Interacting Strings and Particles

The 3D Ising Model

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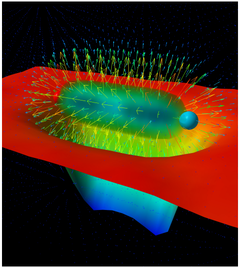
A broader program for effective strings

Toward more realistic effective string theory

- Long open/closed strings and universal EST [Polchinski, Strominger], [Aharony, Komargodski],[Dubovsky et al.] [Aharony, Field]
- Precision spectra and interface free energy [Billò, Caselle, Ferro, Hasenbusch, Panero], [Brandt]
- Junctions [Pfeuffer, Bali, Panero], [Komargodski, Zhong]
- Boundaries [Aharony, Field], [Hellerman, Swanson]
- Dynamical endpoints/string length [Hellerman, Swanson], [Aharony, Field]
- String breaking and mixing with hadronic states
- **Interactions with bulk massive modes** [Athenodorou, Dubovsky, Luo, Teper], [Brandt]

Color code

- Established / well developed
- Wishlist
- Main topic of this talk



Lattice QCD simulation of a meson. Source: physics.adelaide.edu.au/theory/sta leinweber/VisualQCD/Nobel/

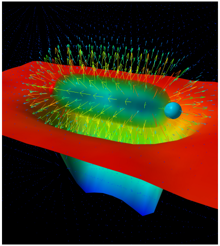
A broader program for effective strings

Toward more realistic effective string theory

- Long strings, precision spectra, interfaces [Polchinski, Strominger], [Aharony, Komargodski], [Dubovsky et al.], [Aharony, Field], [Billò, Caselle, Ferro, Hasenbusch, Panero], [Brandt]
- Junctions and boundaries [Pfeuffer, Bali, Panero], [Komargodski, Zhong], [Aharony, Field], [Hellerman, Swanson]
- Dynamical endpoints, string breaking, hadronic mixing [Hellerman, Swanson], [Aharony, Field]
- **Interactions with bulk massive modes** [Athenodorou, Dubovsky, Luo, Teper], [Brandt]

Legend

- Established
- Open problems
- This talk



Lattice QCD flux tube

Plan

1. Naive local couplings fail for fluctuating strings.
2. We resum string fluctuations and keep the bulk coupling perturbative.
3. The resulting EFT predicts one- and two-point observables.
4. We test this on 3D Ising domain walls and connect it to particle-string interactions in gauge theory.

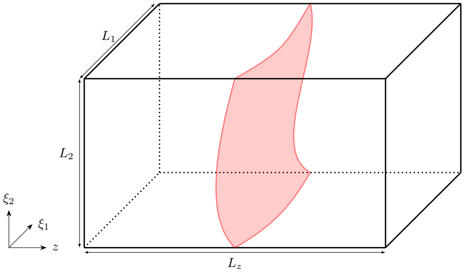
3D Ising: A clean analogue of flux tubes and glueballs

The same structural problem appears in both systems:

- **Confining gauge theory:** a fluctuating flux tube coexists with gapped bulk states (glueballs).
- **Broken phase of 3D Ising:** a fluctuating domain wall coexists with a light bulk massive mode.
- In both cases, long-distance string dynamics is described by EST.
- The question is the same: **how does a fluctuating string dress bulk observables, and how can one extract the particle–string coupling?**

Why interfaces in the 3D Ising first?

- One transverse Goldstone mode,
- No boundaries: use a toroidal domain wall,
- Clean access to odd/even observables and to the wall-induced background.
- Delocalized domain wall generated by a twist.



Flux tube $\leftrightarrow$ Domain wall

Glueball $\leftrightarrow$ Massive bulk particle

Effective string theory in one slide

- A long confining string spontaneously breaks $SO(3) \times \mathbb{R}^3 \to SO(2) \times \mathbb{R}^2$.
- The universal gapless fields are the transverse Goldstone modes.
- The leading infrared action is Nambu–Goto, understood as an **effective field theory**:

$$S_{\mathrm{EST}} = \sigma \int d^2\xi \, \sqrt{\det h_{\alpha\beta}} \; + \; \cdots \, , \qquad h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu .$$

- In the static gauge $X^\mu = (\xi_1, \xi_2, \pi(\xi))$ at large area:

$$S_{\mathrm{EST}} = \sigma A + \frac{1}{2} \int d^2\xi \, (\partial_\alpha \pi)^2 + \cdots$$

Key point: non-critical Nambu–Goto is not a fundamental UV theory, but it is the correct universal *low-energy* EFT.
Long-distance dynamics $\Longrightarrow$ collective transverse fluctuations only

What EST got right

- **Low-energy universality.** For closed strings and interfaces, the first genuinely non-universal operator is

$$\gamma_3 K^4 \qquad \sim \qquad \gamma_3 \left(\partial_\mu \partial_\nu \pi \, \partial^\mu \partial^\nu \pi\right)^2 .$$

- **Long-string spectrum.** The Nambu–Goto / GGRT prediction is

$$E(k, k'; R) = \sigma R \sqrt{1 + \frac{4\pi}{\sigma R^2}\left(k + k' - \frac{D-2}{24}\right) + \left(\frac{2\pi(k-k')}{\sigma R^2}\right)^2} .$$

- **Universal sector and integrability.** The universal worldsheet dynamics is described by an integrable massless scattering theory:

$$S(s) = \exp\left(i \frac{\ell_s^2}{4} s + i\gamma_3 s^3 + \cdots\right) .$$

- **$T\bar{T}$ viewpoint.** The GGRT finite-volume spectrum can be understood as the gravitational dressing ($T\bar{T}$-type flow) of free transverse bosons.

Flux-tube width: the key warning sign

- A universal consequence of EST is **logarithmic broadening**:

$$w^2(R) \equiv \left\langle (\pi - \langle \pi \rangle)^2 \right\rangle = \frac{D-2}{2\pi\sigma} \log\left(\frac{R}{R_0}\right) + \cdots .$$

- So the string is not a rigid classical object: its transverse fluctuations grow with the size of the system.

Takeaway. Because the width grows with the system size, couplings to a fluctuating wall cannot always be treated as a small expansion around a flat background.

Why the naive flat-wall expansion fails

- The bulk mode couples to the wall through the pullback

$$\int d^2x_\parallel \sqrt{h}\phi\big(x_\parallel, \pi(x_\parallel)\big)\,.$$

- Expanding around a flat embedding gives

$$\phi\big(x_\parallel, \pi\big) = \phi\big(x_\parallel, 0\big) + \pi\,\partial_z\phi\big(x_\parallel, 0\big) + \frac{\pi^2}{2}\,\partial_z^2\phi\big(x_\parallel, 0\big) + \cdots$$

- But the transverse fluctuation is massless, and its variance grows:

$$d^2(x_\parallel) \equiv \Big\langle \big(\pi(x_\parallel) - \pi(0)\big)^2 \Big\rangle \sim \frac{1}{2\pi\sigma}\log|x_\parallel|.$$

- Bulk exchange probes transverse momenta of order the bulk mass,

$$k_\perp \sim m,$$

so the relevant expansion parameter is

$$k_\perp^2\, d^2(x_\parallel) \;\sim\; m^2 d^2(x_\parallel).$$

- Once $m^2 d^2(x_\parallel) \sim 1$, truncating the series in π is not controlled.

Takeaway. The failure is geometric, not a breakdown of EST itself: the wall fluctuates so much in the infrared that the pullback coupling must be treated nonperturbatively in π.

EFT strategy: fixed wall first, then average

The difficulty is the expansion in the fluctuating embedding, not the wall–bulk coupling itself.

We proceed in two steps:

1. For a fixed wall profile $\pi(x_\parallel)$, compute the bulk observable in that background.
2. Average the result over branon fluctuations:

$$\langle \mathcal{O} \rangle = \int \mathcal{D}\pi \, e^{-S_{\mathrm{EST}}[\pi]} \, \langle \mathcal{O} \rangle_\pi .$$

For observables depending on the transverse difference

$$w \equiv \pi(x_\parallel) - \pi(y_\parallel),$$

the averaging is encoded in the universal kernel

$$K(w; x_\parallel - y_\parallel) \equiv \Big\langle \delta\big(w - \pi(x_\parallel) + \pi(y_\parallel)\big) \Big\rangle_\pi ,$$

so that

$$\big\langle F(\pi(x_\parallel) - \pi(y_\parallel)) \big\rangle_\pi = \int dw \, F(w) \, K(w; x_\parallel - y_\parallel).$$

Perturbative in the wall–bulk coupling. Nonperturbative in the fluctuating embedding.

Universal kernel from branon fluctuations

Using the Fourier representation of the delta function,

$$K(w; x_\parallel - y_\parallel) = \int \frac{dk}{2\pi}\, e^{ikw} \left\langle e^{-ik(\pi(x_\parallel)-\pi(y_\parallel))} \right\rangle_\pi .$$

For the Gaussian branon field,

$$\left\langle e^{-ik(\pi(x_\parallel)-\pi(y_\parallel))} \right\rangle_\pi = \exp\left[-\frac{k^2}{2}\, d^2(x_\parallel - y_\parallel) \right],$$

with

$$d^2(x_\parallel - y_\parallel) = \left\langle \left(\pi(x_\parallel) - \pi(y_\parallel) \right)^2 \right\rangle .$$

Therefore,

$$K(w; x_\parallel - y_\parallel) = \frac{1}{\sqrt{2\pi\, d^2(x_\parallel - y_\parallel)}} \exp\left[-\frac{w^2}{2\, d^2(x_\parallel - y_\parallel)} \right].$$

Takeaway. Averaging over embeddings produces a universal Gaussian smearing of the transverse separation, controlled by $d^2(x_\parallel - y_\parallel)$.

Minimal EFT and regime of control

$$S_{\text{eff}} = \frac{\sigma}{2}\int d^2x\,(\partial\pi)^2 \;+\; S_{\text{bulk}}[\phi] \;+\; \lambda_0 \int d^2x\;\phi\big(x,\pi(x)\big)$$

- **Bulk input:** we do not need a detailed microscopic bulk EFT, only the bulk 2pt function

$$\langle\phi\phi\rangle \;\sim\; \int_0^\infty ds\,\rho(s)\,\frac{1}{k^2+s}, \qquad \rho(s) = Z\delta(s-m^2) + \tilde{\rho}(s).$$

- **Controlled regime:** the EFT is used for observables dominated by

$$p^\mu = (q, \pm i\sqrt{m^2+q^2}), \qquad q^2 \ll m^2,$$

so momentum *along* the wall is soft even if the exchanged bulk mode is nearly on shell.

- **In 3D Ising:** $\mathbb{Z}_2$ symmetry organizes the observables: odd sector isolates wall geometry, even sector probes the wall-bulk coupling.

EFT predictions for bulk observables

- **1pt function**

$$\delta\langle\phi\rangle \equiv \langle\phi\rangle_{\mathrm{AP}} - \langle\phi\rangle_{\mathrm{P}} = \frac{\lambda_0}{L_z}\int ds\, \frac{\rho(s)}{s},$$

  which depends on the full bulk spectral density.

- **Nearby 2pt function**

$$\frac{\delta\langle\phi(0)\phi(x)\rangle}{[\delta\langle\phi\rangle]^2} \approx \frac{L_z}{\sqrt{2\pi d^2(x_\parallel)}}\exp\left[-\frac{x_\perp^2}{2d^2(x_\parallel)}\right],$$

  a universal Gaussian in $x_\perp$, with width set by $d^2(x_\parallel)$.

- **Asymptotic 2pt function**

$$\delta\langle\phi(0)\phi(x)\rangle \approx \frac{\lambda^2 m}{64\pi\,\chi^2}\, e^{-m|x_\perp|}(m|x_\parallel|)^{2\chi}\frac{|x_\perp|}{L_z},$$

  controlled by the renormalized coupling λ.

- **Finite-L_z correction**

$$\frac{\sigma_{\mathrm{eff}}(L_z)-\sigma}{\sigma} \approx -\frac{\lambda^2\, 2^\chi\Gamma(\chi)}{8\pi}(mL_z)^\chi e^{-mL_z}, \qquad mL_z \gg 1.$$

Key point: the same wall fluctuations control both the nearby Gaussian smearing and the asymptotic branon dressing.

The EFT parameters χ and λ

- The universal fluctuation exponent is

$$\chi \equiv \frac{m^2}{4\pi\sigma},$$

fixed by the bulk mass and the string tension, not by a new coupling.

- It controls the logarithmic growth of the wall width,

$$m^2 d^2(x_\parallel) \sim 4\chi \log(|x_\parallel|/r_0),$$

and therefore the same dressing exponents that appear in the EFT predictions.

- The bare coupling λ_0 is not physical by itself, because the logarithmic wall fluctuations introduce the UV scale r_0. The proper dimensionless coupling is

$$\lambda^2 \equiv \lambda_0^2 \frac{m}{\sigma^2} \frac{1}{(mr_0)^{2\chi}}.$$

- χ controls the universal branon dressing, while λ measures the strength of the particle-wall interaction.

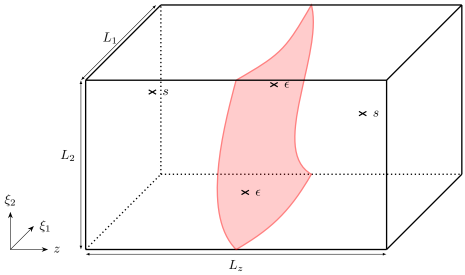
Takeaway: χ is fixed by bulk data and governs the infrared dressing; λ is the physical coupling to be extracted.

Universal correlator properties

- $\mathbb{Z}_2$-**odd sector:** the wall mainly acts by *flipping the sign* of the order parameter across the interface.
- $\mathbb{Z}_2$-**even sector:** the wall is seen as a localized *bump* or core profile, which is then Gaussian-smeared by the wall fluctuations. In the Ising application, the generic even bulk field ϕ is realized by ϵ.
- Therefore:

$$\text{odd correlators} \Rightarrow \text{geometry } d^2(x_{\|});$$

$$\text{even correlators} \Rightarrow \text{coupling / wall core.}$$



$\mathbb{Z}_2$-odd sector: a direct probe of wall geometry

- For odd operators (take s as the example), the wall mainly flips the sign. In the thin-wall regime, for a fixed wall profile,

$$\delta\langle s(x)s(0)\rangle \approx \langle s\rangle^2 \left\langle \operatorname{sgn}\big(x_\perp - \pi(x_\parallel)\big)\operatorname{sgn}\big(-\pi(0)\big)\right\rangle_{\pi_0|w}.$$

- Averaging first over the zero mode with $w \equiv \pi(x_\parallel) - \pi(0)$, gives

$$\Big\langle \operatorname{sgn}\big(x_\perp - \pi(x_\parallel)\big)\operatorname{sgn}\big(-\pi(0)\big)\Big\rangle_{\pi_0\,|\,w} = \frac{L_z - 2|w - x_\perp|}{L_z}.$$

- Since w is Gaussian with variance $d^2(x_\parallel)$,

$$\frac{\delta\langle s(x)s(0)\rangle}{\langle s\rangle^2} \approx 1 - \frac{2\,d(x_\parallel)}{L_z}\, f\left(\frac{|x_\perp|}{d(x_\parallel)}\right), \quad f(y) = y\,\operatorname{erf}\left(\frac{y}{\sqrt{2}}\right) + \sqrt{\frac{2}{\pi}}\, e^{-y^2/}$$

Takeaway. The odd sector is controlled by the wall position alone. It provides a direct, coupling-independent determination of $d^2(x_\parallel)$.

$\mathbb{Z}_2$-even sector: the wall as a smeared core profile

- For even operators, the defect does *not* flip a sign. Instead, after subtracting the periodic background, the effect is localized in the wall core:

$$p_\epsilon\big(x_\perp - \pi(x_\parallel)\big) \equiv \langle \epsilon(x_\perp)\rangle_\pi - \langle \epsilon\rangle_{\mathrm{P}}.$$

- Its integrated strength is

$$T_\epsilon \equiv \int dx_\perp\, p_\epsilon(x_\perp) \Longrightarrow \delta\langle\epsilon\rangle \equiv \langle\epsilon\rangle_{\mathrm{AP}} - \langle\epsilon\rangle_{\mathrm{P}} = \frac{T_\epsilon}{L_z}.$$

- For the connected 2pt function,

$$\delta G_\epsilon(x) \equiv \langle\epsilon(0)\epsilon(x)\rangle^{\mathrm{conn}}_{\mathrm{AP}} - \langle\epsilon(0)\epsilon(x)\rangle^{\mathrm{conn}}_{\mathrm{P}},$$

the thin-wall regime gives

$$\delta G_\epsilon(x) \approx \int \frac{dw}{\sqrt{2\pi d^2(x_\parallel)}} e^{-w^2/(2d^2(x_\parallel))} \int_0^{L_z} \frac{dz}{L_z} p_\epsilon(-z) p_\epsilon(x_\perp - z - w) \tag{1}$$

$$\approx \frac{T_\epsilon^2}{L_z} \frac{e^{-x_\perp^2/(2d^2(x_\parallel))}}{\sqrt{2\pi d^2(x_\parallel)}} \qquad \big(L_z \gg d(x_\parallel) \gg 1/m\big). \tag{2}$$

Monte Carlo strategy: what we want to test

- **Goal 1:** measure the wall geometry directly from $\mathbb{Z}_2$-odd correlators,

$$d^2(x_\parallel) = \langle (\pi(x_\parallel) - \pi(0))^2 \rangle.$$

- **Goal 2:** use the $\mathbb{Z}_2$-even 1pt function to fix the overall wall-core amplitude.
- **Goal 3:** with no extra shape parameter, test the EFT prediction that the even 2pt function is a Gaussian smearing in $x_\perp$ with width set by the measured $d^2(x_\parallel)$.
- **Goal 4:** look for consistency with the longer-distance EFT pattern, while keeping in mind that the asymptotic regime and the free-energy test are not yet fully resolved.
- *Dataset:* 3D Ising in the broken phase, comparing periodic and anti-periodic sectors; benchmark 2pt ensemble $\beta = 0.222136$, $V = 384^2 \times 256$.

Monte Carlo: 1pt function fixes the normalization

- Measure the wall-induced shift

$$\delta\langle\epsilon\rangle \equiv \langle\epsilon\rangle_{\mathrm{AP}} - \langle\epsilon\rangle_{\mathrm{P}}.$$

- EFT expectation:

$$\delta\langle\epsilon\rangle = \frac{T_\epsilon}{L_z} \qquad \left(\text{equivalently } \delta\langle\phi\rangle = \frac{\lambda_0}{L_z}\int ds\,\frac{\rho(s)}{s}\right).$$

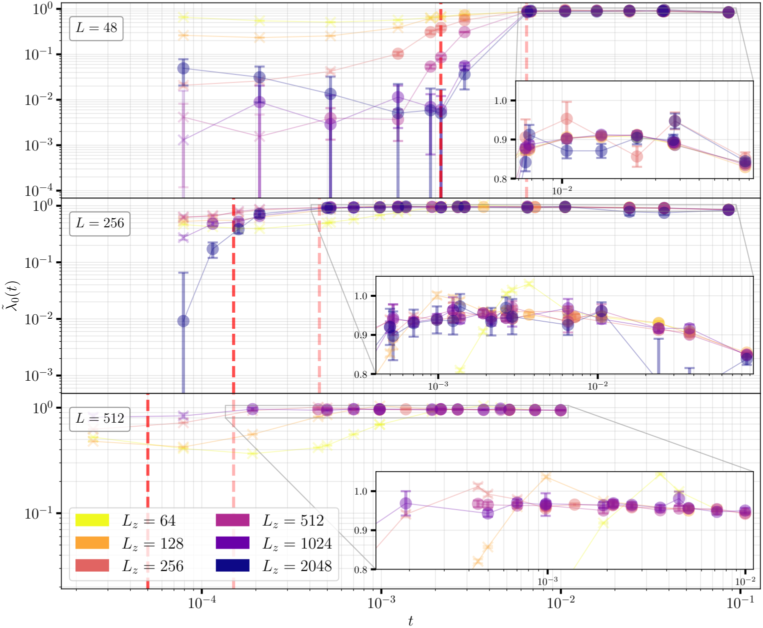
- This observable is **not universal**: it depends on the full bulk spectral density and on lattice UV details.
- But it is operationally useful: after normalizing by the periodic 2pt overlap N_ϵ,

$$\tilde{\lambda}_0 \equiv \frac{T_\epsilon^{\mathrm{lat}}}{N_\epsilon},$$

it fixes the overall wall-core amplitude.

- So the 1pt function is mainly an **input** for the 2pt tests, not the main place where universality is tested.

1pt Function



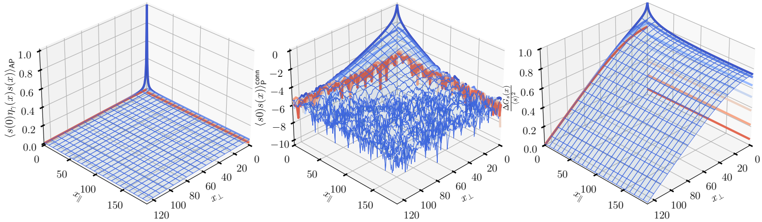
Monte Carlo: odd 2pt extracts the wall variance

EFT prediction:

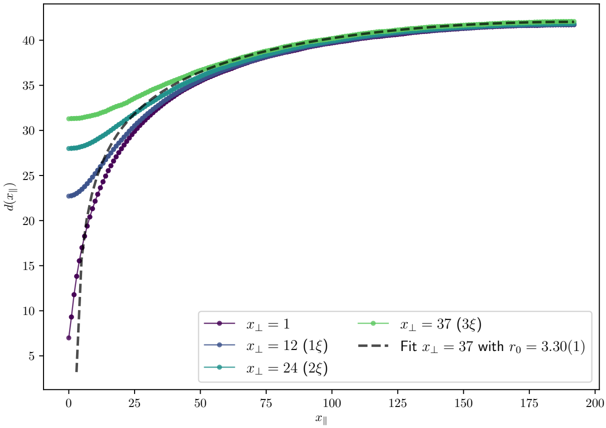
$$\frac{\delta\langle s(x)s(0)\rangle}{\langle s\rangle^2} \approx 1 - \frac{2\,d(x_\parallel)}{L_z}\, f\left(\frac{|x_\perp|}{d(x_\parallel)}\right),$$

$$f(y) = y\,\mathrm{erf}\left(\frac{y}{\sqrt{2}}\right) + \sqrt{\frac{2}{\pi}}\,e^{-y^2/2}.$$

No new coupling enters: the shape is controlled only by the wall variance $d^2(x_\parallel)$.



Monte Carlo: Wall variance $d^2(x_\parallel)$



$$d^2(x_\parallel) = 2\Big(\langle \pi^2 \rangle - \langle \pi(x_\parallel)\pi(0)\rangle\Big) = \alpha' \ln \left| \vartheta_1 \left(\frac{\pi x_\parallel}{L} \middle| e^{i\pi\tau} \right) \right|^2 + 2\alpha' C$$

Monte Carlo: even 2pt tests the Gaussian smearing

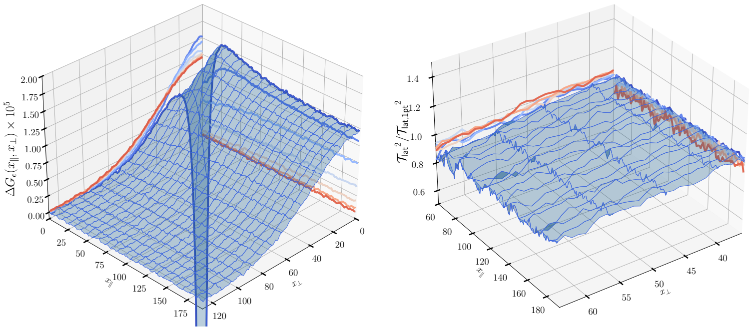
- EFT prediction in the nearby regime:

$$\delta G_\epsilon(x) \approx \frac{T_\epsilon^2}{L_z} \frac{e^{-x_\perp^2/(2d^2(x_\parallel))}}{\sqrt{2\pi d^2(x_\parallel)}}, \qquad L_z \gg d(x_\parallel) \gg \xi.$$

- Inputs are already fixed:

T_ϵ from the 1pt function, $\qquad d^2(x_\parallel)$ from the odd 2pt function.

- Therefore this is a **parameter-free shape test** of the EFT prediction for the transverse dependence.



Takeaway so far: geometry first, coupling next

So far: odd correlators fix $d^2(x_{\|})$, and nearby even correlators are consistent with Gaussian smearing. **Next question: can we extract the particle-wall coupling λ?**

Even 2pt function: asymptotic target

For

$$|x_\perp| \gg md^2(x_\parallel) \gg m^{-1},$$

the correlator is dominated by the lightest bulk pole. In this controlled limit, the EFT predicts

$$\delta G_\phi(x) \approx \frac{\lambda^2 m}{64\pi\,\chi^2}\,\frac{(m|x_\parallel|)^{2\chi}\,|x_\perp|}{L_z}\,e^{-m|x_\perp|}.$$

This gives four characteristic signatures:

- $e^{-m|x_\perp|}$: Yukawa decay in the transverse direction,
- $(m|x_\parallel|)^{2\chi}$: branon dressing along the wall,
- $|x_\perp|/L_z$: zero-mode geometry,
- λ: overall interaction strength.

The next slides test how close the simulations are to this asymptotic regime.

Asymptotic even 2pt: transverse diagnostic

If the asymptotic regime is reached, then at fixed $x_\parallel$

$$\delta G_\phi(x) \propto (m|x_\parallel|)^{2\chi}\, |x_\perp|\, e^{-m|x_\perp|},$$

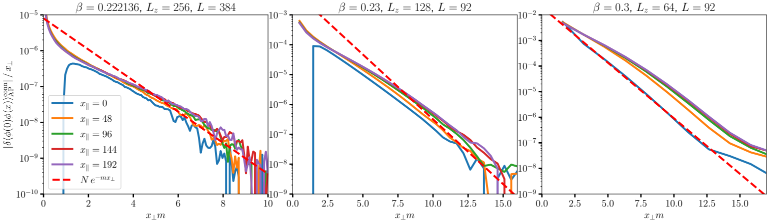
so

$$\frac{|\delta G_\phi(x)|}{|x_\perp|} \propto e^{-m|x_\perp|}.$$

As a first diagnostic, we plot

$$|\delta G_\phi(x)|/|x_\perp|$$

against $mx_\perp$ for several fixed values of $x_\parallel$.



Status: the data show the expected Yukawa trend, but residual pre-asymptotic effects are still visible.

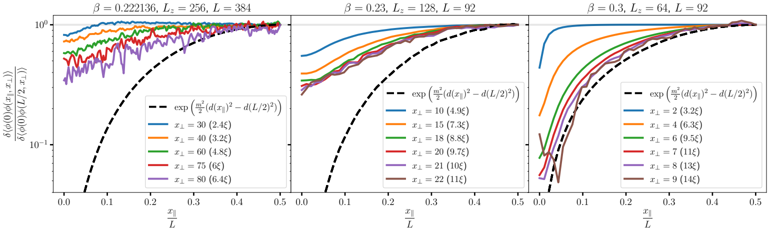
Asymptotic even 2pt: longitudinal diagnostic

To isolate the $x_\parallel$ dependence, consider the ratio

$$\frac{\delta\langle\phi(0)\phi(x_\parallel,x_\perp)\rangle}{\delta\langle\phi(0)\phi(L/2,x_\perp)\rangle} \approx \exp\left[-\frac{1}{2}m^2\left(d^2(L/2)-d^2(x_\parallel)\right)\right].$$

This removes the overall normalization and the leading $x_\perp$ factor, leaving a direct test of the branon dressing along the wall.

We then plot this ratio for several fixed values of $x_\perp$.



Status: the trend is consistent with the predicted longitudinal dressing, but the asymptotic regime is not yet fully clean.

Finite-L_z free-energy correction

Keeping only the leading winding sector,

$$\Delta F \equiv F(A, L_z) - F(A, \infty) \simeq -\lambda_0^2 \int d^2x\, d^2y \left\langle G\big(x-y, \pi(x)-\pi(y)+L_z\big)\right\rangle_\pi .$$

For $A \to \infty$, the large-mL_z EFT prediction is

$$\frac{\sigma_{\text{eff}}(L_z) - \sigma}{\sigma} = \frac{\Delta F}{\sigma A} \simeq -\frac{\lambda^2\, 2^\chi \Gamma(\chi)}{8\pi} (mL_z)^\chi e^{-mL_z}, \qquad \chi = \frac{m^2}{4\pi\sigma},$$

valid for $mL_z \gg 1$.

Using the measured $d^2(r)$, we can also evaluate the full smeared integral numerically and compare it with the asymptotic prediction.
At $mL_z \simeq 30$, the full result is still about 24% below the asymptotic estimate: the asymptotic regime is becoming visible, but has not yet been reached.

Conclusions

- **Obstacle:** logarithmic broadening makes the naive flat-wall expansion fail.
- **EFT cure:** compute at fixed embedding, then average over embeddings.
- **Universal result:** embedding fluctuations produce Gaussian smearing.
- **Odd vs even:**

$$\mathbb{Z}_2\text{-odd} \;\rightarrow\; d^2(x_\parallel), \qquad \mathbb{Z}_2\text{-even} \;\rightarrow\; \text{amplitude / coupling.}$$

- **Monte Carlo supports the EFT separation between geometry and dynamics:** odd correlators determine $d^2(x_\parallel)$, nearby even correlators are consistent with the predicted Gaussian smearing, and the asymptotic λ-sensitive observables show the expected trend but are not yet fully asymptotic.

Outlook

- Sharpen the λ-sensitive observables: asymptotic even 2pt and finite-L_z free energy.
- Go beyond the minimal EFT: subleading interactions, heavier bulk channels, non-Gaussian corrections.
- Apply the same logic to gauge-theory defects and flux-tube systems, where the geometric observables are less direct.