Refined topological string for Omega background

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Motivation

- To understand gauge theory better

  (Low energy effective action, instanton, wall crossing)
- To understand string/quantum gravity better

  (Counting black hole microstates, holomorphic anomaly)
- To understand math/geometry better

  (GW, DT theory, Calabi-Yau, wall crossing)

All are related via (refined) topological string

Refinement in the simplest

We know Euler characteristic very well

$$\mathrm{Tr}(-1)^F = b_0 - b_1 + b_2 + \cdots = \chi_X$$

Refined version:

$$\mathrm{Tr}_{BPS}(-1)^F y^J = b_0 - b_1 y + b_2 y^2 + \cdots = P_X(y)$$

Typically, much harder problem

Physicists approach to math

$$\mathrm{Tr}(-1)^F = b_0 - b_1 + b_2 + \cdots = \chi_X$$

Vacuum counting = SUSY path integral

$$\mathrm{Tr}(-1)^F e^{-\beta H} = \int \mathcal{D}X\mathcal{D}\psi e^{-S_{SUSY}} = \int R^n$$

The above example is closed within field theory.

duality

$$\mathrm{Tr}_1(-1)^F = \int \mathcal{D}X_1 e^{-S_1} = \int \mathcal{D}X_2 e^{-S_2} = \mathrm{Tr}_2(-1)^F$$

Refined mathematics

- "Motivic" DT invariant
- "quantum" Langlands correspondence
- "beta-deformed" matrix model
- "Double periodic (elliptic)" multiple Gamma/Zeta function
- "quantum" Liouville theory / integrable system (AGT conjecture)
- "Refined" Chern-Simons theory (Khovanov homology)
- "Refined" counting of BPS states

Everything String theory (M-theory)

Without refinement, all the connections are explained (even proved?)

by using duality in string theory

→ How about the refinement?

Each piece is understood, and experimentally they agree with each other…

But it lacks the blueprint!

String definition is lacking!!!

Mystery of BPS counting

Consider M-theory on Calabi-Yau X

"Count BPS particles": $N_{\beta_i}^{J_1,J_2} \in \mathbf{Z}$

β: $H_2(X,Z)$,

J_1 , J_2 : "spin" of BPS particles (Lefschetz action)

$$F_{\text{ref}} = \sum_i N_i^{j_L,j_R} \int \frac{dt}{t} \mathrm{Tr}_{R_i}(-1)^{j_L+j_R} \frac{e^{-\mu_i t + i\epsilon_- j_L t + i\epsilon_+ j_R t}}{\sinh(\epsilon_1 t)\sinh(\epsilon_2 t)}$$

This formula must be the holy grail, or is it??

Unrefined holy grail

$$F_{unref} = \int \frac{ds}{s} \mathrm{Tr}_R \frac{(-1)^{J_L^3 + J_R^3} e^{-sm^2} e^{-2s\epsilon J_L^3}}{(2\sinh(s\epsilon/2))(-2\sinh(s\epsilon/2))}$$

- This is the graviphoton corrected prepotential (2 graviton + 2g-2 graviphoton)
- SUSY version of Schwinger integral
- The prepotential is computed from topological string (= Gromov-Witten theory)
- This is also the Nekrasov partition function

Worldsheet instanton counting in 2D
= gauge instanton counting in 4D

Aim of the talk

I would like to propose
a worldsheet formulation of
the refined topological string theory

Stay tuned!

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Nekrasov partition function

- Consider 5-dimensional N=2 gauge theory

$$Z(\epsilon_+, \epsilon_-) = \mathrm{Tr}\ (-1)^F e^{-\epsilon_- J^3_-} e^{-\epsilon_+ (J^3_+ + J^3_R)}$$

$$Z(\epsilon_+, \epsilon_-) = \exp\left(-\frac{1}{\epsilon_+^2 - \epsilon_-^2} \sum_{g,n=0}^{\infty} \epsilon_-^{2g} \epsilon_+^{2n} F_{g,n} \right).$$

- $F_{0,0}$ is identified with Seiberg-Witten prepotential
- Computed by localization (on instanton moduli space)
- What is the interpretation of $F_{g,n}$?

Six dimensional Omega background

- Compactify heterotic string on K_3 x T_2
  (= six-dimensioanl (1,0) theory on T_2).
- Dual to type IIA (or M) on Calabi-Yau X

$$ds^2 = (dx^\mu + \Omega^\mu dz + \bar{\Omega}^\mu d\bar{z})^2 + dz d\bar{z}$$

- Locally trivial
- We need further R-symmetry twist to preserve SUSY
- Compared with the KK ansatz

$$ds^2 = dx^2 + (dz + 2\bar{A}_\mu dx^\mu)(d\bar{z} + 2A_\mu dx^\mu).$$

- Gauge field for STU, and graviphoton.

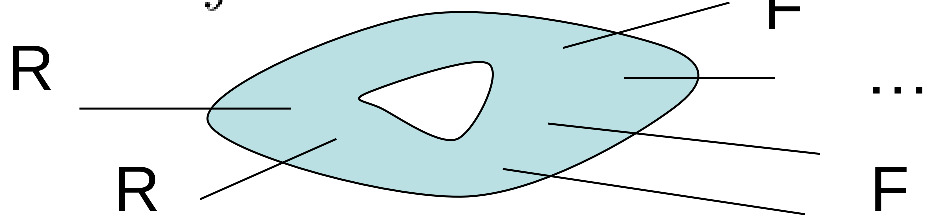
$$F^G_{\alpha\beta} = F^T_{\alpha\beta} = \frac{1}{2} F_{\alpha\beta}, \qquad \epsilon_-^2 = \det F_{\alpha\beta}, \quad \epsilon_+^2 = \det F_{\dot{\alpha}\dot{\beta}}.$$

$$F^{\bar{U}}_{\dot{\alpha}\dot{\beta}} = F^{\bar{S}}_{\dot{\alpha}\dot{\beta}} = \frac{1}{2} F_{\dot{\alpha}\dot{\beta}}.$$

Relation with (unrefined) topological string

- Compute 2 self-dual graviton, (2g-2) self-dual graviphoton at heterotic 1-loop

$$\sum_g \epsilon^{2g-2} F_g = \int d\tau^2 \langle e^{\epsilon_- \int X^1 \bar\partial X^2 + \bar X^1 \bar\partial \bar X^2} \rangle_{R^4} \times Z_{T^2 \times K3}$$



- LHS is the gravitphoton corrected prepotential (topological string amplitude)

$$\sum_g \epsilon_-^{2g-2} F_g = (-1)^{2j} \int_0^\infty \frac{dt}{t} \frac{\mathrm{tr}\ e^{-4t\epsilon_- J_-^3}}{(\sinh \epsilon_- t)^2} e^{-t\mu},$$

- RHS is sum of $\mathrm{tr}(-1)^{2j} e^{-2\epsilon J_-^3} e^{-\beta H}$
- Topological string = Nekrasov partition function

Refined case?

- Refined Nekrasov partition function

$$F_{\rm ref} = \sum_i N_i^{j_L, j_R} \int \frac{dt}{t} {\rm Tr}_{R_i} (-1)^{j_L + j_R} \frac{e^{-\mu_i t + i\epsilon_- j_L t + i\epsilon_+ j_R t}}{\sinh(\epsilon_1 t) \sinh(\epsilon_2 t)}$$

- Counting BPS particles with right SU(2) spin?

- What is the corresponding string background?
  - ASD graviphoton? (No SUSY?)
  - ASD vectormultiplet? (Antoniadis et al 2010)
    - Computes higher derivative F-terms
    - Interesting proposal but does not agree with Nekrasov's result…
- These approaches lack R-symmetry twist…

Flux + FI-term background in SUGRA

SUSY transformation

$$\delta\psi^{i}_{\alpha\dot{\alpha}\beta} = \nabla_{\alpha\dot{\alpha}}\zeta^{i}_{\beta} + (F^{G}_{\alpha\beta}\delta^{i}_{j} + \epsilon_{\alpha\beta}P^{Gi}_{j})\zeta^{j}_{\dot{\alpha}},$$
$$\delta\psi^{i}_{\alpha\dot{\alpha}\dot{\beta}} = \nabla_{\alpha\dot{\alpha}}\zeta^{i}_{\dot{\beta}} + (F^{G}_{\dot{\alpha}\dot{\beta}}\delta^{i}_{j} + \epsilon_{\dot{\alpha}\dot{\beta}}P^{Gi}_{j})\zeta^{j}_{\alpha},$$
$$\delta\lambda^{Ai}_{\alpha} = \partial_{\alpha\dot{\alpha}}t^{A}\zeta^{\dot{\alpha}i} + (F^{A}_{\alpha\beta}\delta^{i}_{j} + \epsilon_{\alpha\beta}P^{Ai}_{j})\zeta^{\beta j},$$
$$\delta\lambda^{\bar{A}i}_{\dot{\alpha}} = \partial_{\alpha\dot{\alpha}}t^{\bar{A}}\zeta^{\alpha i} + (F^{\bar{A}}_{\dot{\alpha}\dot{\beta}}\delta^{i}_{j} + \epsilon_{\dot{\alpha}\dot{\beta}}P^{\bar{A}i}_{j})\zeta^{\dot{\beta}j},$$

$SU(2)_L : \alpha\ ,\quad SU(2)_R : \dot{\alpha}\ ,\quad SU(2)_I : i$ A: 1… n_v

- Hypermultiplet decouples
- Looking for SUSY configuration

1. Purely graviphoton

This is the original topological string configuration

$$\delta\psi^i_{\alpha\dot{\alpha}\beta} = \partial_{\alpha\dot{\alpha}}\zeta^i_\beta + F^G_{\alpha\beta}\zeta^i_{\dot{\alpha}},$$

$$\delta\psi^i_{\alpha\dot{\alpha}\dot{\beta}} = \partial_{\alpha\dot{\alpha}}\zeta^i_{\dot{\beta}}.$$

$$\zeta^i_\alpha = \zeta^{(0)i}_\alpha - F^G_{\alpha\beta}x^{\beta\dot{\beta}}\zeta^{(0)i}_{\dot{\beta}},$$

$$\zeta^i_{\dot{\alpha}} = \zeta^{(0)i}_{\dot{\alpha}},$$

- Preserves 4 constant, 4 non-const SUSY

$$\{Q^i_\alpha, Q^j_{\dot{\beta}}\} = 2\epsilon^{ij}P_{\alpha\dot{\beta}},$$

$$\{P_{\alpha\dot{\beta}}, Q^i_{\dot{\beta}}\} = 2\epsilon_{\dot{\alpha}\dot{\beta}}F^G_{\alpha\beta}Q^{\beta i},$$

$$\{Q^i_{\dot{\alpha}}, Q^j_{\dot{\beta}}\} = 4\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{ij}F^G_{\alpha\beta}M^{\alpha\beta},$$

- Computes graviphoton corrected prepotential

2. SD graviphoton + ASD vector

Deformed topological string proposed by Antoniadis et al (2010).

$$\delta\psi^i_{\alpha\dot\alpha\beta} = \partial_{\alpha\dot\alpha}\zeta^i_\beta + F^G_{\alpha\beta}\zeta^i_{\dot\alpha},$$

$$\delta\psi^i_{\alpha\dot\alpha\dot\beta} = \partial_{\alpha\dot\alpha}\zeta^i_{\dot\beta},$$

$$\delta\lambda^{\bar{A}i}_{\dot\alpha} = F^{\bar{A}}_{\dot\alpha\dot\beta}\zeta^{\dot\beta i}.$$

$$\zeta^i_\alpha = \zeta^{(0)i}_\alpha, \zeta^i_{\dot\alpha} = 0$$

- Preserves 4 constant supercharges
- Computes higher derivative F-terms in ASD background

3. SD graviphoton + SD vector

Self-dual Omega background is an example

$$\delta\psi^i_{\alpha\dot{\alpha}\beta} = \partial_{\alpha\dot{\alpha}}\zeta^i_\beta + F^G_{\alpha\beta}\zeta^i_{\dot{\alpha}},$$
$$\delta\psi^i_{\alpha\dot{\alpha}\dot{\beta}} = \partial_{\alpha\dot{\alpha}}\zeta^i_{\dot{\beta}},$$
$$\delta\lambda^{Ai}_\alpha = \partial_{\alpha\dot{\alpha}}t^A\zeta^{\dot{\alpha}i} + F^A_{\alpha\beta}\zeta^{\beta i}.$$

$$\zeta^i_\alpha = -F^G_{\alpha\beta}x^{\beta\dot{\beta}}\zeta^{(0)i}_{\dot{\beta}},$$
$$\zeta^i_{\dot{\alpha}} = \zeta^{(0)i}_{\dot{\alpha}},$$

$$t^A = t^A_0 + A^{G\mu}A^A_\mu.$$

- Preserves 4 non-const supercharges
- Nekrasov's Omega background:

$$\begin{aligned} ds^2 &= (dx^\mu + \Omega^\mu dz)^2 + dzd\bar{z} \\ &= dx^2 + 2\Omega_\mu dx^\mu dz + 4\Omega_\mu\Omega^\mu dzdz + dzd\bar{z}. \end{aligned}$$

R-twist and FI-term

The key object in refined topological string (refined Nekrasov's theory) is FI-term (R-twist)

- Only defined for non-compact limit ($M_{pl} \to \infty$)
- R-twist needs (of course!) R-symmetry
- R-symmetry presents only when theory is non-compact
- So is FI-term
- Twisting by R-symmetry ⇔ introduction of FI-term

4. Refined Omega background

R-twist ⇔ FI-term is needed to realize refined Omega background

$$\delta\psi^i_{\alpha\dot\alpha\beta} = \nabla_{\alpha\dot\alpha}\zeta^i_\beta + F^G_{\alpha\beta}\zeta^i_{\dot\alpha},$$
$$\delta\psi^i_{\alpha\dot\alpha\dot\beta} = \nabla_{\alpha\dot\alpha}\zeta^i_{\dot\beta},$$
$$\delta\lambda^{Ai}_\alpha = \partial_{\alpha\dot\alpha}t^A\zeta^{\dot\alpha i} + (F^A_{\alpha\beta} + P^{Ai}_j)\zeta^j_\alpha,$$
$$\delta\lambda^{\bar A i}_{\dot\alpha} = (F^{\bar A}_{\dot\alpha\dot\beta}\delta^i_j + \epsilon_{\dot\alpha\dot\beta}P^{\bar A i}_j)\zeta^{\dot\beta j},$$

$$\zeta^i_\alpha = A^G_{\alpha\dot\alpha}\zeta^{(0)\dot\alpha i}\ , \quad \zeta^i_{\dot\alpha} = \zeta^{(0)i}_{\dot\alpha}. \qquad (F^{\bar A} + P^A)\zeta^{(0)} = 0$$

- Preserves 2 non-const supercharges
- Nekrasov computed partition function with this geometry

5. Refined topological string

We can preserve the additional 2 const SUSY

$$\delta\psi^i_{\alpha\dot{\alpha}\beta} = \nabla_{\alpha\dot{\alpha}}\zeta^i_\beta + F^G_{\alpha\beta}\zeta^i_{\dot{\alpha}},$$

$$\delta\psi^i_{\alpha\dot{\alpha}\dot{\beta}} = \nabla_{\alpha\dot{\alpha}}\zeta^i_{\dot{\beta}},$$

$$\delta\lambda^{Ai}_\alpha = \partial_{\alpha\dot{\alpha}}t^A\zeta^{\dot{\alpha}i} + (F^A_{\alpha\beta} + P^{Ai}_j)\zeta^j_\alpha,$$

$$\delta\lambda^{\bar{A}i}_{\dot{\alpha}} = (F^{\bar{A}}_{\dot{\alpha}\dot{\beta}}\delta^i_j + \epsilon_{\dot{\alpha}\dot{\beta}}P^{\bar{A}i}_j)\zeta^{\dot{\beta}j},$$

$$\zeta^i_\alpha = \zeta^{(0)i}_\alpha - A^G_{\alpha\dot{\alpha}}\zeta^{(0)\dot{\alpha}i}\ , \quad \zeta^i_{\dot{\alpha}} = \zeta^{(0)i}_{\dot{\alpha}}.$$

$$(F^{\bar{A}} + P^A)\zeta^{(0)} = 0$$

$$(F^A + P^A)\zeta^{(0)} = 0$$

- Preserves 2 const and 2 non-const SUSY
- Analogue of pure graviphoton (topological string) for unrefined Omega background.

Summary of SUGRA background

fields turned on	$\zeta_{\alpha}^{(0)i}$	$\zeta_{\dot{\alpha}}^{(0)i}$	
$F^{G}_{\alpha\beta}$	1	1	topological string ($\epsilon_+ = 0$)
$F^{G}_{\alpha\beta}, F^{\bar{A}}_{\dot{\alpha}\dot{\beta}}$	1	0	discussed in [AHTN]
$F^{G}_{\alpha\beta}, F^{A}_{\alpha\beta}, \partial t^A$	0	1	Omega ($\epsilon_+ = 0$)
$F^{G}_{\alpha\beta}, F^{A}_{\alpha\beta}, F^{\bar{A}}_{\dot{\alpha}\dot{\beta}}, P^A, \partial t^A$	0	1/2	Omega ($\epsilon_+ \neq 0$)
$F^{G}_{\alpha\beta}, F^{A}_{\alpha\beta}, F^{\bar{A}}_{\dot{\alpha}\dot{\beta}}, P^A, \partial t^A$	1/2	1/2	topological string ($\epsilon_+ \neq 0$)

- Topological string background is different from Nekrasov's Omega background
- Nevertheless we may extract the same physical quantity (prepotential vs partition function)

Heterotic worldsheet construction

Heterotic string on $T_2 \times K_3$

- Consider generic (4,0) compactification

We have universal 3 (+1) vector multiplets

- Complex structure (T), Kahler moduli (U), and dilaton (S)

$$F^G_- = (\partial_{[\mu,}g_{\nu]z} + \partial_{[\mu,}B_{\nu]z})_-, \quad F^G_+ = (\partial_{[\mu,}g_{\nu]\bar{z}} + \partial_{[\mu,}B_{\nu]\bar{z}})_+,$$
$$F^T_- = (\partial_{[\mu,}g_{\nu]z} - \partial_{[\mu,}B_{\nu]z})_-, \quad F^T_+ = (\partial_{[\mu,}g_{\nu]\bar{z}} - \partial_{[\mu,}B_{\nu]\bar{z}})_+,$$
$$F^U_- = (\partial_{[\mu,}g_{\nu]\bar{z}} - \partial_{[\mu,}B_{\nu]\bar{z}})_-, \quad F^U_+ = (\partial_{[\mu,}g_{\nu]z} - \partial_{[\mu,}B_{\nu]z})_+,$$
$$F^S_- = (\partial_{[\mu,}g_{\nu]\bar{z}} + \partial_{[\mu,}B_{\nu]\bar{z}})_-, \quad F^S_+ = (\partial_{[\mu,}g_{\nu]z} + \partial_{[\mu,}B_{\nu]z})_+,$$

- Note that SD part and ASD part has a different combination

R-symmetry and FI terms

- Assume K3 has an R-symmetry (must be non-compact)

J_I^3 : Bosonic R-current in left mover (SUSY side)

$J_{N=4}^3$: SU(2) current in N=4 SCA

$J_I^3 + J_{N=4}^3$: physical (BRST invariant)

We can construct the vertex operator for the FI terms from any right-moving current $\mathcal{J}_A$

	$V_{D_A} = (J_I^3 + J_{SU(2)}^3)\mathcal{J}_A$	FI-term
vector multiplet:	$A_\mu \partial X^\mu \mathcal{J}_A$	gauge field
	$\partial Z \mathcal{J}_A$	scalar field

FI terms in universal multiplets

$$(A_\mu^T)_-\partial X^\mu\bar\partial Z,\quad (A_\mu^T)_+\partial X^\mu\bar\partial\bar Z,$$
$$(A_\mu^U)_-\partial X^\mu\bar\partial\bar Z,\quad (A_\mu^U)_+\partial X^\mu\bar\partial Z.$$

→ $(\mathrm{A}_\mu^T)_-$ and $(\mathrm{A}_\mu^U)_+$ have the same FI-term coupled to $(J_I^3+J_{N=4}^3)\bar\partial Z$

FI-terms are not independent

Similarly $(A_\mu^T)_+$ and $(A_\mu^U)_-$ have the same FI-term coupled to $(J_I^3+J_{N=4}^3)\bar\partial\bar Z$

Refined topological string bcg

1. Metric deformation

$$\epsilon_- = (\partial_{[\mu,} g_{\nu]z})_- = F_-^G + F_-^T,$$
$$\epsilon_+ = (\partial_{[\mu,} g_{\nu]z})_+ = F_+^U + F_+^S.$$

2. To preserve 2 const, 2 non-const SUSY

$$\mathrm{Det} P^A = \mathrm{Det} F^A$$

3. FI terms are not independent $P^T = P^{\bar{U}}$
4. We don't introduce P^S

→ Unique choice

$$F_-^G = \epsilon_- - \epsilon_+,$$
$$F_-^T = F_+^U = \epsilon_+.$$

$$P^T = P^{\bar{U}} = \epsilon_+$$

Test of the proposal

Refined topological string bcg

We have computed the refined topological string amplitude in the zero slope limit (from heterotic string theory)

For a hypermultiplet

$$\int_0^\infty \frac{dt}{t} \frac{e^{-\mu t}}{\sinh(\epsilon_- + \epsilon_+)t \, \sinh(\epsilon_- - \epsilon_+)t},$$

For a vectormultiplet

$$\int_0^\infty \frac{dt}{t} \frac{-2\cosh(2\epsilon_+ t) e^{-\mu t}}{\sinh(\epsilon_- + \epsilon_+)t \, \sinh(\epsilon_- - \epsilon_+)t}.$$

Features of amplitudes

- FI-term gives the correct R-twist in the partition function → $-2\cos(\epsilon_+ t)$ factor in vector multiplet (missing in Antoniadis et al)

- It has the symmetry $\epsilon_\pm \longrightarrow -\epsilon_\pm$
- This must be a symmetry of Nekrasov's partition function (not at all manifest in other approaches: beta deformed matrix model etc)

Summary and Outlook

Summary

- Proposed SUSY background for refined topological string theory
- Computed in the heterotic string in zero-slope limit.
- Agreed with Nekrasov's refined partition function
- Full string computation?
- Type II setup?
- Holomorphic anomaly?