, 1) = g(x) \circ \mathcal{A}(x_\mu, 0)$$

with $[g(x)] \in \pi_4(SU(2))$ the large gauge transformation and

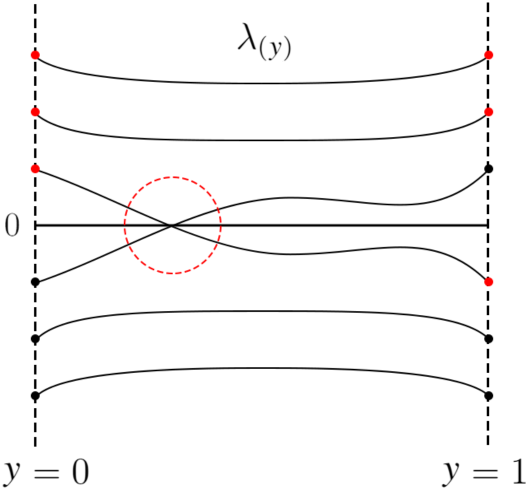
$$g \circ \mathcal{A} = g(\mathcal{A} - \mathbb{i}d)g^{-1}.$$



- Suppose that N positive (negative) eigenvalue crosses zero over to negative (positive) side, we then find

$$Z[\mathcal{A}_\mu^a(y=1)] = (-1)^N Z[\mathcal{A}_\mu^a(y=0)]$$

- When N is odd, they theory is inconsistent since the two are related by gauge transformation



Mapping Torus and Mod-2 Index

- Let us glue $y = 0$ and $y = 1$ and consider the compact manifold $\hat{\mathbb{Y}}_5$. The 4D counting translates to:

  $$(-1)^N = (-1)^{n_{5D}}, \quad \text{where } n_{5D} \text{ is \# of } \psi_{5D} \text{ s.t. } \mathbb{i}\not{\mathscr{D}}\psi_{5D} = 0$$

  The Witten index is detected by the 5D mod-2 index $(-1)^{n_{5D}}$.
- To see how they are related, consider the Dirac equation on the bulk $\hat{\mathbb{Y}}_5$

  $$\mathbb{i}\boldsymbol{\gamma}^y\left(\partial_y + \mathbb{D}_{(y)}\right)\Psi = 0\,, \qquad \mathbb{D}_{(y)} \equiv \boldsymbol{\gamma}^y\gamma^a\mathscr{D}_a\Big|_y$$

  where $\mathbb{D}_{(y)}$ share the same set of eigenvalues with the 4D Dirac operators $\mathbb{i}\gamma^\mu\mathscr{D}_\mu$.

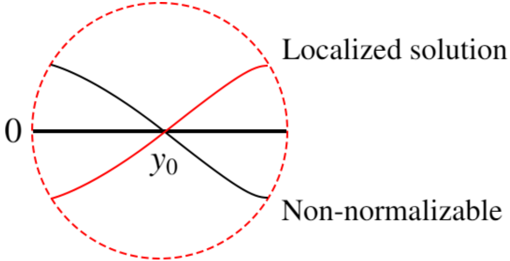
- Using the 4D eigen-equation

$$\mathbb{D}_{(y)}\Psi^{(\lambda_{(y)})}_{(y)} = \lambda_{(y)}\Psi^{(\lambda_{(y)})}_{(y)}$$

  At the crossing point y_0, the 5D equation $\left(\partial_y + \mathbb{D}_{(y)}\right)\Psi = 0$ is solved by

$$\Psi \;\simeq\; e^{-\int_{y_0}^{y}\lambda_{(y)}dy} \times \Psi^{(\lambda_{(y_0)})}_{(y_0)}$$

- We can conclude that each eigenvalue crossing of the pair $(\lambda, -\lambda)$ implies a single zero mode on $\hat{\mathbb{Y}}_5$.



Witten Anomaly
and
Fermioinc Instanton Zero Modes

Why Problematic [Witten, 1985]

- Denote π as the operator that performs the *large gauge transformation on the spatial slice* $S^3 = \mathbb{R}^3 \bigcup \infty$, which changes the winding number of vacuum (and create an instanton)

$$\pi|\mathcal{A}(\vec{x}), \Psi(\vec{x}), \cdots\rangle = |\mathcal{A}^g(\vec{x}), \Psi^g(\vec{x}), \cdots\rangle$$

  where $[g(\vec{x})] \in \pi_3(SU(2))$ is the generator.
- Let J be the generator of a rotation about an arbitrary axis on S^3 and consider the operator

$$G_\varphi = \pi^{-1} \exp(-\mathbb{i}\varphi J)\pi \exp(\mathbb{i}\varphi J)\,, \quad (\varphi \in [0, 2\pi])$$

  which is a family of *small gauge transformation* on S^3.
- Obviously $G_0 = 1$, and $G_{2\pi} = \pi^{-1}(-1)^F\pi(-1)^F$.

- If the instanton is equipped with n_{4D}^{Inst} number of fermionic zero modes, then π will create n_{4D}^{Inst} fermions so that

$$(-1)^F \pi = (-1)^{n_{4D}^{\text{Inst}}} \pi (-1)^F$$

and one has

$$G_0 = 1, \quad G_{2\pi} = (-1)^{n_{4D}^{\text{Inst}}}.$$

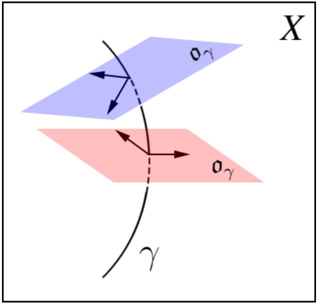
- But since G_φ are small gauge transformation on the spatial slice, by Gauss law one should have $G_\varphi = 1$. When n_{4D}^{Inst} is odd, $G_{2\pi} = -1$ and we have a conflict.

Then the question is

$$\text{Why } (-1)^{n_{5D}} = (-1)^{n_{4D}^{\text{Inst}}} \text{ ?}$$

Pontryagin-Thom Construction

- We will not present the general construction but instead, let us focus on $\pi_4(S^3) = \pi_4(SU(2))$.
- Given a line γ in X, a framing of γ is a basis $\mathfrak{o}_\gamma$ of the normal bundle N_γ. The line is called **framed** if it carries a framing structure. We will denote $(\gamma, \mathfrak{o}_\gamma)$.



- (**Pontryagin-Thom**) The homotopy classes $\pi_4(S^3)$, which are equivalence classes of maps from S^4 to S^3, are one-to-one correspondence to the equivalence classes of **framed** circle in S^4.
- A framing is equivalent as choosing an SO(3) rotation $R(\beta)$ for each point $\beta \in [0, 1)$ along γ

$$R(\beta) : S^1 \to \mathrm{SO}(3), \quad R(\beta + 1) = R(\beta)$$

- Since $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$, the class of $R(\beta)$ and thus frame circle is also characterized by $\mathbb{Z}_2 = \{e, a\}$.
- If we lift $R(\beta)$ to SU(2) group as $\tilde{R}(\beta)$, the two classes are classified by

$$e : \tilde{R}(1) = \tilde{R}(0), \quad a : \tilde{R}(1) = -\tilde{R}(0)$$

From framed circle to $\pi_4(S^3)$

- Consider the tubular region $\gamma(\beta) \times \mathbb{R}^3_d \in S^4$, where $\mathbb{R}^3_d$ is 3-dimension ball with d the radius:

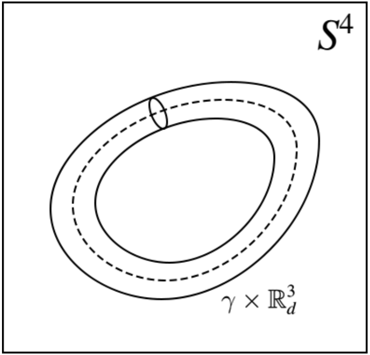
$$(x_1(\beta))^2 + (x_2(\beta))^2 + (x_3(\beta))^2 \leq d^2$$

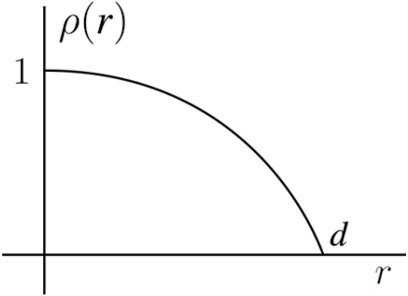
  with $x_1(\beta), x_2(\beta), x_3(\beta)$ are coordinates associated to the frame $\mathfrak{o}_\gamma(\beta)$.

- Also, let us think of the target S^3 as $S^3 = \{\mathbb{R}^3 \bigcup \infty\}$ where $\mathbb{R}^3$ is parametrized by $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$.
- The homotopy map from S^4 to S^3 is constructed as following

$$(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \begin{cases} \frac{(x_1(\beta), x_2(\beta), x_3(\beta))}{\rho(r)} & \text{if } \vec{x}(\beta) \in \gamma(\beta) \times \mathbb{R}^3_d \\ \infty \quad \text{if } x \notin \gamma(\beta) \times \mathbb{R}^3_d \end{cases}$$

  where $\rho(r)$ is the cut-off function. The map on each $\mathbb{R}^3_d$ slice carries a winding number on S^3 $(\pi_3(S^3))$.


S^4
$\gamma \times \mathbb{R}^3_d$


$\rho(r)$
1
d
r

5D mod-2 index revisit

- Consider the 5D mapping torus built by

$$\hat{\mathbb{Y}}_5 = S^1_y \times \underbrace{S^3 \times S^1_\beta}_{4D \text{ spacetime}}$$

where $y = 1$ is glued with $y = 0$ via a large gauge transformation $[g(\vec{x}, \beta)] \in \pi_4(SU(2))$,

$$\mathcal{A}(\beta, y = 1) = g(\vec{x}, \beta) \circ \mathcal{A}(\beta, y = 0)$$

and periodic along the β direction

$$\mathcal{A}(\beta = 1, y) = \mathcal{A}(\beta = 0, y) .$$

Here

$$g(\vec{x}, \beta) = \left(e^{\mathbb{i}2\pi\beta\frac{\sigma^3}{2}} g(\vec{x}) e^{-\mathbb{i}2\pi\beta\frac{\sigma^3}{2}} \right)$$

where S^1_β as the framed circle and $[g(\vec{x})] \in \pi_3(SU(2))$

- Now let us perform a singular gauge transformation

$$g' = e^{-2\pi \mathbb{i} \beta \frac{\sigma^3}{2}}, \quad \mathcal{A}' = g' \circ \mathcal{A}.$$

- After that, the new gauge fields satisfy

$$\mathcal{A}'(\beta, y = 1) = g(\vec{x}) \circ \mathcal{A}'(\beta, y = 0)$$

  along y-direction and

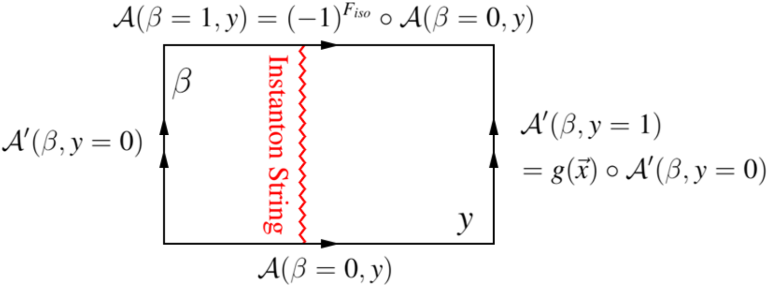
$$\mathcal{A}(\beta = 1, y) = (-1)^{F_{iso}} \circ \mathcal{A}(\beta = 0, y)$$

  along β direction, with $(-1)^{F_{\text{iso}}} = e^{-2\pi i \frac{\sigma_3}{2}}$.
- When the isospin is half-integer $(k + \frac{1}{2})$, one has

$$(-1)^{F_{iso}} = (-1)^F$$

  and the fermions satisfy periodic condition due along β-direction due to $(-1)^F$.

If we switch the role between S^1_β and S^1_y, we have



- The 4D fermionic instanton zero modes can be parallel transported along β-direction and becomes 5D zero modes.
- If we assume all 5D zero modes are obtained by transporting instanton zero mode, then one has

$$(-1)^{n_{5D}}=(-1)^{n_{4D}^{\text{Inst}}}\,.$$

It leaves a puzzle for $SU(N)$ groups. Since

$$\pi_4(SU(N)) = 0\,, \quad (N \geq 3)$$

there is no Witten anomaly. However, the fermionic zero modes in the instanton background could still be odd.

Why $SU(N)$ is safe?

Before talking about Witten anomaly, we need to make sure the perturbative anomaly is already canceled.

- The perturbative anomaly is contributed by

$$\sum \pm \frac{1}{3!(2\pi)^3} \mathrm{Tr}\left[\mathcal{F}^{\mathbf{R}} \wedge \mathcal{F}^{\mathbf{R}} \wedge \mathcal{F}^{\mathbf{R}}\right]$$

  with $\mathrm{Tr}\left[\mathcal{F}^{\mathbf{R}} \wedge \mathcal{F}^{\mathbf{R}} \wedge \mathcal{F}^{\mathbf{R}}\right] = C_3^{\mathbf{R}} \times \mathrm{Tr}\left[\mathcal{F}^{def} \wedge \mathcal{F}^{def} \wedge \mathcal{F}^{def}\right]$.
- The anomaly-free condition is

$$\sum \pm C_3^{\mathbf{R}} = 0.$$

- On the other hand, for minimal instanton, the fermionic zero modes are give by $\mathrm{Tr}\left[\mathcal{F}^{\mathbf{R}} \wedge \mathcal{F}^{\mathbf{R}}\right] = D_2^{\mathbf{R}} \times \mathrm{Tr}\left[\mathcal{F}^{def} \wedge \mathcal{F}^{def}\right]$
- The total number of fermionic zero modes is

$$\sum \pm D_2^{\mathbf{R}}.$$

- D_2, C_3 of k-th fundamental representations of $SU(N)$ are separately

$$D_2^{[k]} = \begin{pmatrix} N-2 \\ k-1 \end{pmatrix}, \qquad C_3^{[k]} = \frac{N-2k}{N-k-1} \cdot \begin{pmatrix} N-3 \\ k-1 \end{pmatrix}.$$

- The key observation is that

$$D_2^{[k]} - C_3^{[k]} = 2 \times \begin{pmatrix} N-3 \\ k-2 \end{pmatrix} \quad \in \quad 2\mathbb{Z}$$

  If $\sum \pm C_3^{\mathbf{R}} = 0$, the total fermionic zero modes $\sum \pm D_2^{\mathbf{R}}$ is always even.

- Higher representations of $SU(N)$ are obtained by tensoring fundamental representations. The same even-ness holds for all higher representations.

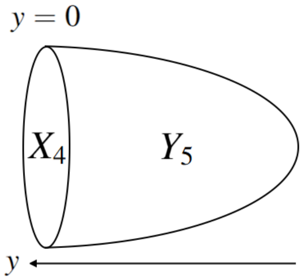
Are $SU(N)$ Theory with Anomaly Inflow Safe?

Witten-Yonekura Setup [Yonekura,Witten,2019]

- Suppose we have a 5D Dirac fermion Ψ with mass M living on Y_5, and impose a local chiral boundary condition on its boundary $X_4 = \partial Y_5$

$$\mathbf{L}: \quad (1+\gamma^y)\Psi|_{y=0} = 0\,.$$

such that Ψ restricts to a 4D chiral fermion on X_4.



- Solving 5D Dirac equation near the boundary gives

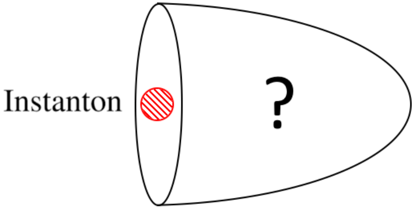
$$\Psi = \psi_{X_4} \exp(My) , \quad (1+\gamma^y)\psi_{X_4} = 0 , \quad (\mathbb{i}\gamma^\mu \mathscr{D}_\mu)|_{y=0}\psi_{X_4} = 0 .$$

  where ψ_{X_4} is a 4D chiral fermion living on X_4, solving the 4D massless Dirac equation.
- If $M > 0$, this mode decays exponentially in the bulk and is localized along X_4.
- Sending $M \to +\infty$, the total partition function is

$$Z_{Y_5} = \lim_{M\to+\infty} \langle \mathbf{L}|Y_5\rangle = |\mathcal{Z}_{X_4}| e^{2\pi i \eta^{\text{APS}}(Y_5)} .$$

  where $\mathcal{Z}_{X_4}$ is the 4D partition function of the chiral fermion, and $\eta^{\text{APS}}(Y_5)$ is the η-invariant on Y_5 with APS boundary conditions on X_4.
- Z_{Y_5} is gauge invariant, and is a *non-perturbative* version of anomaly inflow.

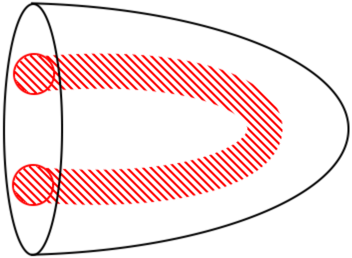
- Since the perturbative anomaly is canceled via inflow, the fermion zero mode counting around a 4D instanton is no longer constrained to be even.
- Does this lead to inconsistency?



- Thankfully, the concern resolves naturally by noticing that an instanton on X_4 would elevate to a instanton string extend into the bulk Y_5.
- There are three logical possibilities.

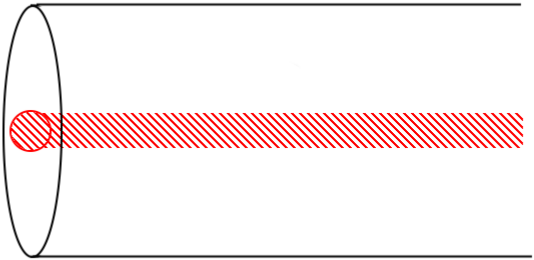
Case I : Y_5 closes off somewhere

- The instanton string with its non-trivial topology has nowhere to end except back to X_4 boundary, and would contribute a pair of instanton and anti-instanton.
- The number of fermion zero modes would be either doubled or lifted altogether.



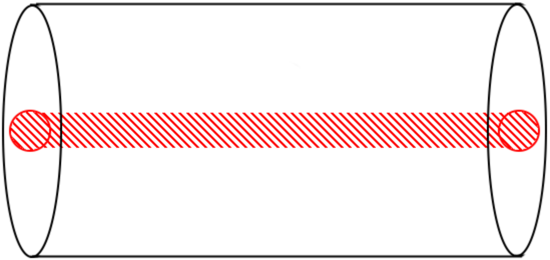
Case II : Y_5 extends to infinity

- If the instanton do not return to X_4, it must extend to the infinity.
- The bulk action is suppressed exponentially due to the infinite length of the instanton string. Therefore a single instanton is forbidden.



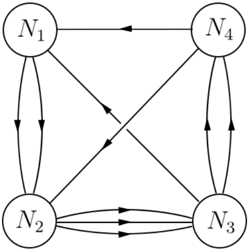
Case III : Y_5 is a finite cylinder

- The instanton string would end on the other boundary, we would find two boundary chiral theories at the two ends.
- The fermion zero modes are also doubled.



Geometrical Engineering

Consider fractional D3-branes probing a $\mathbb{C}^3/\Gamma$ singularity, or toric singularity, where the 4D theory is encoded in the quiver diagram. For example the cone over dP_1 geometry:



- Each node is a combination of D5-branes and D7-branes wraping 2/4-cycles
- The incoming and outgoing arrow stands for chiral and anti-chiral bi-fundamental hypermultiplets due to the intersection of D-branes

- The gauge anomaly can be canceled via *I-brane inflow*, relied on the CS-coupling on the worldvolume of D-brane [Green, Harvey, Moore, 1996]

$$S_{CS} \sim \mu_p \int_{D_p} C \wedge \mathrm{Tr}\left(e^{i\frac{F}{2\pi}} \sqrt{\frac{\hat{A}(\mathcal{T})}{\hat{A}(\mathcal{N})}} \right)$$

  Then the question of odd number of fermionic zero modes seems remains. For example, for $SU(N_1)$ instanton when $N_3 + N_4$ is odd.

- The soluton is, the RR-tadpole cancellation conditions imply non-abelian gauge anomaly cancellation. [Leigh,Rozali,1998] [Franco,Uranga,2013]

Thank you