1. Motivation : Wall crossing phenomena of moduli of semistable sheaves on $\mathbb{P}^2$

Moduli of semistable sheaves on a smooth projective surface X are constructed by GIT

$R \circlearrowleft G$ quotient by group G action

We can not treat all points of R, but good points (semistable points)

defined by a stability condition
(Gieseker - Maruyama stability)

$$\cancel{R/G} \implies R^{ss} \subset R \ , \ R^{ss}/G$$

" Wall crossing phenomena "

= " How does the moduli change

as a stability condition changes ? "

Wall

$$R^{ss} \subset R \qquad R_0^{ss} \subset R \implies R^{ss}/G \xleftarrow{\text{birational}} \dashrightarrow R'^{ss}/G$$

$$R'^{ss} \subset R$$

stability change $\quad (R'^{ss}, R^{ss} \subset R_0^{ss})$

$$\begin{tikzcd} R^{ss}/G \arrow[rr, dashed, leftrightarrow, "\text{birational}"] \arrow[dr] & & R'^{ss}/G \arrow[dl] \\ & R_0^{ss}/G & \end{tikzcd}$$

In case of X = ruled surface
these phenomena have some applications

- computing invariants, Betti number
  Donaldson polynomial
  ⋮
- studying the birational property of the moduli

But in case of $X = \mathbb{P}^2$, these phenomena do not occur in usual way

<u>Def</u> (Gieseker-Maruyama stability)

X : smooth projective surface
H : ample line bundle on X
E : torsion free coherent sheaf on X

E is H-semistable
$\Longleftrightarrow$ for all subsheaf $F \subset E$

$$\frac{\chi(F \otimes \mathcal{O}(nH))}{r(F)} \leqq \frac{\chi(E \otimes \mathcal{O}(nH))}{r(E)} \quad (n \gg 0)$$

$M_X(\alpha, H)$:= moduli of H-semistable torsion free sheaves E with $ch(E) = \alpha \in H^{2*}(X, \mathbb{Q})$

$\mathrm{Pic}\,\mathbb{P}^2 = \mathbb{Z}[H]$, H : ample line bundle on $\mathbb{P}^2$

$\Rightarrow$ $M_{\mathbb{P}^2}(\alpha, H) = M_{\mathbb{P}^2}(\alpha, 2H) = \cdots$

Gieseker – Maruyama stability CAN NOT detect the wall crossing phenomena on $\mathbb{P}^2$

$\mathbb{R}_{>0}[H] = \circ\!\longrightarrow$

space parametrizing Gieseker - Maruyama stability on $\mathbb{P}^2$

Idea : Consider the derived category $D^b(\mathbb{P}^2)$ of $\mathrm{Coh}(\mathbb{P}^2)$

$\left(\mathrm{Coh}(\mathbb{P}^2) \subset D^b(\mathbb{P}^2)\right)$

and use Bridgeland stability conditions on $D^b(\mathbb{P}^2)$

$\Rightarrow$ We can change stability conditions widely !

In the process of this direction of our research, we have another proof of Le Potier's result and wall crossing phenomena as desired

2. Notation

$$B = \left(\begin{tikzcd} \overset{v_0}{\bullet} & \overset{v_1}{\bullet} \arrow[l, shift right=2, "a_0"'] \arrow[l, "a_1"] \arrow[l, shift left=2, "a_2"'] & \overset{v_2}{\bullet} \arrow[l, shift right=2, "b_0"'] \arrow[l, "b_1"] \arrow[l, shift left=2, "b_2"'] \end{tikzcd} \right) \Big/ (a_i b_j = a_j b_i)$$

path algebra of quiver with relation

M : f.g right B- module

$\Longrightarrow$ $M = M_0 \oplus M_1 \oplus M_2$ (M_i : eigenspace of v_i^*)

$$\begin{tikzcd} M_0 \arrow[loop above, "v_0^* = id_{M_0}"] \arrow[r, shift left=2, "a_0^*"] \arrow[r, "a_1^*"] \arrow[r, shift right=2, "a_2^*"'] & M_1 \arrow[loop above, "v_1^* = id_{M_1}"] \arrow[r, shift left=2, "b_0^*"] \arrow[r, "b_1^*"] \arrow[r, shift right=2, "b_2^*"'] & M_2 \arrow[loop above, "v_2^* = id_{M_2}"] \end{tikzcd}, \quad b_j^* a_i^* = b_i^* a_j^*$$

$\underline{\dim}(M) := (\dim M_0, \dim M_1, \dim M_2) \in \mathbb{N}^3$

Fix $v = (v_0, v_1, v_2) \in \mathbb{N}^3$: dimension vector

$v^\perp := \{ \theta = (\theta_0, \theta_1, \theta_2) \in \mathbb{R}^3 \mid \theta_0 v_0 + \theta_1 v_1 + \theta_2 v_2 = 0 \}$

Take $\theta \in v^\perp$

<u>Def</u> (θ-stability)

M = B-module with $\underline{\dim}(M) = v \in \mathbb{N}^3$

M is θ-semistable

$\iff$ for all submodules $N \subset M$

$$\theta(N) = \theta_0 \dim N_0 + \theta_1 \dim N_1 + \theta_2 \dim N_2 \geqq 0$$

$M_B(v, \theta) :=$ moduli of θ-semistable B-modules M with $\underline{\dim}(M) = v$

By Bondal's theorem

abelian category of right B-modules

$$\begin{array}{ccc} {}^\exists\Phi : D^b(\mathbb{P}^2) & \simeq & D^b(\text{mod-}B) \\ \cup & & \cup \\ \mathcal{B} & \simeq & \text{mod-}B \end{array}$$

$$\begin{array}{ccccccc} \mathcal{B} = \langle & \mathcal{O}_{\mathbb{P}^2}[2], & \Omega^1_{\mathbb{P}^2}(2)[1], & \mathcal{O}_{\mathbb{P}^2}(1) & \rangle & \subset & D^b(\mathbb{P}^2) \\ & \updownarrow & \updownarrow & \updownarrow & & & \wr\!\wr \\ \text{mod-}B = \langle & \mathbb{C}v_0, & \mathbb{C}v_1, & \mathbb{C}v_2 & \rangle & \subset & D^b(\text{mod-}B) \end{array}$$

3. Main theorem

Main theorem (equivalent to Le Potier's result)

Take $\alpha = (r, sH, u) \in H^{2*}(\mathbb{P}^2, \mathbb{Q})$

with $r > 0$, $0 < s \leq r$

Then Φ gives the isomorphism

$$\begin{array}{ccc} M_{\mathbb{P}^2}(\alpha, H) & \simeq & M_B(V_B, \theta_B) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ E & \longmapsto & \Phi(E[1]) \end{array}$$

where $V_B = (V_0, V_1, V_2) \in \mathbb{N}^3$ such that

$$E[1] \simeq \left(\mathcal{O}_{\mathbb{P}^2}^{\oplus V_0} \longrightarrow \left(\Omega^1_{\mathbb{P}^2}(2)\right)^{\oplus V_1} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus V_2} \right) \text{ in } \mathcal{B}$$

(quasi isomorphism)

$V_B^{\perp}$

$\theta(1,1,1) > 0$ | $\theta(1,1,1) < 0$

$\ell = \{\theta(1,1,1) = 0\} \cap V_B^{\perp}$

θ_B •

$(1,1,1) = \underline{\dim}(\Phi(\mathcal{O}_x))$

$\mathcal{O}_x$: skyscraper sheaf at $x \in \mathbb{P}^2$

θ_B is enough near from line ℓ

4. Application : Wall crossing phenomena

$$\text{mod-}B = \langle \mathbb{C}v_0, \mathbb{C}v_1, \mathbb{C}v_2 \rangle \subset D^b(\text{mod-}B)$$

$$\begin{tikzcd}
\text{mod-}B'' & \text{mod-}B \arrow[l, bend right, "\text{tilting at } \mathbb{C}v_0"'] \arrow[r, bend left, "\text{tilting at } \mathbb{C}v_2"] & \text{mod-}B'
\end{tikzcd}$$

$$\Downarrow$$

$$M_{\mathbb{P}^2}(\alpha, H)$$
$$\text{Le Potier} \; \wr\!\wr$$

$$\begin{tikzcd}[column sep=small]
M_{B''}(V_{B''}, \theta_{B''}^-) \arrow[d] & M_{B''}(V_{B''}, \theta_{B''}) \arrow[l, dashed] \arrow[dl] \arrow[r, phantom, "\underset{\psi}{\simeq}"] & M_B(V_B, \theta_B) \arrow[r, phantom, "\underset{\varphi}{\simeq}"] & M_{B'}(V_{B'}, \theta_{B'}) \arrow[r, dashed] \arrow[dr] & M_{B'}(V_{B'}, \theta_{B'}^-) \arrow[d] \\
M_{B''}(V_{B''}, \theta_{B''}^0) & & & & M_{B'}(V_{B'}, \theta_{B'}^0)
\end{tikzcd}$$

ψ, φ directly follow from the property of Bridgeland stability (invariance of $\widetilde{GL}^+(2, \mathbb{R})$ action)

In case of rank 1, (after normalization)

$$\begin{tikzcd}
M_{B''}(V_{B''}, \theta_{B''}^-) \arrow[dr] & \mathbb{P}(V)^{[n]} \arrow[l, dashed, "\text{flip}"'] \arrow[d] \arrow[dr, "\pi"] & \mathbb{P}(V^*)^{[n]} \arrow[d, "\pi^*"'] \\
& Z & S^n(\mathbb{P}(V))
\end{tikzcd}$$

$V = \mathbb{C}^3$

π^*, π : Hilbert-Chow morphism

5. Proof of Main theorem (rough sketch)

$$\mathrm{Stab}(\mathbb{P}^2) := \{\, \sigma : \text{Bridgeland stability on } D^b(\mathbb{P}^2) \,\}$$

$$\Downarrow$$

$$\mathcal{G} := \{\, \text{"Gieseker - Maruyam stability" on } \mathbb{P}^2 \,\} \subset \mathrm{Stab}(\mathbb{P}^2)$$

$$\Sigma := \{\, \text{"}\theta\text{-stability" for } B\text{-modules} \,\} \subset \mathrm{Stab}(\mathbb{P}^2)$$

$$\Downarrow$$

We find $\sigma \in \mathcal{G} \cap \Sigma$ and consider moduli $M_{D^b(\mathbb{P}^2)}(\alpha, \sigma)$ of σ-semistable objects E with $\mathrm{ch}(E) = \alpha$

$$\Downarrow$$

$$\begin{array}{ccccc}
M_{\mathbb{P}^2}(\alpha, H) & \simeq & M_{D^b(\mathbb{P}^2)}(\alpha, \sigma) & \simeq & M_B(V_B, \theta_B) \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
E & \longmapsto & E[1] & \longmapsto & \Phi(E[1])
\end{array}$$

6, Bridgeland stability

<u>Def</u> (Bridgeland stability)

$\mathcal{T}$ = triangulated category

a pair $\sigma = (Z, \mathcal{A})$ is a stability condition on $\mathcal{T}$

$\Longleftrightarrow$

- $Z : K(X) \longrightarrow \mathbb{C}$: group hom (central charge)
- $\mathcal{A} \subset \mathcal{T}$: heart of bounded t-structure on $\mathcal{T}$
- ${}^{\forall}E \in \mathcal{A},\ E \neq 0 \Rightarrow Z(E) \in \mathbb{R}_{>0} \exp \sqrt{-1}\pi\phi(E)$

$$0 < \underbrace{\phi(E)}_{\text{phase of } E} \leq 1$$

$E \in \mathcal{A}$: σ-(semi)stable

$\overset{\text{def}}{\Longleftrightarrow}$ ${}^{\forall}F \subset E$ in $\mathcal{A}$, $\phi(F) \underset{(\leq)}{<} \phi(E)$

- Harder Narasimhan property

$$\mathrm{Stab}(\mathcal{T}) := \{\text{"good" stability conditions on } \mathcal{T}\}$$

<u>Fact</u>

$\mathrm{Stab}(\mathcal{T})$ = complex mfd

When $\mathcal{T} = D^b(X)$ $\quad X$ = smooth proj variety

$\mathrm{Stab}(X) := \mathrm{Stab}(D^b(X))$

Example 1, C = smooth proj curve, $\mathcal{T} = D^b(C)$

$$\mathcal{A} := Coh(C) \subset D^b(C), \quad Z : K(C) \longrightarrow \mathbb{C}$$

$$E \longmapsto -\deg E + \sqrt{-1}\,\mathrm{rank}\,E$$

$\Rightarrow \sigma = (Z, \mathcal{A}) \in Stab(C)$ (classical slope stability)

Example 2. X = smooth proj surface, $\mathcal{T} = D^b(X)$

$\beta, \omega \in NS(X) \otimes \mathbb{R}$ (ω = ample) give a pair $(Z_{(\beta,\omega)}, \mathcal{A}_{(\beta,\omega)})$

$\mathcal{A}_{(\beta,\omega)} \subset D^b(X)$ = tilting of $Coh(X)$

$$Z_{(\beta,\omega)} : K(X) \longrightarrow \mathbb{C}$$

$$E \longmapsto -\int \exp(-\beta - \sqrt{-1}\omega) \cdot ch(E)$$

For simplicity, we assume $X = \mathbb{P}^2$ $\quad$ n(ot

<u>Prop</u>

$$\overline{\alpha = (r, c_1, ch_2) \in H^{2*}(\mathbb{P}^2, \mathbb{Q})}^{\,0 < c_1 \cdot \omega \leq r}$$

then $\omega \cdot \beta \uparrow \frac{c_1 \cdot \omega}{r} \Rightarrow M_{D^b(\mathbb{P}^2)}(\alpha, \sigma_{(\beta,\omega)}) \simeq M_{\mathbb{P}^2}(\alpha, H)$

$\widetilde{g \in GL^+(2, \mathbb{R})}$

$$\wr\ [1]$$

$$M_{D^b(\mathbb{P}^2)}(\alpha, \sigma g) \qquad D^b(X) \supset Coh(X)$$

$$\wr$$

$$M_B(V_B, \theta_B)$$

7. Key Lemma

<u>Fact</u>

- $\exists$ right group action $\quad \mathrm{Stab}(X) \curvearrowleft \widetilde{GL}^+(2,\mathbb{R})$
- This action does not change semistable objects

  E = σ-semistable $\Longrightarrow$ E = σg-semistable

<u>Def</u> (Geometric stability)

$\sigma_{(\beta,\omega)}$ up to $\widetilde{GL}^+(2,\mathbb{R})$ action are called geometric stability condition

<u>Key lemma</u> $\qquad X$ = smooth proj surface

$\sigma \in \mathrm{Stab}(X)$ is geometric

$\Longleftrightarrow$

1. $\forall x \in X$, skyscraper sheaf $\mathcal{O}_x$ is σ-stable
2. $\forall \beta \in K(X)$, $Z(\beta) = 0 \Longrightarrow \underbrace{c_1(\beta)^2 - 2r(\beta)\mathrm{ch}_2(\beta)}_{\text{Bogomolov-Gieseker inequality}} \cancel{\leq} 0$ $\quad \geq 0$

E. Macrì $\quad \left(\mathcal{O}(-1) \to \cancel{\mathcal{O}^{\oplus 2}} \to \mathcal{O}_{\mathbb{P}^2} \right) \simeq \mathcal{O}_x$

$\qquad \Omega_{\mathbb{P}^2}(2)$

p

$\mathcal{B}$

8, Main theorem (restatement)

$$\mathcal{B} = \langle \mathcal{O}_{\mathbb{P}^2}[2],\ \Omega^1_{\mathbb{P}^2}(1)[1],\ \mathcal{O}_{\mathbb{P}^2}(1) \rangle \subset D^b(\mathbb{P}^2)$$

$\mathcal{B} \simeq$ mod-B, $\quad E_0 = \mathcal{O}_{\mathbb{P}^2}[2]$, $\quad E_1 = \Omega^1_{\mathbb{P}^2}(1)[1]$, $\quad E_2 = \mathcal{O}_{\mathbb{P}^2}(1)$

$$Z : K(\mathbb{P}^2) = \mathbb{Z}[E_0] \oplus \mathbb{Z}[E_1] \oplus \mathbb{Z}[E_2] \longrightarrow \mathbb{H}$$

$$E_i \longmapsto x_i + \sqrt{-1}\, y_i \in \mathbb{H}$$

$(i = 0, 1, 2)$

$$\Longrightarrow \sigma = (Z, \mathcal{B}) \in \mathrm{Stab}(\mathbb{P}^2)$$

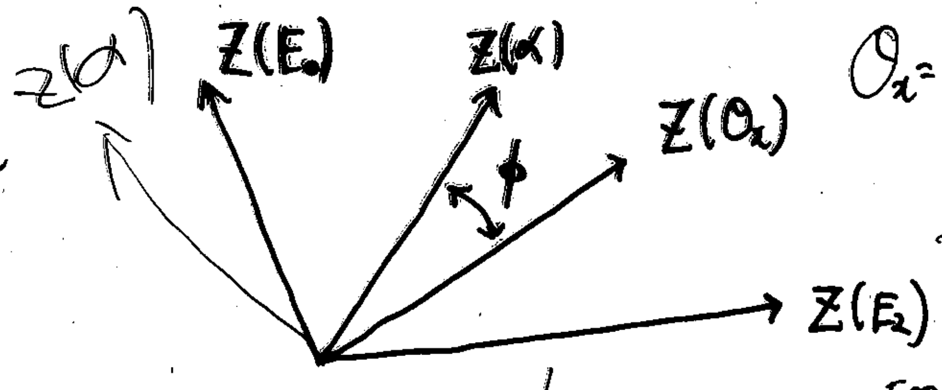
$$M_{D^b(\mathbb{P}^2)}(\alpha, \sigma) \simeq M_B(V_B, \theta_B)$$

$K(\mathbb{P}^2) \simeq$

<u>Main theorem</u>

$\alpha = \cancel{?} \in H^{2*}(\mathbb{P}^2; \mathbb{Q})$ = fixed

We take



$\mathcal{O}_x$ = skyscraper sheaf $x \in \mathbb{P}^2$

$[\mathcal{O}_x] = [\mathcal{O}_{x'}]$ in K

then ϕ = small $\Longrightarrow M_{D^b(\mathbb{P}^2)}(-\alpha, \sigma) \simeq M_{\mathbb{P}^2}(\alpha, H)$

ϕ

$[\mathcal{O}_x] = [E_0] + [E_1] + [E_2]$

$E[1] \longleftrightarrow E$

9. $X = \mathbb{P}^1 \times \mathbb{P}^1$

Similar results are obtained in case of $X = \mathbb{P}^1 \times \mathbb{P}^1$

<u>Thm</u>

$$\omega = \mathcal{O}(1,1) = p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1) \in \mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$$

$$(p_i = i\text{-th projection},\ i = 1, 2)$$

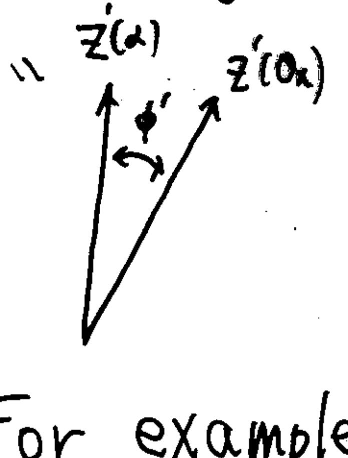
$$\alpha \in K(\mathbb{P}^1 \times \mathbb{P}^1),\quad ch(\alpha) = (r, c_1, ch_2)$$

with $r > 0$, $-2r < c_1 \cdot \omega \leq 0$

$$\mathcal{B}' = \langle \underset{E_0}{\underset{\parallel}{\mathcal{O}(-1,-1)[2]}},\ \underset{E_1}{\underset{\parallel}{\mathcal{O}(0,-1)[1]}},\ \underset{E_2}{\underset{\parallel}{\mathcal{O}(-1,0)[1]}},\ \underset{E_3}{\underset{\parallel}{\mathcal{O}}} \rangle \subset D^b(\mathbb{P}^1 \times \mathbb{P}^1)$$

$$Z'(E_i) = x_i' + \sqrt{-1}\, y_i' \in \mathbb{H} \quad (i = 0, 1, 2, 3)$$

$$\sigma = (Z', \mathcal{B}') \in \mathrm{Stab}(\mathbb{P}^1 \times \mathbb{P}^1)$$



$\phi' :$ small $\implies M_{D^b(\mathbb{P}^1 \times \mathbb{P}^1)}(\alpha, \sigma) \simeq M_{\mathbb{P}^1 \times \mathbb{P}^1}(\alpha, \omega)$

For example, in case of $\alpha = (1, 0, -n) \in H^{2*}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Q})$

$Z(E_0) = \sqrt{-1}$

$Z(E_1) = Z(E_2) = \frac{287}{645} + \sqrt{-1}$

$Z(E_3) = \frac{28}{645} + \frac{\sqrt{-1}}{14}$

$\sigma = (Z, \mathcal{B})$

$\implies M_{D^b(\mathbb{P}^2)}(-\alpha, \sigma) \simeq (\mathbb{P}^1 \times \mathbb{P}^1)^{[n]}$