AdS$_3$ quantum gravity and finite-N expansions

Wei Li

Institute of Theoretical Physics, Chinese Academy of Sciences

[[arxiv 2511.00636] with Ji Hoon Lee]

Kavli-IPMU Mathematics and String seminar

Kashiwa, Feb 10, 2026

General motivation

- Understand finite N quantum gravity (via AdS/CFT)

  - Recall: there are 3 levels of AdS/CFT, depending on how N appears

- Weakest form: $N \to \infty$ (Planar limit ~ tree-level string)

$$\text{planar QFT} \longleftrightarrow \text{classical gravity} \qquad (\text{vN algebra: III}_1)$$

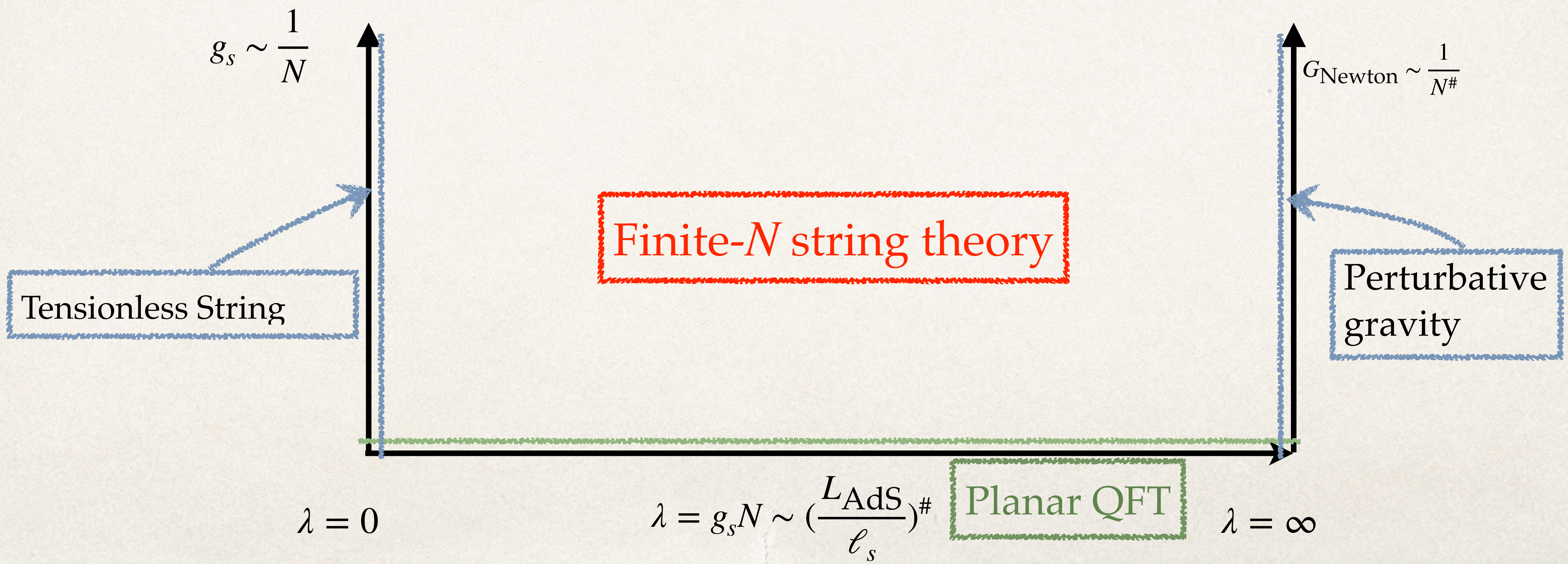
- Intermediate: large-N, with asymptotic $\frac{1}{N}$ expansion

$$\text{non-planar QFT} \longleftrightarrow \text{semi-classical gravity} \qquad (\text{vN algebra: II}_\infty)$$

(non-planar correction in QFT ~ loop corrections in gravity)

- Strongest form: finite-N

$$\text{finite-}N\text{ QFT} \longleftrightarrow \text{quantum gravity} \qquad (\text{vN algebra: I})$$



Today: finite-N from gravity point of view

- On QFT side, can easily set N finite
- The question is how to reproduce this finite-N QFT result from gravity side
- Not as $\frac{1}{N}$ expansion !
- Namely: how to realize the stringy exclusion principle?

Stringy exclusion principle

[Maldacena-Strominger '98, de Boer '98]

- Boundary dual CFT at finite-N: finite-N CFT should have fewer states than large-N CFT, and the number of states should decreases as N decreases (for fixed energy).
  - AdS_5/CFT_4: 4D $\mathcal{N} = 4$ $U(N)$ SYM
    - Single-trace operators are all orthogonal at $N \to \infty$
    - At finite-N, single-trace operators become inter-dependent due to **Trace Relations** (i.e. $\mathrm{Tr}[X^{n>N+1}]$ can be written in terms of $\mathrm{Tr}[X^{n\leq N}]$)
  - AdS_3/CFT_2: $\mathrm{Sym}^N(\mathcal{M}_4)$
    - Total cycle length cannot be bigger than N
- Bulk non-perturbative gravity: *naively* should have more states than perturbative gravity, and the number of states should increase as g_s increases (for fixed energy).
  - black holes
  - bulk brane, bulk defects, etc…
- Stringy exclusion principle:

Bulk non-perturbative effects should cut-down number of states!

To understand finite-N quantum gravity, we need to understand the mechanism of Stringy Exclusion Principle.

Today's strategy: finite-N expansion (of GCE partition function).

Finite-N expansion of GCE partition function

- GCE partition function:

$$\mathcal{Z}^{\mathrm{GCE}}(\zeta;\mathfrak{q}) := \sum_{N=0}^{\infty} \zeta^N Z_N(\mathfrak{q})$$

- Given $\mathcal{Z}^{\mathrm{GCE}}(\zeta;\mathfrak{q})$, two ways to obtain CE partition function $Z_N(\mathfrak{q})$.

  1. Finite-N from QFT: as coefficient of ζ^N term in ζ-expansion of $\mathcal{Z}^{\mathrm{GCE}}(\zeta;\mathfrak{q})$
  2. Finite-N from gravity: start from contour integral

$$Z_N(\mathfrak{q}) = \oint_{\zeta=0} \frac{d\zeta}{2\pi i} \zeta^{-N-1} \mathcal{Z}^{\mathrm{GCE}}(\zeta;\mathfrak{q})$$

  - Deform the contour

$$Z_N(\mathfrak{q}) = -\sum_{\{\zeta_k\}} \mathrm{Res}_{\zeta=\zeta_k}[\zeta^{-N-1}\mathcal{Z}^{\mathrm{GCE}}(\zeta;\mathfrak{q})]$$

    1. Choice of contour (i.e. the set of poles $\{\zeta_k\}$) depends on regime of $\mathfrak{q}$
    2. Find bulk interpretation for contribution from each pole
    3. $\exists$ negative modes: analytical continuation $\Longrightarrow$ alternating signs $(-1)^k$

Stringy exclusion principle is realized by cancellation among $\hat{Z}_k(\mathfrak{q})$ from different k, at each order of $\mathfrak{q}$.

Today's goal

- using finite-N expansion to understand mechanism of stringy exclusion principle for AdS_3

Today's goal

- using finite-N expansion to understand mechanism of stringy exclusion principle for AdS_3

Let us first review the AdS_5 case.

Example: 1/2-BPS partition function of 4D $\mathcal{N} = 4$ SYM

[Imamura '21, Gaiotto-Lee '21]

- GCE partition function:

$$\mathcal{Z}^{\text{GCE}}(\zeta; q) = \prod_{n=0}^{\infty} \frac{1}{1 - \zeta q^n} \qquad \text{(only simple poles: } \zeta = q^{-k}\text{, for } k \in \mathbb{N}_0\text{)}$$

- CE partition function:

1. Finite-N from QFT: as coefficient of ζ^N term:

$$Z_N(q) = \prod_{n=1}^{N} \frac{1}{1 - q^n}$$

2. Finite-N from Gravity: as (deformed) contour integral:

$$Z_N(q) = Z_\infty(q) \sum_{k=0}^{\infty} q^{kN} \hat{Z}_k(q)$$

$$\text{with } \hat{Z}_k(q) = \prod_{n=1}^{k} \frac{1}{1 - q^{-n}} = \frac{(-1)^k q^{\frac{k(k+1)}{2}}}{\prod_{n=1}^{k} \left(1 - q^n\right)}$$

Boundary explanation of finite-N expansion: trace relations

[Imamura '21, Gaiotto-Lee '21]

$$Z_N(q) = Z_\infty(q) \sum_{k=0}^{\infty} q^{kN} \hat{Z}_k(q) \qquad \text{with} \quad \hat{Z}_k(q) = \prod_{n=1}^{k} \frac{1}{1-q^{-n}} = \frac{(-1)^k q^{\frac{k(k+1)}{2}}}{\prod_{n=1}^{k}(1-q^n)}$$

- $Z_N(q) = \prod_{n=1}^{N} \frac{1}{1-q^n}$: counting # of independent gauge-invariant operators $\prod \mathrm{Tr}[X^n]$ and derivatives

  $X: N \times N$ matrix

  - $Z_\infty(q) = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$: as $N \to \infty$, all $\mathrm{Tr}[X^n]$ become independent
  - From $Z_\infty(q)$ to $Z_N(q)$: **Trace Relations** ($\mathrm{Tr}[X^{n \geq N+1}]$ can be expressed in terms of $\mathrm{Tr}[X^{n \leq N}]$)

    $N = 1$: $\mathrm{Tr}[X^2] = \mathrm{Tr}[X]^2$

    $N = 2$: $\mathrm{Tr}[X^3] = \frac{1}{2}(3\mathrm{Tr}[X]^2\mathrm{Tr}[X] - \mathrm{Tr}[X]^3)$

    $\vdots$ $\qquad\vdots$

- $\sum_{k=0}^{\infty} q^{kN} \hat{Z}_k(q)$: implementing trace relations (and relations of relations…)
  - Trace relations starts at q^N
  - Cancelation due to $(-1)^k$

Bulk explanation of finite-N expansion: giant gravitons

[Imamura '21, Gaiotto-Lee '21]

$$Z_N(q) = Z_\infty(q) \sum_{k=0}^{\infty} q^{kN} \hat{Z}_k(q) \qquad \text{with} \quad \hat{Z}_k(q) = \prod_{n=1}^{k} \frac{1}{1-q^{-n}} = \frac{(-1)^k q^{\frac{k(k+1)}{2}}}{\prod_{n=1}^{k}(1-q^n)}$$

- $Z_\infty(q) = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$: $\frac{1}{2}$−BPS S^5 KK modes (background close-string spectrum)

- $\hat{Z}_k(q) = Z_{N=k}(q^{-1})$: $\frac{1}{2}$−BPS open-string excitation on k giant graviton D3-branes (on $\mathbb{R} \times S^3 \subset \mathrm{AdS}_5 \times S^5$)

  - Finite-N effect starts at q^N
  - Negative modes + analytical continuation

    $\Longrightarrow$ alternating signs $(-1)^k$ and energy upshift $\sim \mathcal{O}(k^2)$

This realizes Stringy Exclusion Principle!

light/medium/heavy regimes

$$Z_N(q) = Z_\infty(q) \sum_{k=0}^{\infty} q^{kN} \hat{Z}_k(q) \qquad \text{with} \;\; \hat{Z}_k(q) = \prod_{n=1}^{k} \frac{1}{1-q^{-n}} = \frac{(-1)^k q^{\frac{k(k+1)}{2}}}{\prod_{n=1}^{k} (1-q^n)}$$

AdS5/CFT4	Light O(1)	Medium O(N)	Heavy O(N^2)
Boundary	single-trace operators	trace relations kick in	trace relations dominate at k~N
Bulk	KK spectrum	• giant gravitons (plus closed-string descendants)	• black holes • Multiple (k~N) GG (plus closed-string descendants)

Let us now look at AdS_3 case.

AdS_3/CFT_2

- In IIB string, consider near-horizon limit of D1-D5 system, with D5 wrapping $\mathscr{M}_4$ ($= \mathbb{T}^4$, K3)

- Bulk: IIB SUGRA on $AdS_3 \times S^3 \times \mathscr{M}_4$ with

  - (Q_1, Q_5) RR background

  - For BPS sector, can use U-duality to move to the point with

$$(Q_1, Q_5) = (N, 1) \text{ NS background, with } N \equiv Q_1 Q_5$$

- Bndy: Symmetric orbifold CFT

$$\mathrm{Sym}^N(\mathscr{M}_4) = (\mathscr{M}_4)^{\otimes N}/S_N$$

- Worldsheet CFT: $\mathfrak{psu}(1,1\,|\,2)_1 \oplus \overline{\mathfrak{psu}(1,1\,|\,2)_1}$ together with $\mathscr{M}_4$ fields (top.twisted)

Symmetric orbifold CFT

$$\mathrm{Sym}^N(\mathscr{M}_4) = (\mathscr{M}_4)^{\otimes N}/S_N$$

- (small) $\mathcal{N} = (4,4)$ superconformal symmetry with

$$c = \bar{c} = 6N$$

- Seed theory:
    - $\mathbb{T}^4$: 4 real scalars + 4 fermions
    - K3: $\mathbb{T}^4/\mathbb{Z}_2$
- Symmetric orbifold: sectors labeled by conjugacy classes of S_N

$$[\rho] \sim (1)^{N_1}(2)^{N_2}\ldots(m)^{N_m}$$

    - Single particle spectrum: $[\rho] = (12\ldots w) \longrightarrow w$-twisted sector

Today: chiral-chiral spectrum

- $\mathcal{N} = 4$ SCA:

$$\text{Virasoro } L_n \text{ with } c = 6N, \quad \text{Supercharges } \; G^{\pm,\pm}, \qquad \mathfrak{su}(2)_R \text{ current } J_n^{3,\pm}$$

- $\mathcal{N} = 4$ SCA has spectral flow automorphism (labeled by w)
  - In particular: $L_0 \;\rightarrow\; L_0 + wJ_0 + \frac{c}{24}w^2\,, \quad J_0 \;\rightarrow\; J_0 + \frac{c}{12}w \qquad\qquad (J_0 := J_0^3)$
  - Spectral flow with $w = 1$: RR $\longrightarrow$ NS

$$\text{R-sector ground-state } (L_0 = \tfrac{c}{24}) \longrightarrow \text{NS sector chiral-primaries } (L_0 = J_0)$$

- Today, will focus on chiral-chiral primaries

$$L_0 = J_0 \qquad \bar{L}_0 = \bar{J}_0$$

- E.g. chiral-chiral spectrum of $\mathbb{T}^4$: for each w-twisted sector ($w \in \mathbb{N}$), a BPS quartet (both left and right)

$$\begin{pmatrix} & L_0 = J_0 = \frac{w-1}{2} & \\ L_0 = J_0 = \frac{w}{2} & & L_0 = J_0 = \frac{w}{2} \\ & L_0 = J_0 = \frac{w+1}{2} & \end{pmatrix} \begin{pmatrix} & \bar{L}_0 = \bar{J}_0 = \frac{w-1}{2} & \\ \bar{L}_0 = \bar{J}_0 = \frac{w}{2} & & \bar{L}_0 = \bar{J}_0 = \frac{w}{2} \\ & \bar{L}_0 = \bar{J}_0 = \frac{w+1}{2} & \end{pmatrix}$$

Chiral-chiral GCE partition function

- CE partition for chiral-chiral primaries in NS sector

$$Z^{\text{CE}}_{\text{NS}}(y,\bar{y}) := \text{Tr}_{\mathscr{H}_{\text{c.c.}} \otimes \mathscr{H}_{\text{c.c.}}}[y^{2L_0}\bar{y}^{2\bar{L}_0}]$$

- GCE partition for chiral-chiral primaries in NS sector : [DMVV '96]

$$\mathcal{Z}^{\text{GCE}}_{\text{NS}}(\zeta; y, \bar{y}) = \prod_{n=1}^{\infty} \prod_{r,\bar{r}=0}^{2} \frac{1}{\left(1-(-1)^{r+\bar{r}}\zeta^n y^{n+r-1}\bar{y}^{n+\bar{r}-1}\right)^{(-1)^{r+\bar{r}}h^{r,\bar{r}}}}$$

$\mathbb{T}^4$:

		1		
	2		2	
1		4		1
	2		2	
		1		

$K3$:

		1		
	0		0	
1		20		1
	0		0	
		1		

Finite-N from boundary perspective

- $Z_N(y, \bar{y}) =$ coeff. of ζ^N term of ζ-expansion of $\mathscr{Z}^{\text{GCE}}(\zeta; y, \bar{y})$
- $Z_\infty(y, \bar{y}) = -\,\text{Res}_{\zeta=1}\mathscr{Z}^{\text{GCE}}(\zeta; y, \bar{y})$

Let's compute $Z_N(y, \bar{y})$ for K3.

For simplicity, set $y = \bar{y}$, let $\Delta \equiv L_0 + \bar{L}_0$

$$Z_N(y, y) = \sum_{\Delta=0} d(\Delta)\, y^{2\Delta}$$

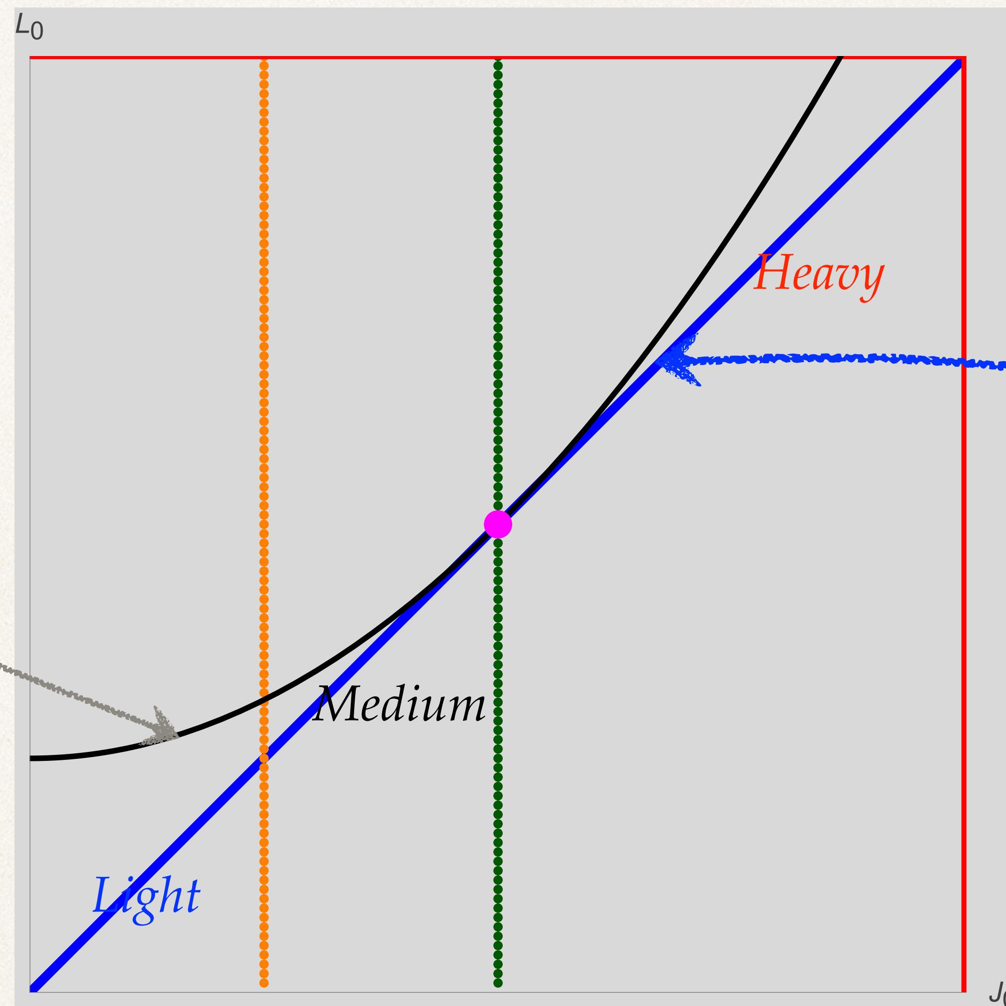
Comparing finite-N with KK spectrum

$$
\begin{aligned}
N = \infty &: \; 1 + 23y^2 + 300y^4 + 2876y^6 + 22450y^8 + 150606y^{10} + 897440y^{12} + \cdots \\
N = 1 &: \; 1 + 22y^2 + y^4 \\
N = 2 &: \; 1 + 23y^2 + 276y^4 + 23y^6 + y^8 \\
N = 3 &: \; 1 + 23y^2 + 299y^4 + 2554y^6 + 299y^8 + 23y^{10} + y^{12} \\
N = 4 &: \; 1 + 23y^2 + 300y^4 + 2852y^6 + 19298y^8 + 2852y^{10} + 300y^{12} + 23y^{14} + y^{16} \\
N = 5 &: \; 1 + 23y^2 + 300y^4 + 2875y^6 + 22127y^8 + 125604y^{10} + 22127y^{12} + 2875y^{14} + \cdots \\
&\;\vdots
\end{aligned}
$$

light/medium/heavy states ($c = 6N$)

cosmic censorship bound

$L_0 \geq \frac{J_0^2}{c/6} + \frac{c}{24}$



chiral primary

$L_0 = J_0$

$\frac{c}{24} = \frac{N}{4}$ $\frac{c}{12} = \frac{N}{2}$

unitarity bound: $\frac{c}{6} = N$

Match with KK spectrum up to $L_0 = J_0 \leq \frac{N}{4}$ (light spectrum)

$N = \infty : \quad 1 + 23y^2 + 300y^4 + 2876y^6 + 22450y^8 + 150606y^{10} + 897440y^{12} + \cdots$

$N = 1 : \quad 1 + 22y^2 + y^4$

$N = 2 : \quad 1 + 23y^2 + 276y^4 + 23y^6 + y^8$

$N = 3 : \quad 1 + 23y^2 + 299y^4 + 2554y^6 + 299y^8 + 23y^{10} + y^{12}$

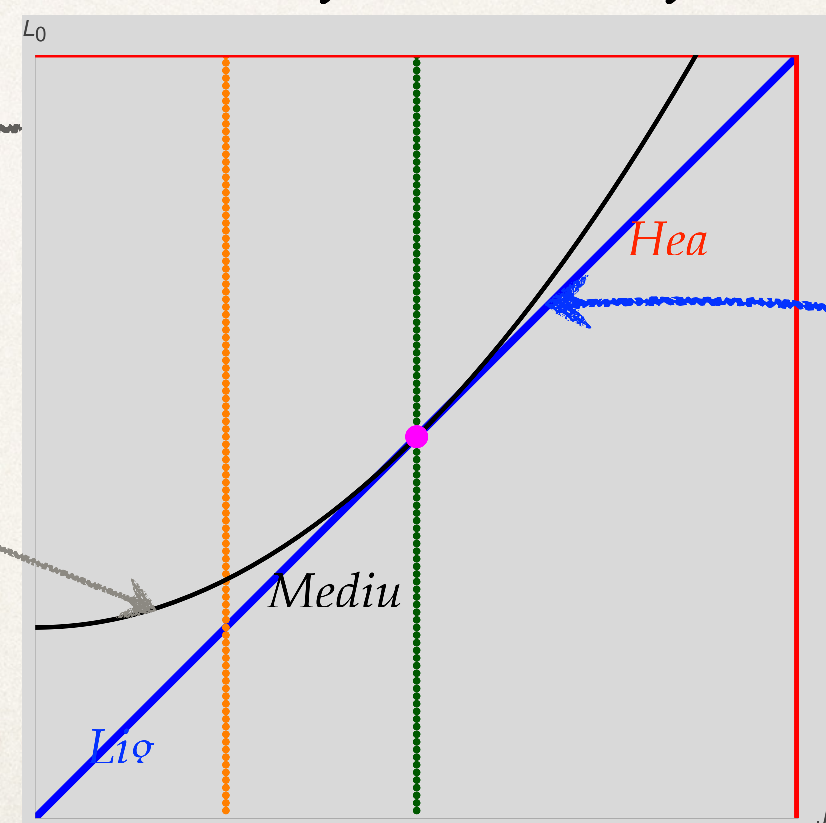
$N = 4 : \quad 1 + 23y^2 + 300y^4 + 2852y^6 + 19298y^8 + 2852y^{10} + 300y^{12} + 23y^{14} + y^{16}$

$N = 5 : \quad 1 + 23y^2 + 300y^4 + 2875y^6 + 22127y^8 + 125604y^{10} + 22127y^{12} + 2875y^{14} + \cdots$

$\vdots$

cosmic censorship bound

$L_0 \geq \frac{J_0^2}{c/6} + \frac{c}{24}$



$\frac{c}{24} = \frac{N}{4}$ $\quad \frac{c}{12} = \frac{N}{2}$

unitarity bound: $\frac{c}{6} = N$

chiral primary

$L_0 = J_0$

Gravitational back reaction starts at

$\frac{N}{4} < L_0 = J_0$

$\frac{N}{2} \leq L_0 = J_0$: heavy spectrum

$$N=\infty: \quad 1+23y^2+300y^4+2876y^6+22450y^8+150606y^{10}+897440y^{12}+\cdots$$

$$N=1: \quad 1+\underline{22}y^2+y^4$$

$$N=2: \quad 1+23y^2+\underline{276}y^4+23y^6+y^8$$

$$N=3: \quad 1+23y^2+299y^4+\underline{2554}y^6+299y^8+23y^{10}+y^{12}$$

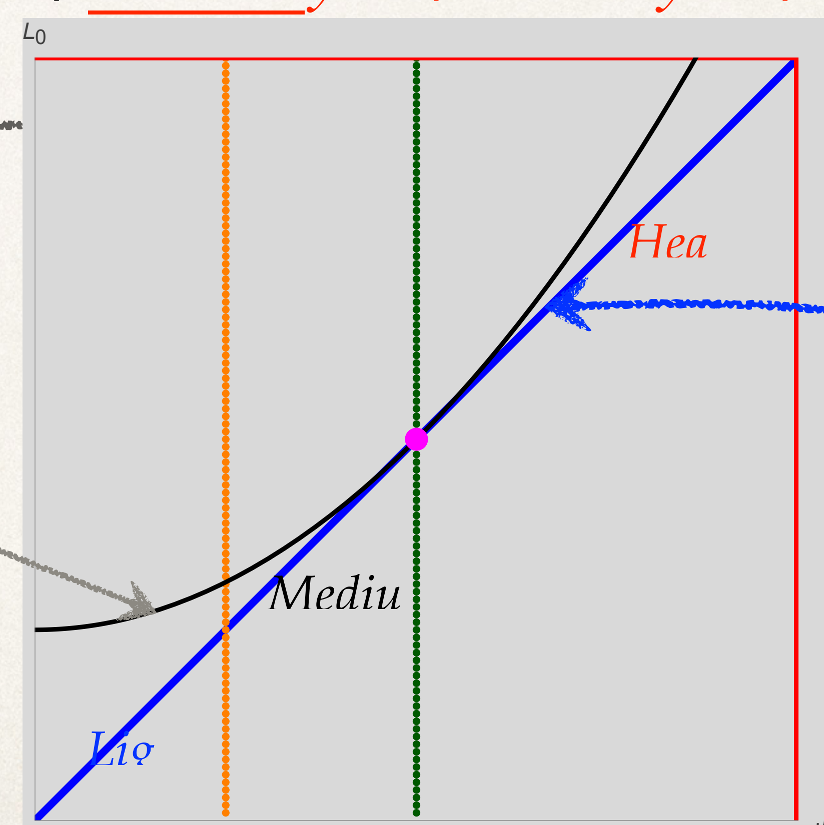
$$N=4: \quad 1+23y^2+300y^4+2852y^6+\underline{19298}y^8+2852y^{10}+300y^{12}+23y^{14}+y^{16}$$

$$N=5: \quad 1+23y^2+300y^4+2875y^6+22127y^8+\underline{125604}y^{10}+22127y^{12}+2875y^{14}+\cdots$$

$$\vdots$$

cosmic censorship bound

$$L_0 \geq \frac{J_0^2}{c/6} + \frac{c}{24}$$



chiral primary
$L_0 = J_0$

$\frac{c}{24} = \frac{N}{4}$ $\quad \frac{c}{12} = \frac{N}{2}$

unitarity bound: $\frac{c}{6} = N$

Gravitational back reaction dominates for

$$\frac{N}{2} \leq L_0 = J_0$$

What are medium spectra?

$N=\infty$: $1+23y^2+300y^4+2876y^6+22450y^8+150606y^{10}+897440y^{12}+\cdots$

$N=1$: $1+\underline{22}y^2+y^4$

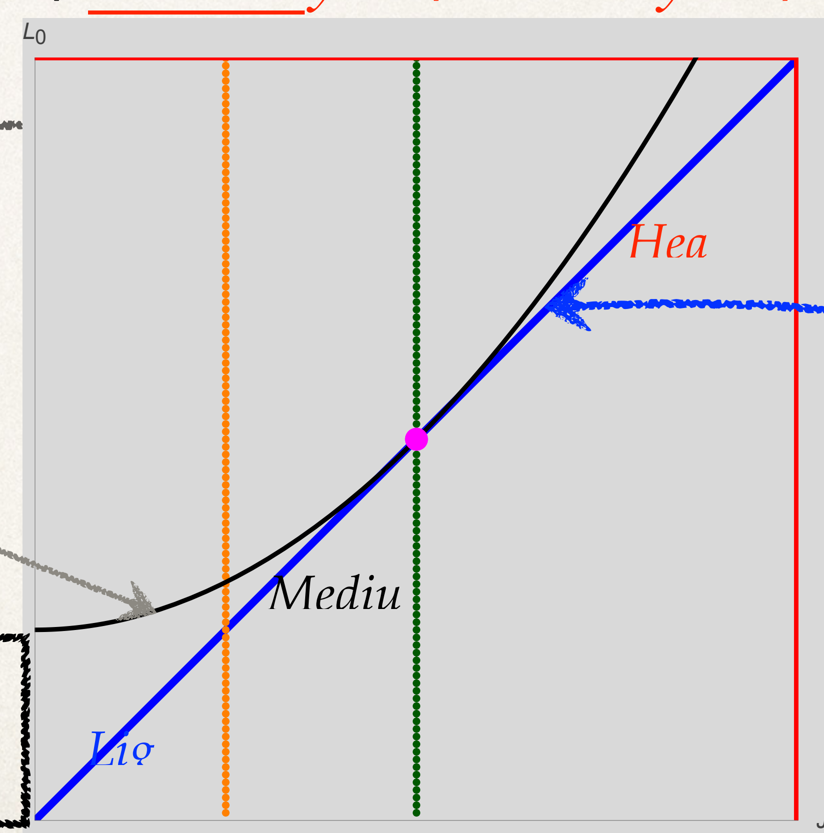
$N=2$: $1+23y^2+\underline{276}y^4+23y^6+y^8$

$N=3$: $1+23y^2+299y^4+\underline{2554}y^6+299y^8+23y^{10}+y^{12}$

$N=4$: $1+23y^2+300y^4+2852y^6+\underline{19298}y^8+2852y^{10}+300y^{12}+23y^{14}+y^{16}$

$N=5$: $1+23y^2+300y^4+2875y^6+22127y^8+\underline{125604}y^{10}+22127y^{12}+2875y^{14}+\cdots$

$\vdots$



cosmic censorship bound

$$L_0 \geq \frac{J_0^2}{c/6} + \frac{c}{24}$$

chiral primary

$L_0 = J_0$

$\frac{c}{24} = \frac{N}{4}$ $\frac{c}{12} = \frac{N}{2}$

unitarity bound: $\frac{c}{6} = N$

Q1: what are the medium spectra?

Q2: How to reproduce all these integers from gravity?

Let us apply the finite-N expansion to this case.

Finite-N from bulk perspective

- GCE partition for chiral-chiral primaries in NS sector :

$$\mathcal{Z}^{\text{GCE}}_{\text{NS}}(\zeta; y, \bar{y}) = \prod_{n=1}^{\infty} \prod_{r,\bar{r}=0}^{2} \frac{1}{\left(1 - (-1)^{r+\bar{r}} \zeta^n y^{n+r-1} \bar{y}^{n+\bar{r}-1}\right)^{(-1)^{r+\bar{r}} h^{r,\bar{r}}}}$$

$\mathbb{T}^4$:

		1		
	2		2	
1		4		1
	2		2	
		1		

K3 :

		1		
	0		0	
1		20		1
	0		0	
		1		

- 4 families of simple poles (from 4 corners of Hodge diamond):

$$\zeta^{\pm,\pm}_{k,m} = e^{2\pi i \frac{m}{k}} y^{-\left(1 \pm \frac{1}{k}\right)} \bar{y}^{-\left(1 \pm \frac{1}{k}\right)} \quad \text{with} \quad \begin{cases} k = 1,2,3\cdots \\ m = 0,1,2\cdots, k-1 \end{cases}$$

- Wall of essential singularities (from center of Hodge diamond):

$$|\zeta| = |y^{-1}\bar{y}^{-1}|$$

- Finite-N from bulk : $$Z_N(y,\bar{y}) = -\sum_{\{\zeta_k\}} \text{Res}_{\zeta=\zeta_k}[\zeta^{-N-1}\mathcal{Z}^{\text{GCE}}(\zeta; y, \bar{y})]$$

Q1: How to choose $\{\zeta_k\}$? Q2: Bulk interpretation of each residue? Q3: How does bulk realize SEP?

Evaluating residues and sum over m

- Simple poles have $\mathbb{Z}_k$ orbifold structures: $\zeta^{\pm,\pm}_{k,m} = e^{2\pi i \frac{m}{k}} y^{-\left(1 \pm \frac{1}{k}\right)} \bar{y}^{-\left(1 \pm \frac{1}{k}\right)}$

$\Longrightarrow$ For each $k \in \mathbb{N}_0$, sum the residues over $\mathbb{Z}_k$ images $m = 0,1,\ldots k-1$:

$$\hat{Z}_k^{(\pm,\pm)}(y,\bar{y}) \equiv -\sum_{m=0}^{k-1} \mathrm{Res}_{\zeta=\zeta^{(\pm,\pm)}_{k,m}} \left[\zeta^{-N-1}\mathscr{Z}(\zeta; y, \bar{y})\right]$$

$$= \frac{1}{k}\sum_{m=0}^{k-1} D_k^{(\pm,\pm)}\left(e^{2\pi i m} y, \bar{y}\right)$$

$$\text{with} \quad D_k^{(\pm,\pm)}(y,\bar{y}) \equiv y^{N\left(1 \pm \frac{1}{k}\right)} \bar{y}^{N\left(1 \pm \frac{1}{k}\right)} \prod_{\substack{n=1 \\ n \neq k}}^{\infty} \frac{1}{(1 - y^{\mp(\frac{n}{k}-1)}\bar{y}^{\mp(\frac{n}{k}-1)})} \cdot \left(\text{remaining}\right)$$

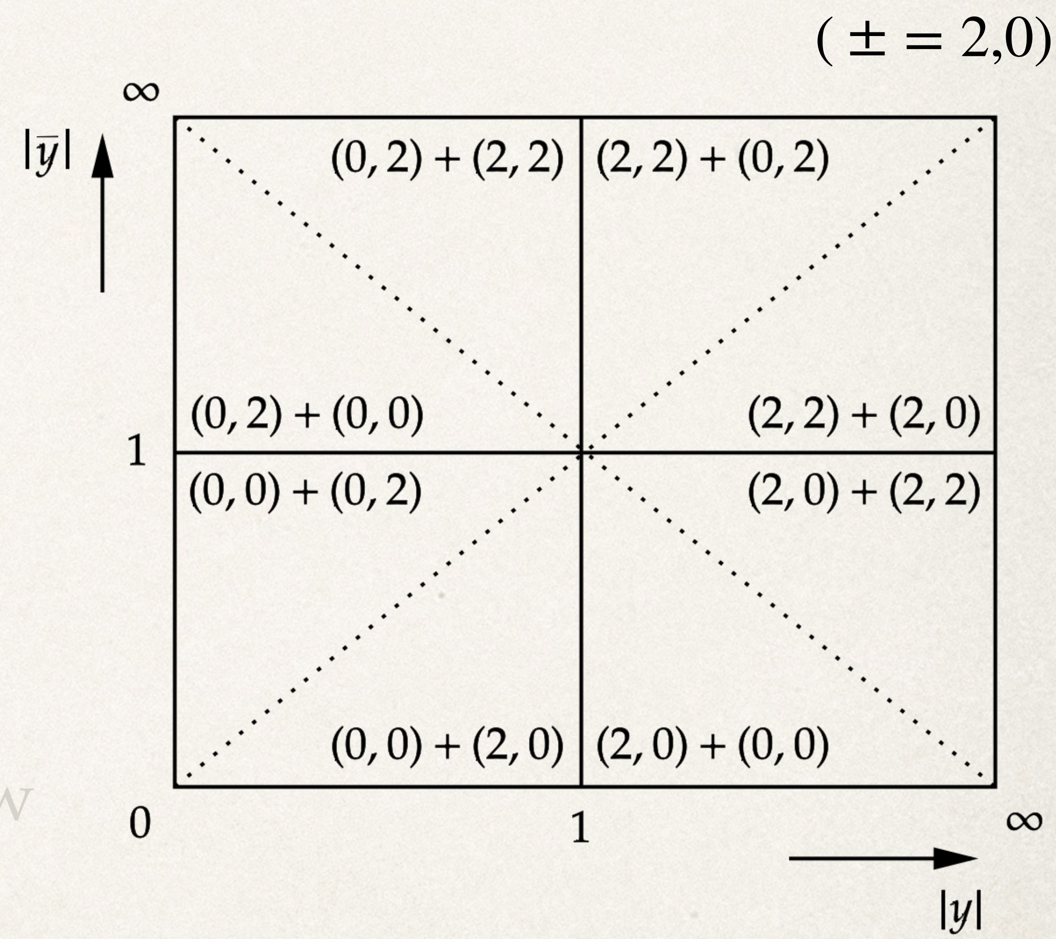
$$\text{e.g. K3:} \qquad \text{remaining} = \prod_{n=1}^{\infty} \frac{(1 - y^{\mp(\frac{n}{k}-1)}\bar{y}^{\mp(\frac{n}{k}-1)})}{\prod_{s,\bar{s}=\pm}(1 - y^{\mp(\frac{n}{k}-s)}\bar{y}^{\mp(\frac{n}{k}-\bar{s})})} \cdot \frac{1}{(1 - y^{\mp\frac{n}{k}}\bar{y}^{\mp\frac{n}{k}})^{20}}$$

Stokes phenomenon

- Wall of essential singularities at $|\zeta| = |y^{-1}\bar{y}^{-1}|$

$\Longrightarrow$ Contributing saddles depend on regime of $(y, \bar{y})$:

$$|y| = |\bar{y}| < 1 : \quad (-,-)$$
$$|\bar{y}| < |y| < 1 : \quad (-,-) \;\&\; (+,-)$$
$$\vdots$$



Let's demonstrate with $|y| = |\bar{y}| < 1$, and show

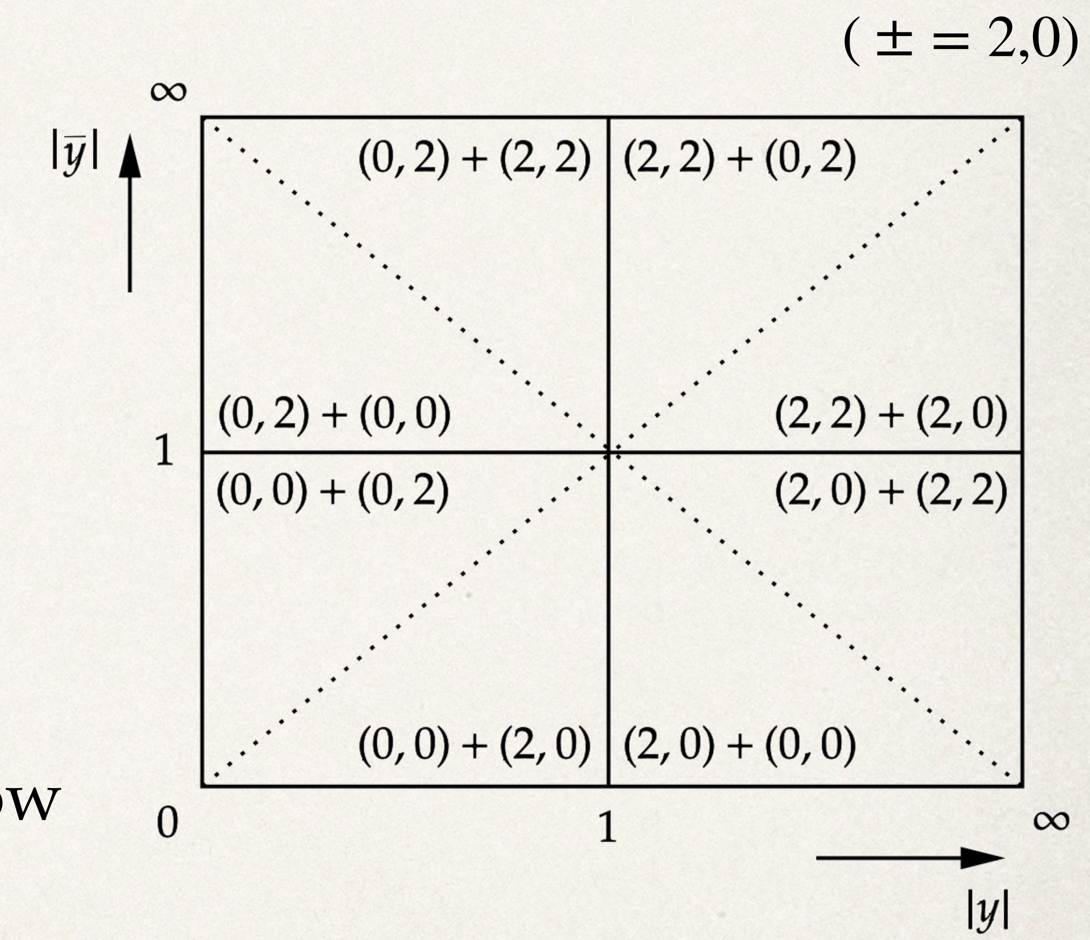
$$Z_N(y, y) = \sum_{k=1}^{\infty} \hat{Z}_k^{(-,-)}(y, y)$$

Stokes phenomenon

- Wall of essential singularities at $|\zeta| = |y^{-1}\bar{y}^{-1}|$

$\implies$ Contributing saddles depend on regime of $(y, \bar{y})$:

$$|y| = |\bar{y}| < 1 : \quad (-,-)$$
$$|\bar{y}| < |y| < 1 : \quad (-,-)\ \&\ (+,-)$$
$$\vdots$$



Let's demonstrate with $|y| = |\bar{y}| < 1$, and show

$$Z_N(y, y) = \sum_{k=1}^{\infty} \hat{Z}_k^{(-,-)}(y, y)$$

Reproducing finite-N by summing over bulk: $N = 1$

$$Z_{N=1}(y, y) = 1 + 22y^2 + y^4 \qquad \text{(K3)}$$

Let's compare with bulk result $\displaystyle\sum_{k=1}^{\text{cutoff } K} \hat{Z}_k^{(-,-)}(y, y)$

$$\begin{array}{lcl}
K = 1 & : & 1 + 23y^2 + 300y^4 + 2876y^6 + 22450y^8 + 150606y^{10} + \cdots \\
K = 2 & : & 1 + 22y^2 - 23y^4 - 22450y^6 - 1025620y^8 - 28954249y^{10} + \cdots \\
K = 3 & : & 1 + 22y^2 + y^4 + 3176y^6 + 4879226y^8 + 604364555y^{10} + \cdots \\
K = 4 & : & 1 + 22y^2 + y^4 - 24y^6 - 1048070y^8 - 2094819829y^{10} + \cdots \\
K = 5 & : & 1 + 22y^2 + y^4 + 0y^6 + 25626y^8 + 609221331y^{10} + \cdots \\
K = 6 & : & 1 + 22y^2 + y^4 + 0y^6 - 24y^8 - 30002319y^{10} + \cdots \\
K = 7 & : & 1 + 22y^2 + y^4 + 0y^6 + 0y^8 + 176232y^{10} + \cdots \\
K = 8 & : & 1 + 22y^2 + y^4 + 0y^6 + 0y^8 - 24y^{10} + \cdots \\
K = 9 & : & 1 + 22y^2 + y^4 + 0y^6 + 0y^8 + 0y^{10} + \cdots \\
\vdots & &
\end{array}$$

Reproducing finite-N by summing over bulk: $N=2$

$$Z_{N=2}(y,y) = 1 + 23y^2 + 276y^4 + 23y^6 + y^8 \qquad \text{(K3)}$$

Let's compare with bulk result $\displaystyle\sum_{k=1}^{\text{cutoff } K} \widehat{Z}_k^{(-,-)}(y,y)$

$$K=1 \quad : \quad 1 + 23y^2 + 300y^4 + 2876y^6 + 22450y^8 + 150606y^{10} + \cdots$$

$$K=2 \quad : \quad 1 + 23y^2 + 276y^4 - 300y^6 - 150606y^8 - 5603634y^{10} + \cdots$$

$$K=3 \quad : \quad 1 + 23y^2 + 276y^4 + 24y^6 + 25326y^8 + 24398685y^{10} + \cdots$$

$$K=4 \quad : \quad 1 + 23y^2 + 276y^4 + 23y^6 - 323y^8 - 5754240y^{10} + \cdots$$

$$K=5 \quad : \quad 1 + 23y^2 + 276y^4 + 23y^6 + y^8 + 175932y^{10} + \cdots$$

$$K=6 \quad : \quad 1 + 23y^2 + 276y^4 + 23y^6 + y^8 - 324y^{10} + \cdots$$

$$K=7 \quad : \quad 1 + 23y^2 + 276y^4 + 23y^6 + y^8 + 0y^{10} + \cdots$$

$$\vdots$$

- Other regimes of $y, \bar{y}$ work similarly.
- We conclude that expansion in terms of $\hat{Z}_k^{\pm,\pm}(y, \bar{y})$ reproduces finite-N result computed from boundary.

- What is the bulk interpretation of $\hat{Z}_k^{(\pm,\pm)}(y, \bar{y})$?

$\hat{Z}_k^{(\pm,\pm)}$ is chiral-chiral partition function of $(\mathrm{AdS}_3 \times S^3)/\mathbb{Z}_k \times \mathcal{M}_4$

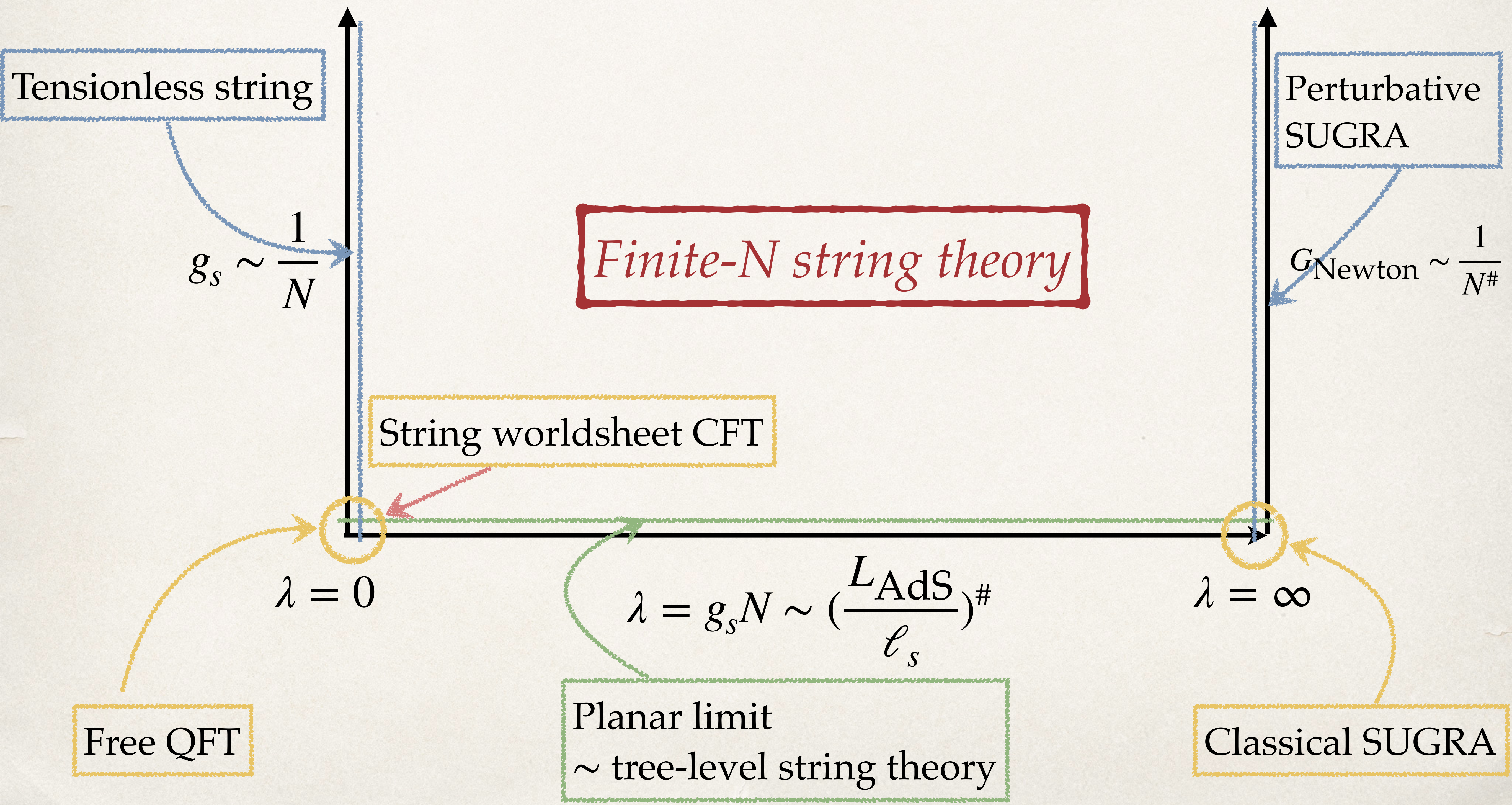
1. Expect the chiral-chiral sector to be independent of the position on moduli space. Therefore, we can consider $(Q_1, Q_5) = (N, 1)$ NS-NS background.

2. With this background, there exists a worldsheet description.

3. We can use this worldsheet description to show that $\hat{Z}_k^{(\pm,\pm)}$ comes from chiral-chiral partition function of $(\mathrm{AdS}_3 \times S^3)/\mathbb{Z}_k \times \mathcal{M}_4$

4. The result applies across the moduli space.

Parameter space of quantum gravity



Orbifold $(\mathrm{AdS}_3 \times S^3)/\mathbb{Z}_k$

$\mathrm{AdS}_3 \times S^3$ metric:

$$ds^2 = \left(-(r^2+1)dt^2 + \frac{dr^2}{r^2+1} + r^2\, d\phi^2\right) + \left(d\theta^2 + \cos^2\theta\, d\psi^2 + \sin^2\theta\, d\chi^2\right)$$

with $\quad (\phi, \psi, \chi) \sim (\phi + 2\pi, \psi, \chi) \sim (\phi, \psi + 2\pi, \chi) \sim (\phi, \psi, \chi + 2\pi)$

- $\mathbb{Z}_k$ orbifold action: $\quad (k \in \mathbb{N},\, s, \bar{s} \in \mathbb{Z})$

$$(\phi, \psi, \chi) \sim (\phi + \frac{2\pi}{k}, \psi - \frac{2\pi(s+\bar{s}+1)}{k}, \chi + \frac{2\pi(s-\bar{s})}{k})$$

- For $\hat{Z}_k^{(\pm,\pm)}$, need to consider $\quad (\pm = -1, 0)$

$$(s, \bar{s}) \in \{(0,0), (-1,0), (0,-1), (-1,-1)\}$$

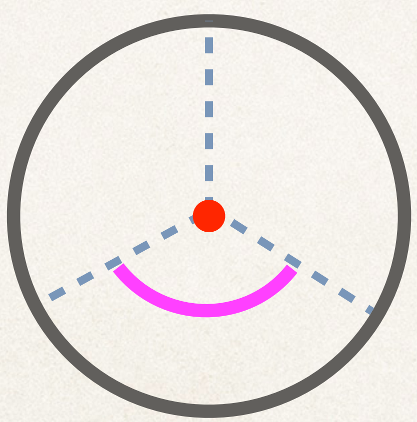
Orbifold $(\mathrm{AdS}_3 \times S^3)/\mathbb{Z}_k$

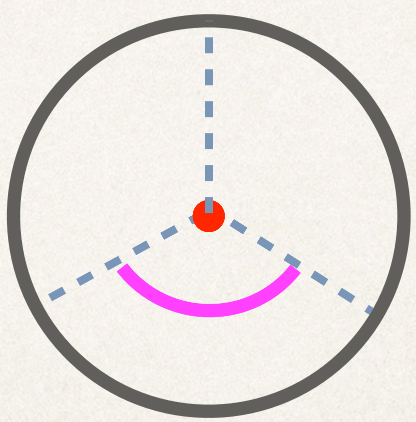
- Let's demonstrate with $s = \bar{s} = 0$

$$\Longrightarrow \mathbb{Z}_k \text{ orbifold action:} \quad (\phi, \psi) \sim (\phi + \frac{2\pi}{k}, \psi - \frac{2\pi}{k})$$

- E.g. $k = 3$ (at fixed t, χ)

(r, ϕ) (θ, ψ)





- These orbifolds are asymptotically $\mathrm{AdS}_3 \times S^3$ but with deficit angle $2\pi(1 - \frac{1}{k})$

Tensionless worldsheet before orbifolding

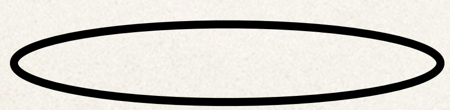
[Eberhardt-Gaberdiel-Gopakumar '18-19]

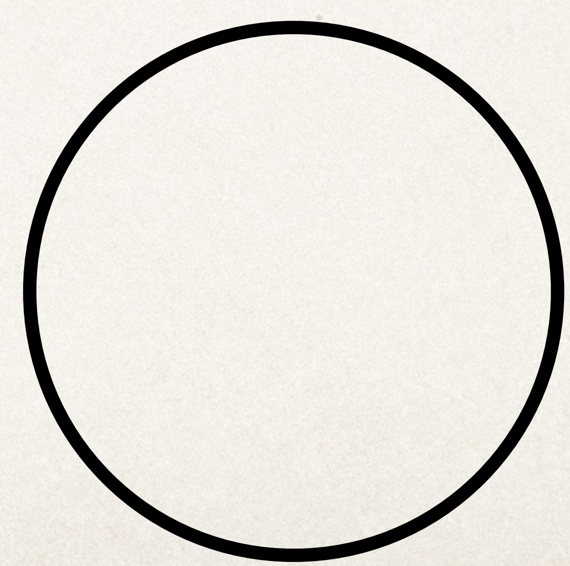
- Worldsheet CFT of tensionless string in $\mathrm{AdS}_3 \times S^3 \times \mathcal{M}_4$: $\mathfrak{psu}(1,1|2)_1 \oplus \overline{\mathfrak{psu}(1,1|2)_1} \oplus \mathcal{M}_4$ fields (top.twisted)
  - It captures the single particle spectrum of spacetime CFT: $\mathrm{Sym}^N(\mathcal{M}_4) = (\mathcal{M}_4)^{\otimes N}/S_N$ as $N \to \infty$
  - Physical spectrum: sum over all integer-spectrally flowed sector

$$\mathrm{z}^{\mathrm{w.s.}}(\mathfrak{q}) = \sum_{w=1}^{\infty} \mathfrak{z}^{(w)}(\mathfrak{q})$$

spacetime CFT (s.p. spectrum)		worldsheet CFT
w-twisted sector	← − − − − − →	w-spectral flowed sector
		(string excitation with winding number w)

  - Vacuum (untwisted): $w = 1$





Tensionless worldsheet before orbifolding

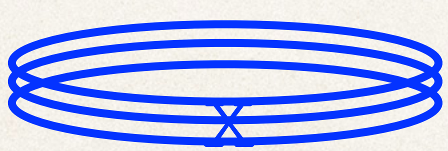
[Eberhardt-Gaberdiel-Gopakumar '18-19]

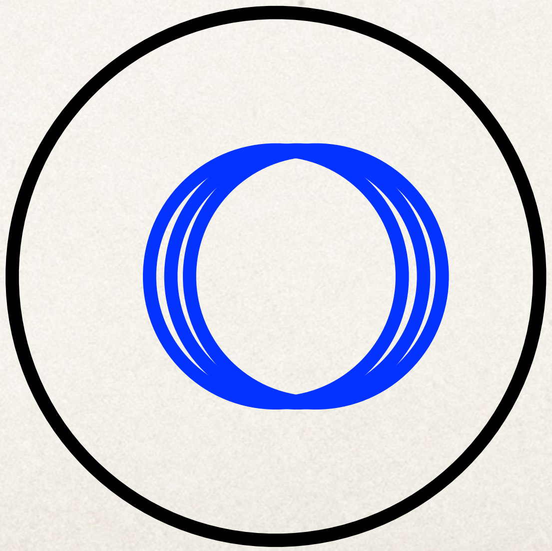
- Worldsheet CFT of tensionless string in $AdS_3 \times S^3 \times \mathscr{M}_4$: $\mathfrak{psu}(1,1|2)_1 \oplus \overline{\mathfrak{psu}(1,1|2)_1} \oplus \mathscr{M}_4$ fields (top.twisted)
  - It captures the single particle spectrum of spacetime CFT: $\mathrm{Sym}^N(\mathscr{M}_4) = (\mathscr{M}_4)^{\otimes N}/S_N$ as $N \to \infty$
  - Physical spectrum: sum over all integer-spectrally flowed sector

$$\mathfrak{z}^{\mathrm{w.s.}}(\mathfrak{q}) = \sum_{w=1}^{\infty} \mathfrak{z}^{(w)}(\mathfrak{q})$$

spacetime CFT (s.p. spectrum)		worldsheet CFT
w-twisted sector	← − − − − − →	w-spectral flowed sector
		(string excitation with winding number w)

  - Twisted sector: $w = 3$





Tensionless worldsheet for orbifolds

[Gaberdiel-Guo-Mathur '24]

- Worldsheet CFT of tensionless string in $(\mathrm{AdS}_3 \times S^3)/\mathbb{Z}_k \times \mathscr{M}_4$
  - Dual CFT: a subsector of $\mathrm{Sym}^N(\mathscr{M}_4)$ with new (non-perturbative) vacuum given by

$$(k)(k)(k)\ldots. \qquad (k) = |0_k^{\pm\pm}\rangle_{\mathrm{NS}}$$

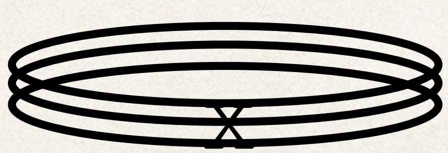
- Same content but spectrally flow becomes quantized by $\frac{1}{k}$

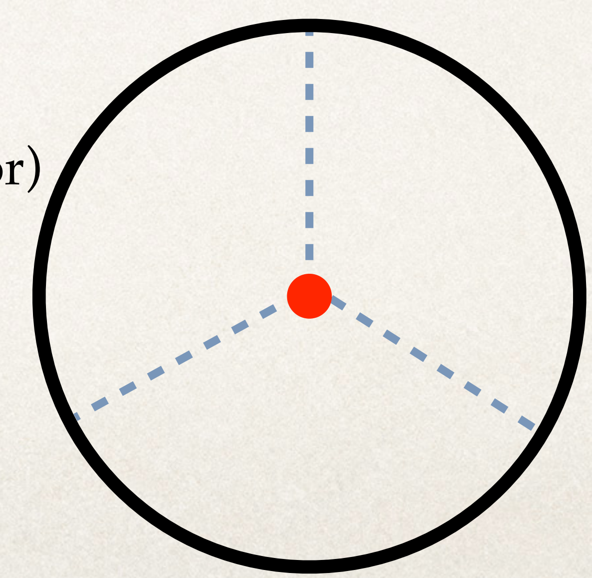
$$\mathfrak{z}^{\mathrm{w.s.}}(\mathfrak{q}) = \sum_{n=1}^{\infty} \mathfrak{z}^{(w=\frac{n}{k})}(\mathfrak{q})$$

(integer w: $\mathbb{Z}_k$-untwisted sector; fractional w: $\mathbb{Z}_k$-twisted sector)

spacetime CFT (s.p. spectrum)		worldsheet CFT
$(w\ k)$ twisted sector	$\leftarrow - - - - - \rightarrow$	w-spectral flowed sector
		(string excitation with winding number w)

- E.g. $k = 3$.
  - Vacuum: $w = 1$ (S_N : 3-twisted sector, $\mathbb{Z}_k$: untwisted sector)





Tensionless worldsheet for orbifolds

[Gaberdiel-Guo-Mathur '24]

- Worldsheet CFT of tensionless string in $(\text{AdS}_3 \times S^3)/\mathbb{Z}_k \times \mathscr{M}_4$
  - Dual CFT: a subsector of $\text{Sym}^N(\mathscr{M}_4)$ with new (non-perturbative) vacuum given by

$$(k)(k)(k)\ldots. \qquad (k) = |0_k^{\pm\pm}\rangle_{\text{NS}}$$

  - Same content but spectrally flow becomes quantized by $\frac{1}{k}$

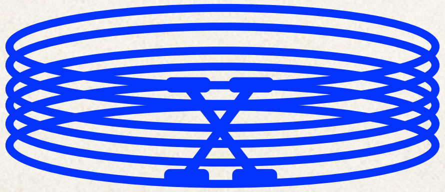
$$\mathcal{Z}^{\text{w.s.}}(q) = \sum_{n=1}^{\infty} \mathfrak{z}^{(w=\frac{n}{k})}(q)$$

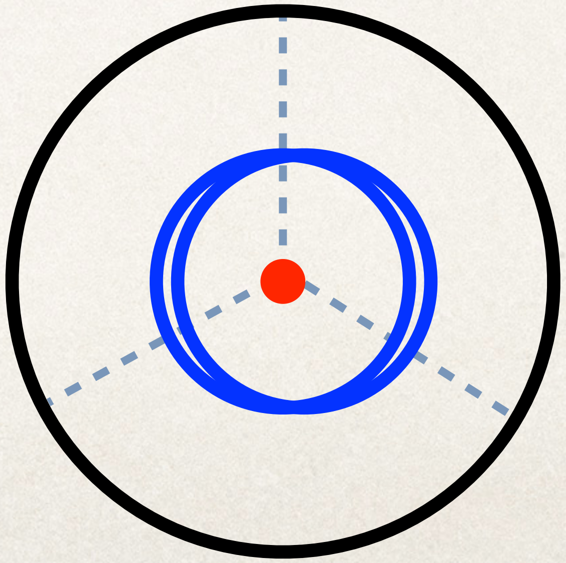
(integer w: $\mathbb{Z}_k$-untwisted sector; fractional w: $\mathbb{Z}_k$-twisted sector)

spacetime CFT (s.p. spectrum)		worldsheet CFT
$(w\ k)$ twisted sector	$\leftarrow - - - - - \rightarrow$	w-spectral flowed sector

(string excitation with winding number w)

- E.g. $k = 3$.
  - $w = 2$ (S_N: 6-twisted sector, $\mathbb{Z}_k$: untwisted sector)





Tensionless worldsheet for orbifolds

[Gaberdiel-Guo-Mathur '24]

- Worldsheet CFT of tensionless string in $(\mathrm{AdS}_3 \times S^3)/\mathbb{Z}_k \times \mathscr{M}_4$
  - Dual CFT: a subsector of $\mathrm{Sym}^N(\mathscr{M}_4)$ with new (non-perturbative) vacuum given by

$$(k)(k)(k)\ldots. \qquad (k) = |0_k^{\pm\pm}\rangle_{\mathrm{NS}}$$

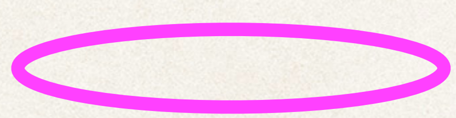
  - Same content but spectrally flow becomes quantized by $\frac{1}{k}$

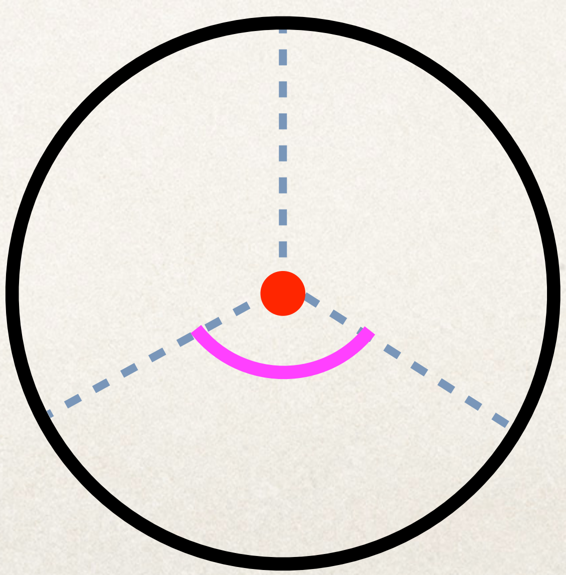
$$\mathfrak{Z}^{\mathrm{W.S.}}(\mathfrak{q}) = \sum_{n=1}^{\infty} \mathfrak{z}^{(w=\frac{n}{k})}(\mathfrak{q})$$

(integer w: $\mathbb{Z}_k$-untwisted sector; fractional w: $\mathbb{Z}_k$-twisted sector)

spacetime CFT (s.p. spectrum)		worldsheet CFT
$(w\ \ k)$ twisted sector	$\leftarrow - - - - - \rightarrow$	w-spectral flowed sector
		(string excitation with winding number w)

  - E.g. $k = 3$.
    - $w = \frac{1}{3}$ (S_N: untwisted sector, $\mathbb{Z}_k$: 1st twisted sector)





Tensionless worldsheet for orbifolds

[Gaberdiel-Guo-Mathur '24]

- Worldsheet CFT of tensionless string in $(\mathrm{AdS}_3 \times S^3)/\mathbb{Z}_k \times \mathscr{M}_4$
    - Dual CFT: a subsector of $\mathrm{Sym}^N(\mathscr{M}_4)$ with new (non-perturbative) vacuum given by

$$(k)(k)(k)\ldots. \qquad (k) = |0_k^{\pm\pm}\rangle_{\mathrm{NS}}$$

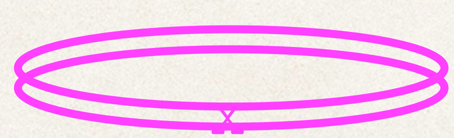
- Same content but spectrally flow becomes quantized by $\frac{1}{k}$

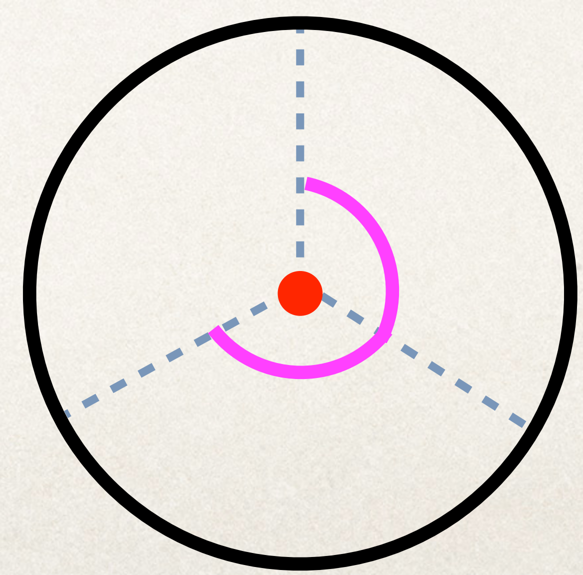
$$\mathfrak{Z}^{\text{w.s.}}(\mathfrak{q}) = \sum_{n=1}^{\infty} \mathfrak{z}^{(w=\frac{n}{k})}(\mathfrak{q})$$

(integer w: $\mathbb{Z}_k$-untwisted sector; fractional w: $\mathbb{Z}_k$-twisted sector)

spacetime CFT (s.p. spectrum)		worldsheet CFT
(wk)-twisted sector	$\leftarrow - - - - - \rightarrow$	w-spectral flowed sector
		(string excitation with winding number w)

- E.g. $k = 3$.
    - $w = \frac{2}{3}$ (S_N: 2-twisted sector, $\mathbb{Z}_k$: 2^{nd} twisted sector)





Reproducing $\hat{Z}_k^{(\pm,\pm)}(y,\bar{y})$ from worldsheet

1. Tree level:

$$L_0 = J_0 = \frac{N}{2}(1 \mp \frac{1}{k})\,, \qquad \bar{L}_0 = \bar{J}_0 = \frac{N}{2}(1 \mp \frac{1}{k})$$

2. One-loop: single-particle spectrum from worldsheet (before $\mathbb{Z}_k$ projection)

$${}^{w.s.}\mathsf{z}_k^{(\pm,\pm)}(y,\bar{y}) = \sum_{n=1}^{\infty} y^{\mp\frac{n}{k}} y^{\mp\frac{n}{k}} \left| y^{-1}(1-2y+y^2) \right|^2$$

3. Multi-particling

$$\mathrm{PE}[{}^{w.s.}\mathsf{z}_k^{(\pm,\pm)}(y,\bar{y}) - 1]$$

4. Project to $\mathbb{Z}_k$-invariant state

$$\hat{Z}_k^{(\pm,\pm)}(y,\bar{y}) = \frac{1}{k}\sum_{m=0}^{k-1} \mathrm{Tr}_{\mathscr{H}^k}\left((e^{\mp i\pi m(1+\delta)}y)^{2J_0} (e^{\mp i\pi m(1-\delta)}\bar{y})^{2\bar{J}_0} \right)$$

Conclusion: $\hat{Z}_k^{(\pm,\pm)}(y,\bar{y})$ compute the 1-loop partition function of chiral-chiral primary excitations on $(\mathrm{AdS}_3 \times S^3)/\mathbb{Z}_k \times \mathscr{M}_4$ orbifold backgrounds!

$\hat{Z}_k^{(-,-)}$ are chiral-chiral partition function of $(\mathrm{AdS}_3 \times S^3)/\mathbb{Z}_k \times \mathscr{M}_4$

$$\mathbb{Z}_k\text{-Projection } P = \frac{1}{k}\sum_{m=0}^{k-1} e^{2\pi i m(J_0+\bar{J}_0)} e^{2\pi i m(J_0-\bar{J}_0)}$$

(after multiplying)

- E.g. for K3

$$\hat{Z}_k^{(-,-)}(y,\bar{y}) = \frac{1}{k}\sum_{m=0}^{k-1} D_k^{(-,-)}\left(e^{2\pi i m}y,\bar{y}\right)$$

Tree-level

Multi-particle 1/2 BPS spectrum

$$D_k^{(-,-)}(y,\bar{y}) = y^{N\left(1-\frac{1}{k}\right)}\bar{y}^{N\left(1-\frac{1}{k}\right)}\Bigg(\prod_{\substack{n=1 \\ n\neq k}}^{\infty}\frac{1}{(1-y^{(\frac{n}{k}-1)}\bar{y}^{(\frac{n}{k}-1)})}$$

$$\times\prod_{n=1}^{\infty}\frac{1}{(1-y^{(\frac{n}{k}-1)}\bar{y}^{(\frac{n}{k}+1)})(1-y^{(\frac{n}{k}+1)}\bar{y}^{(\frac{n}{k}-1)})(1-y^{(\frac{n}{k}+1)}\bar{y}^{(\frac{n}{k}+1)})(1-y^{\frac{n}{k}}\bar{y}^{\frac{n}{k}})^{20}}\Bigg)$$

Final result: only positive integer modes after analytical continuation!

light+medium gives heavy

- Semi-classically, these orbifolds correspond to medium states in (large-c) CFT.

- However, there are negative modes in the one-loop partition function on these geometries.

    - From $\mathrm{Sym}^N(\mathscr{M}_4)$ viewpoint: cycle (k) can break into smaller ones in the $\mathbb{Z}_k$ twisted sector.

Two effects of negative modes

- E.g. for $|y|, |\bar{y}| < 1$, need to rewrite the factor

$$\frac{1}{\prod_{n=1}^{k-1}(1 - y^{\frac{n}{k}-1}\bar{y}^{\frac{n}{k}-1})} = \frac{(-1)^{k-1}y^{\frac{1}{2}(k-1)}\bar{y}^{\frac{1}{2}(k-1)}}{\prod_{n=1}^{k-1}(1 - y^{1-\frac{n}{k}}\bar{y}^{1-\frac{n}{k}})}$$

- Two effects:

1. Negative modes become positive, pushing medium states $L_0 = \frac{N}{2}(1-\frac{1}{k})$

   to heavy states $L_0 = \frac{N}{2}(1-\frac{1}{k})+\frac{1}{4}(k-1)$

2. Produce oscillation factor $(-1)^{k-1}$

   - Same mechanism as AdS_5/CFT_4

$$\hat{Z}_k(q) = \frac{1}{\prod_{n=1}^{k}(1 - q^{-n})} = \frac{(-1)^k q^{\frac{k(k+1)}{2}}}{\prod_{n=1}^{k}\left(1 - q^n\right)}$$

AdS5/CFT4 v.s. AdS3/CFT2

AdS5/CFT4	Light O(1)	Medium O(N)	Heavy O(N^2)
Boundary	single-trace operators	Trace relations kick in	trace relations dominate at k~N
Bulk	KK spectrum	• Giant gravitons (+ closed-string descendants)	• black holes • Multiple (k~N) GG (plus closed-string descendants)

AdS3/CFT2	Light L< N/4	Medium N/4 < L < N/2	Heavy N/2 < L
Boundary	Single-cycle operators	“Constraints” kick in	“Constraints” dominate at k~N
Bulk	KK spectrum	• Orbifold geometry (+ closed-string descendants)	• Black holes • Orbifolds with k~N (+closed-string descendants)

Summary

1. Finite-N expansion of chiral-chiral sector of AdS_3/CFT_2

2. For AdS_3: bulk saddles from finite-N expansion correspond to

$$(AdS_3 \times S^3)/\mathbb{Z}_k \times \mathscr{M}_4 \text{ orbifolds}$$

3. Alternating signs in 1-loop partition function of orbifolds produce truncation to finite-N

$\Longrightarrow$ realizing Stringy Exclusion Principle

$\longrightarrow$ These orbifolds can be viewed as AdS_3 analogue of Giant Gravitons

Further questions

- A direct computation of 1-loop partition function on the orbifold saddles
- A systematic analysis of analogue of trace relations
- Extend to $\frac{1}{4}$-BPS sector

Thank you very much!