IPMU Colloquium, Nov.17th, 2011

An Introduction to AdS/CFT Correspondence (for Non-Experts)

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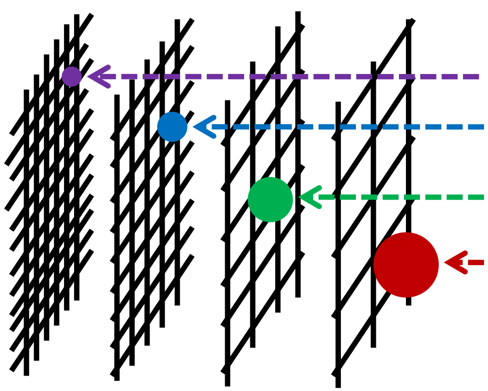
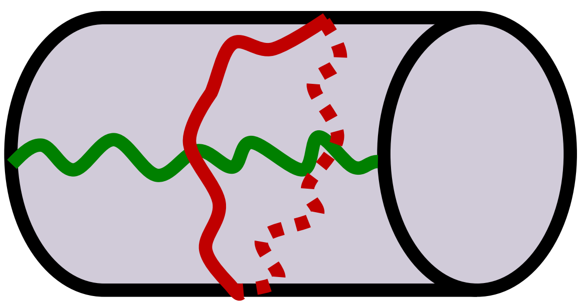




Contents

Note: I will try to give a conceptual guide to AdS/CFT
, rather than a practical one.

① Introduction

The AdS/CFT correspondence argues the equivalence:

AdS = CFT . [1997 Maldacena]

More precisely,

Gravity in Anti de-Sitter Space = Conformal Field Theory .

d+1 dimension d dimension

(1-1) What is the AdS ?

Our spacetime is well described by the 3+1 dimensional Minkowski spacetime $R^{1,3}$.

(Space dim. + Time dim.)

If we express its coordinate by (t, x_1, x_2, x_3) , then its metric is given by

$$ds^2 = -dt^2 + (dx_1)^2 + (dx_2)^2 + (dx_3)^2.$$

It is clear that this spacetime has the SO(1,3) symmetry.

i.e. Lorentz symmetry

This is easily extended to higher dimensions i.e. $R^{1,d}$, which has the symmetry SO(1,d).

These are examples of flat spacetime. In other words, they have the vanishing curvature $R_{\mu\nu} = 0$.

To define AdS (anti-de Sitter) space, we start from $R^{2,d}$:

$$ds^2 = -(dx_0)^2 - (dx_{d+1})^2 + (dx_1)^2 + (dx_2)^2 + \cdots + (dx_d)^2.$$

This has the symmetry SO(2,d). To eliminate one time, we consider the hypersurface defined by

$$(x_0)^2 + (x_{d+1})^2 = (x_1)^2 + (x_2)^2 + \cdots + (x_d)^2 + R^2 \quad .$$

This defines the d+1 dimensional AdS space AdS_{d+1} .

(=Lorentzian version of d+1 dim. hyperbolic space)

Global AdS

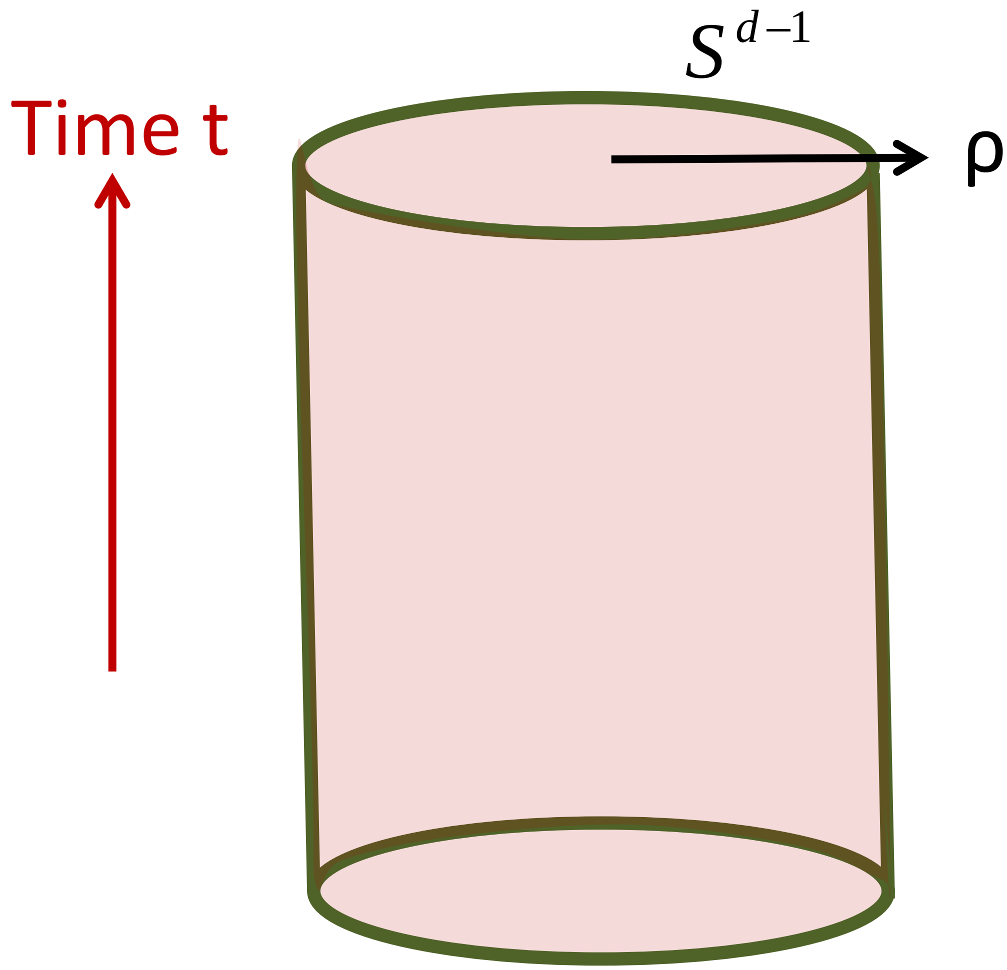
$$(x_0)^2 + (x_{d+1})^2 = (x_1)^2 + (x_2)^2 + \cdots + (x_d)^2 + R^2$$

$$\Rightarrow \quad \begin{cases} x_0 = R\cosh\rho\cos t, \\ x_{d+1} = R\cosh\rho\sin t, \\ x_i = R\sinh\rho \cdot \Omega_i \quad (i = 1,2,..,d) \end{cases}$$

$$\Rightarrow \quad ds^2 = R^2\left(-(\cosh\rho)^2 dt^2 + d\rho^2 + (\sinh\rho)^2 d\Omega^2\right).$$

S^{d-1}

A Sketch of Global AdS



$$\partial(AdS_{d+1}) = R \times S^{d-1}$$

$$ds^2 = R^2\left(-(\cosh\rho)^2 dt^2 + d\rho^2 + (\sinh\rho)^2 d\Omega^2\right)$$

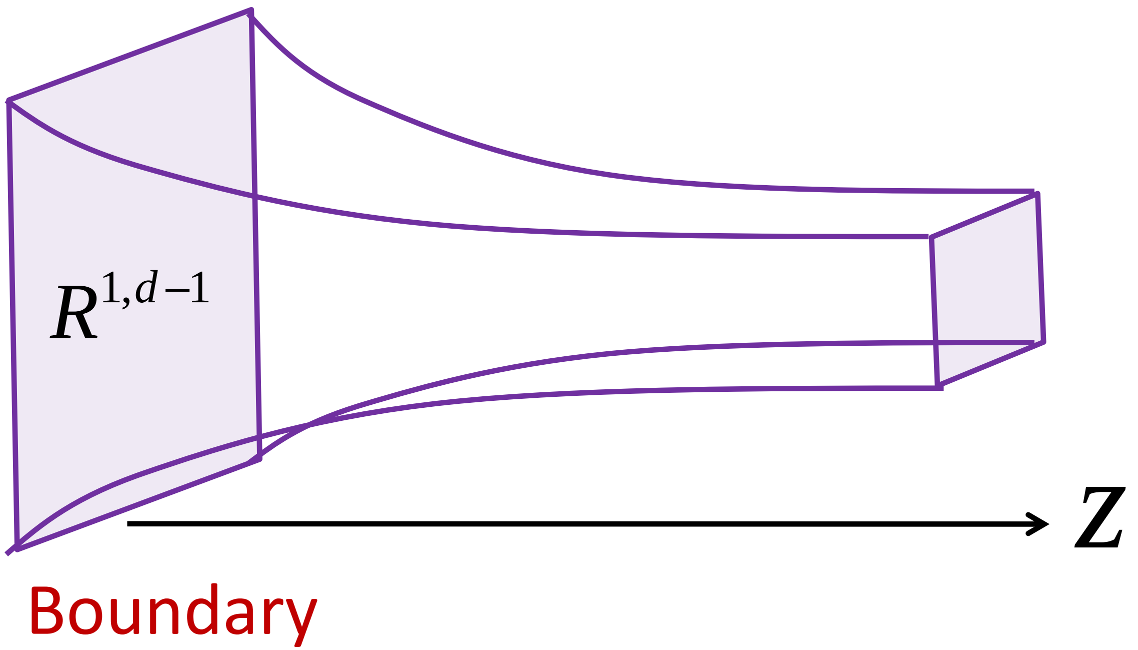
Poincare AdS

$$(x_0)^2 + (x_{d+1})^2 = (x_1)^2 + (x_2)^2 + \cdots + (x_d)^2 + R^2$$

$$\Rightarrow \begin{cases} x_\mu = \dfrac{R y_\mu}{z} \qquad (\mu = 0,1,2,..,d-1), \\ x_d = \dfrac{z}{2}\left(1 - \dfrac{R^2 - y^\mu y_\mu}{z^2}\right), \\ x_{d+1} = \dfrac{z}{2}\left(1 + \dfrac{R^2 + y^\mu y_\mu}{z^2}\right), \end{cases}$$

$$\Rightarrow \qquad ds^2 = R^2 \cdot \frac{dz^2 + dy^\mu dy_\mu}{z^2} \quad .$$

Poincare AdS$_{d+1}$



$$\partial(AdS_{d+1}) = R^{1,d-1}$$

$$ds^2 = R^2 \cdot \frac{dz^2 + dy^\mu dy_\mu}{z^2} \quad .$$

Summary:

(1) AdS_{d+1} has the SO(2,d) symmetry.

(2) AdS_{d+1} is the maximally symmetric space with the negative curvature:

$$R_{\mu\nu} = -\left(\frac{d}{R^2}\right) g_{\mu\nu}.$$

(3) AdS_{d+1} has a d dim. time-like boundary.

(1-2) What is the CFT ?

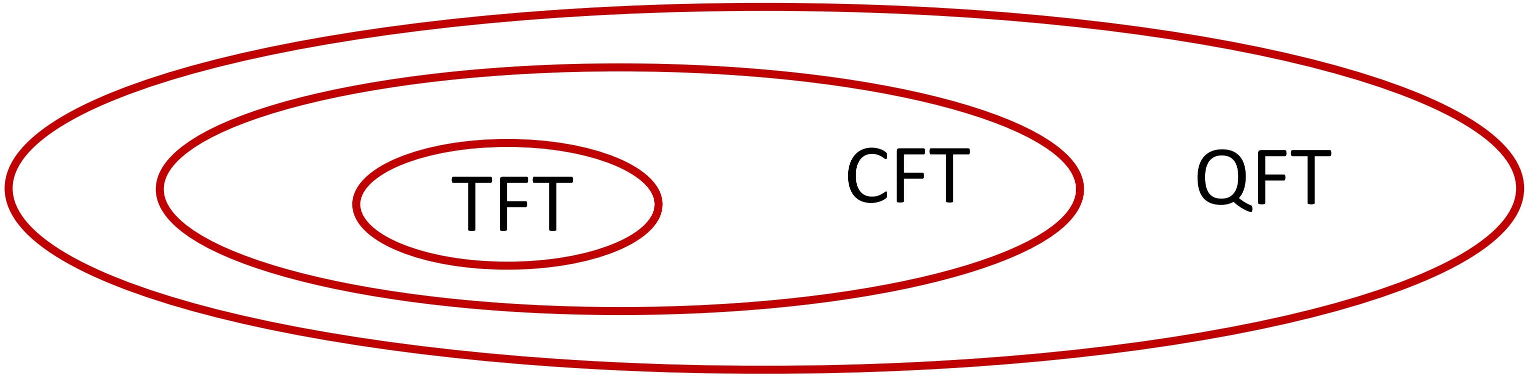
CFT (conformal field theory)

= quantum field theory which has the conformal sym.

``Scale invariant theory''

E.g. Electromagnetism (Maxwell theory)

$$(\vec{E}(x), \vec{B}(x)) \Leftrightarrow A(x) = A_\mu(x) dx^\mu$$



- Gravity = Theory of the dynamical metric $g_{\mu\nu}$.

- QFT = Theory of particles with a fixed metric $g_{\mu\nu}$.

$$\cup$$

- CFT = QFT which is invariant under $g_{\mu\nu} \to e^{\phi} g_{\mu\nu}$

$$\cup$$

- TFT = QFT which does not depend on the metric $g_{\mu\nu}$.

Conformal Symmetry CFT_d on $R^{1,d-1}$

Diffeomorphism $X^{\mu} = X^{\mu}(x^{\nu})$ such that

$$g(X)_{\mu\nu} = \Lambda(x) \cdot g(x)_{\mu\nu}.$$

$$\Rightarrow \begin{cases} \text{Translation}: \quad X^{\mu} = x^{\mu} + a^{\mu} \\ \text{Lorentz trf.}: X^{\mu} = L^{\mu}{}_{\nu} \cdot x^{\nu} \\ \text{Dilatation}: \; X^{\mu} = \lambda \cdot x^{\mu} \\ \text{Special cft.}: X^{\mu} = \dfrac{x^{\mu} - b^{\mu} \cdot x^2}{1 - 2(b \cdot x) + b^2 x^2} \end{cases}$$

SO(1,d-1)

Total conformal sym. = SO(2,d)

② What is AdS/CFT ?

The AdS/CFT is summarized as

Gravity (String Theory) on AdS_{d+1}

$$= \mathrm{CFT}_d \quad \text{on} \quad \partial(\mathrm{AdS}_{d+1}) = \begin{cases} R^{1,d-1} & \text{(Poincare)} \\ R \times S^{d-1} & \text{(Global)} \end{cases}$$

Note: Both has the same symmetry SO(2,d).

Remarks:

(1) Quantum gravity effects are suppressed if $R >> l_{pl}$

⇔ CFT = large N gauge theory (SU(N) Yang-Mills theory).

(2) Stringy effects are suppressed if $R >> l_{string}$

(if so, we can approximate the gravity by general relativity)

⇔ CFT gets strongly interacting.

(= the gauge theory coupling constant gets large)

If the conditions (1) and (2) are satisfied, then

String Theory (Quantum Gravity) ⇒ General Relativity.

(classical diff. geometry)

What does the `equivalence’ mean ?

AdS/CFT argues the equivalence of partition function:

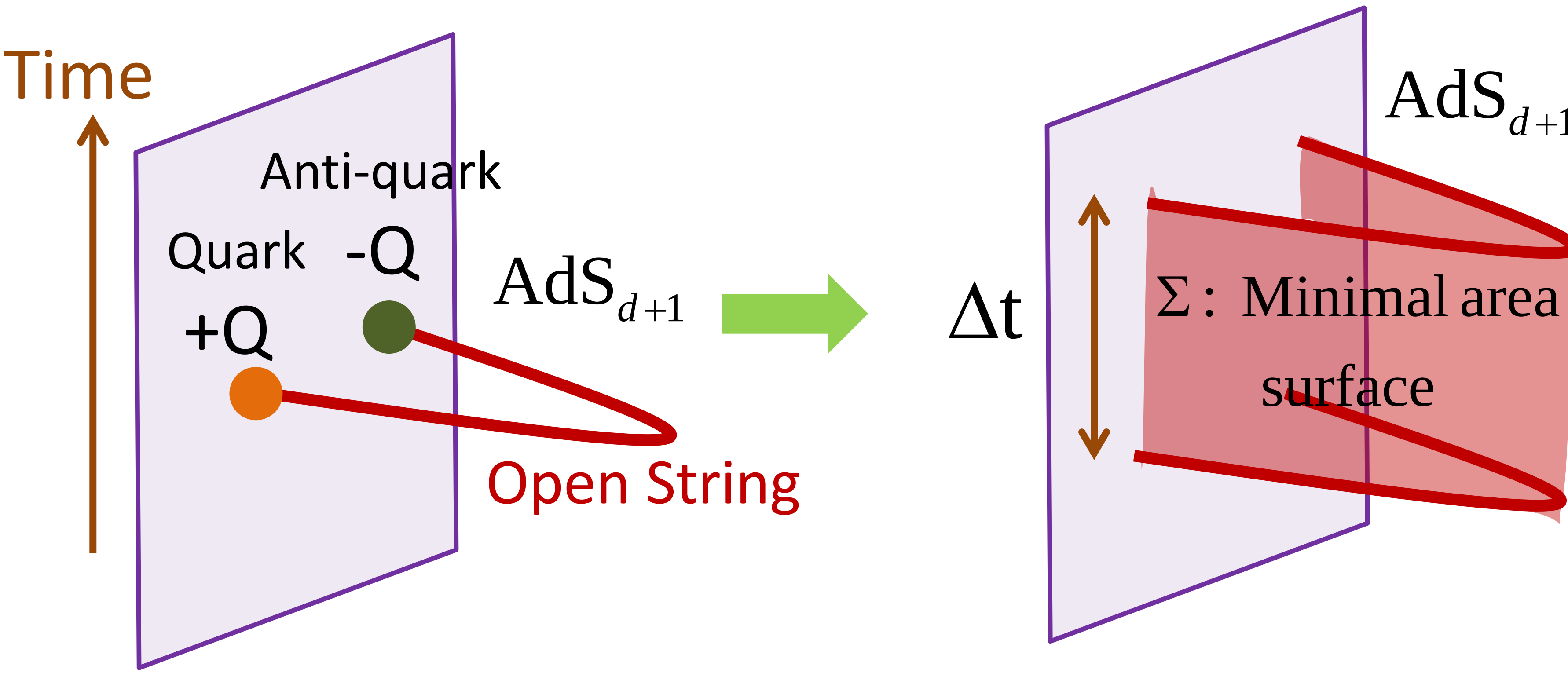
$$Z_{Gravity} = Z_{CFT} \quad .$$

In other words, the (free) energy agrees with each other.

The partition functions are functionals of boundary metric etc. :

$$Z_{Gravity}[g^{(0)}_{\mu\nu}, A^{(0)}_{\mu}, \phi, \ldots] = Z_{CFT}[g^{(0)}_{\mu\nu}, A^{(0)}_{\mu}, \phi, \ldots] \quad .$$

Ex. Wilson loop (Quark-anti quark potential)



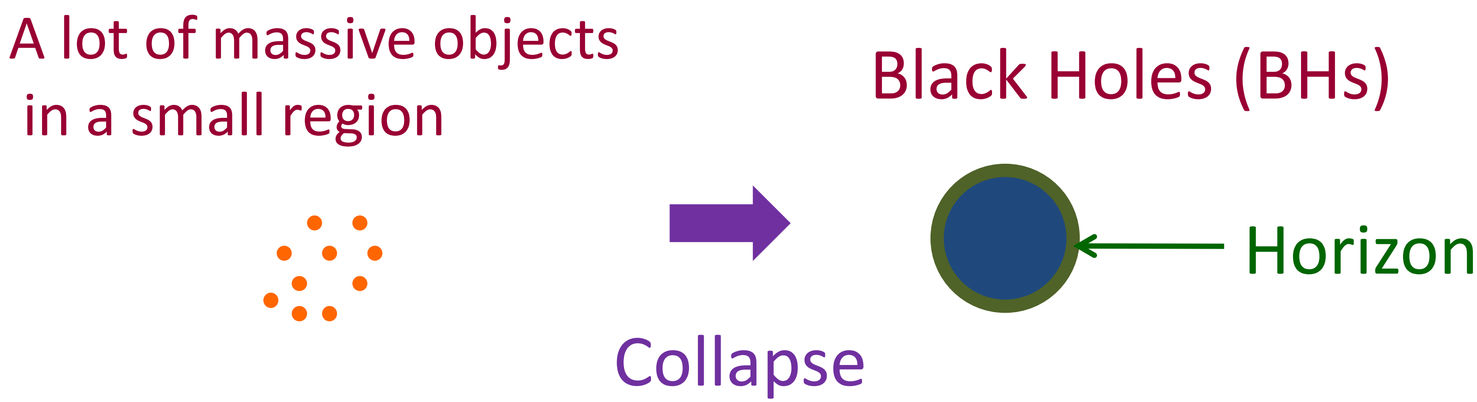
$$Z_{Gravity} = e^{-\text{Area}(\Sigma)} = e^{-E_q \cdot \Delta t} = \langle W \rangle_{CFT}$$

$\Rightarrow$ Quark - anti quark energy E_q

③ How does the AdS/CFT work ?

To understand this, we explain the holographic principle.

Q. How much information can we keep in a given region in the presence of gravity ?



In physics, the amount of (hidden) information is measured by entropy. (= entropy in thermodynamics)

The entropy of a black hole is given by the so called Bekenstein-Hawking formula

$$S_{BH} = \frac{\text{Area(Horizon)}}{4G_N}$$

G_N : Newton constant

It is known that black holes follow the laws of thermodynamics.

Temperature T ⇔ Strength of surface gravity on a BH

Energy E ⇔ Mass of a BH

Entropy S ⇔ Area of a BH

The first law : $TdS=dE$

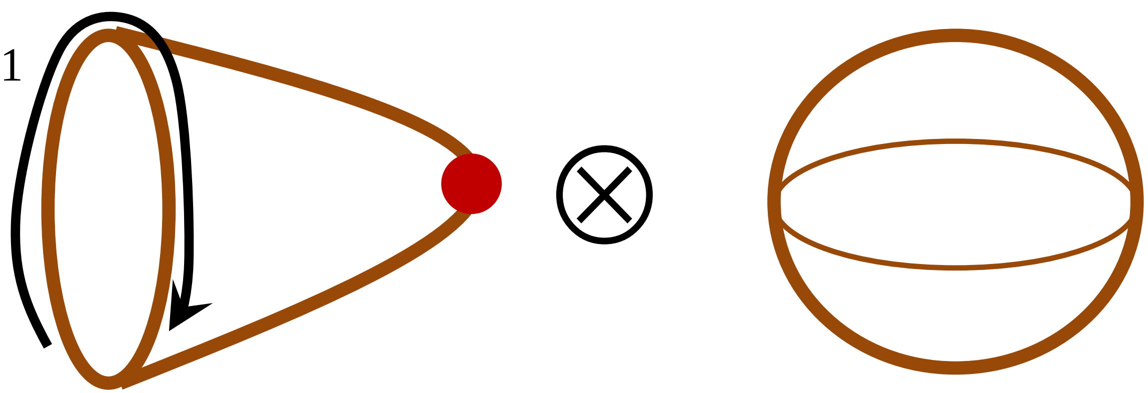
The second law : $\Delta S \geqq 0$

Geometrical origin of BH entropy (ex. 4D BH)

$$ds^2 = -\left(1-\frac{m}{r}\right)dt^2 + \left(1-\frac{m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$$\xrightarrow[WickRotation]{} ds_E^2 = \left(1-\frac{m}{r}\right)dt_E^2 + \left(1-\frac{m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2.$$

$\beta = T^{-1}$



$\text{Disk}: \quad \chi = 1$

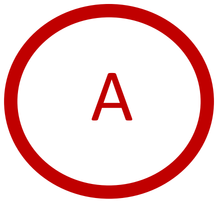
S^2

cf. In flat space, we have a cylinder instead of disk.

The black hole is the maximally entropic state in the presence of gravity.

This leads to the idea of entropy bound:

$$S(A) \leq \frac{\text{Area}(\partial \text{A})}{4G_N}$$

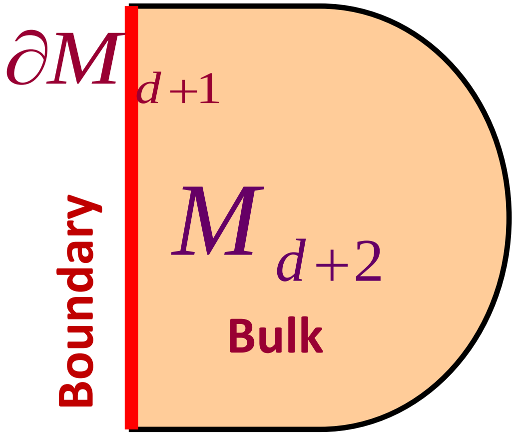


The degrees of freedom in gravity are proportional to the **area**, instead of the volume !

(Cf. In the absence of gravity, they are proportional to the volume.)

Motivated by this, holographic principle has been proposed: ['t Hooft 93 ,Susskind 94]

d+1 dim. Gravity on M
= d dim. non-gravitational theory on ∂Md+1

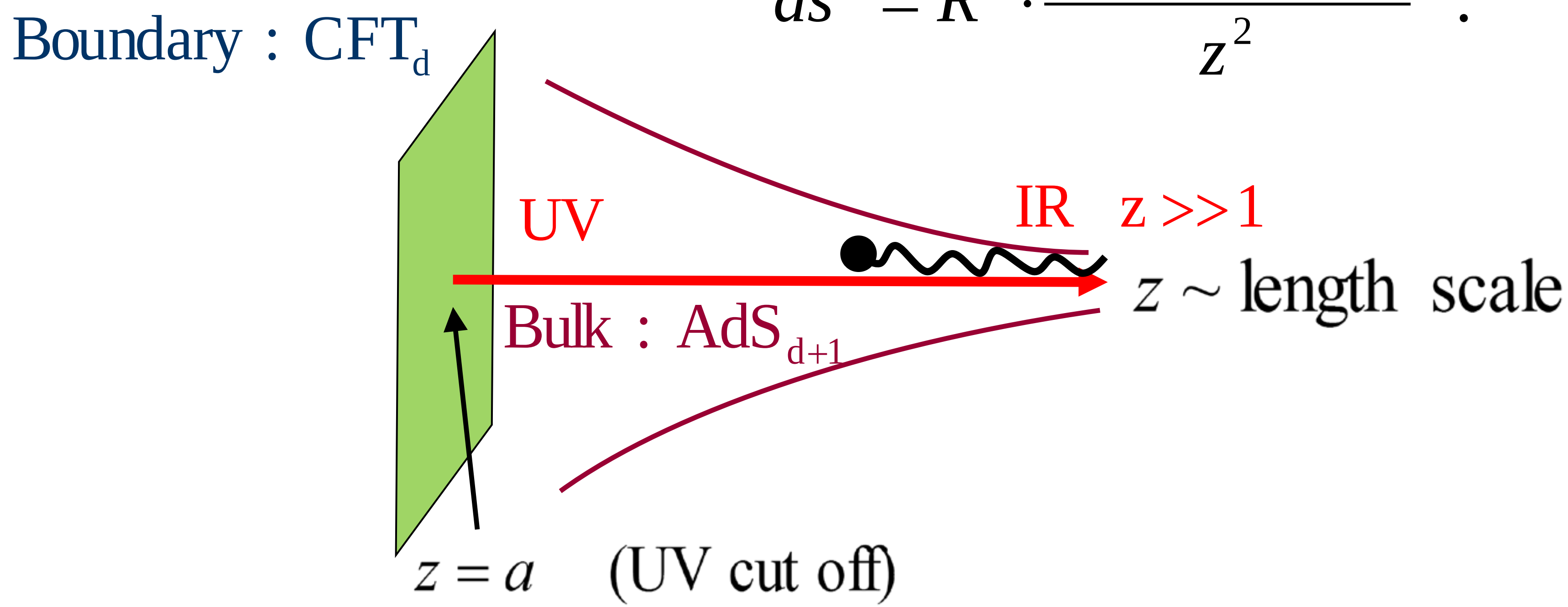


This is a very intuitive argument, but later, an explicit example is provided by the AdS/CFT.

What is the meaning of the extra one dimension ?

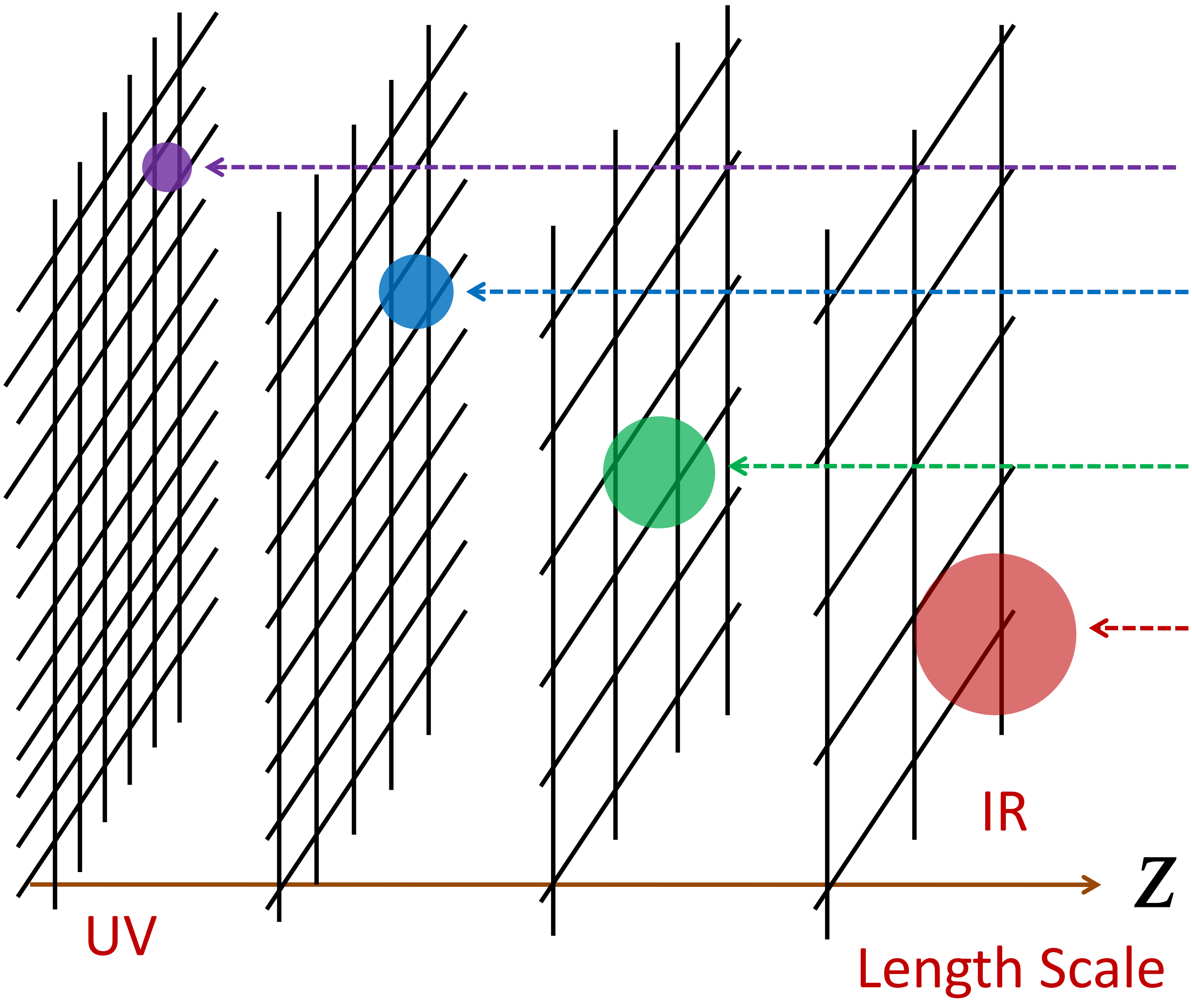
Consider the AdS/CFT:

$$ds^2 = R^2 \cdot \frac{dz^2 + dy^\mu dy_\mu}{z^2} \quad .$$

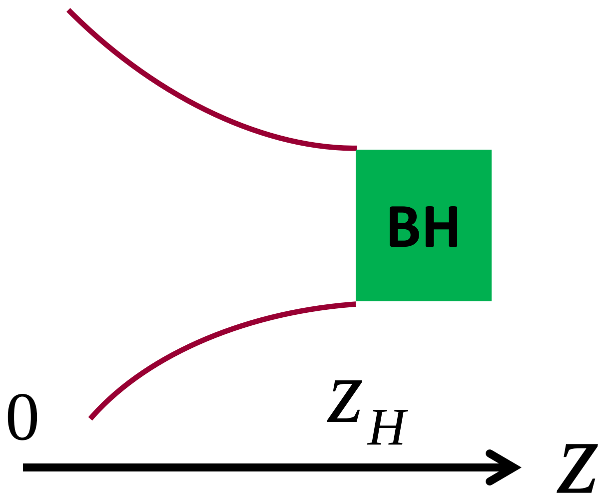


The radial direction z corresponds to the length scale in CFT under the RG flow. (1/z ~ Energy Scale)

AdS/CFT as a ``Gravitational Sieve''



Black holes and AdS/CFT



AdS BH solution

$$ds^2 = R^2\left(-\frac{f(z)}{z^2}dt^2 + \frac{dz^2}{z^2 f(z)} + \frac{dy^i dy_i}{z^2}\right) .$$

$$f(z) \equiv 1-(z/z_H)^d .$$

This is dual to a CFT at finite temperature T.

T is the same as the black hole temperature .

$$T_{BH} \propto \frac{1}{z_H}.$$

The AdS/CFT argues

BH thermodynamics = CFT thermodynamics

BH entropy = thermal entropy

Remark:

It is well-known that in flat spacetime, the black hole typically has the *negative* specific heat (→unstable).

However, the AdS BH (=BH in a box) has the *positive* specific heat.

④ AdS/CFT from String Theory ?

(4-1) Large N Limit and String Theory

It is useful to look at a toy model of non-abelian gauge theories. Consider a **matrix model**:

$$A_{ab} = [A_\mu(x)]_{ab}\, dx^\mu \quad \Rightarrow \quad \Phi_{ab} \qquad (a, b = 1, 2, .., N)$$

SU(N) Connection (1-form)
(SU(N) gauge field)

N × N Hermitian Matrix
(0 dim. matrix model)

Note: This simplification still keeps essential combinatorics.

In this matrix model, the partition function is given by the integral:

$$Z_{matrix} = \int [d\Phi] \exp[-S(\Phi)],$$

$$S(\Phi) = \frac{1}{g^2} \mathrm{Tr}[\Phi^2 + \Phi^3].$$

g : coupling constant (gauge coupling)

Now we take so called the large N limit:

$$N \to \infty \quad \text{with} \quad \lambda \equiv Ng^2 = \text{finite}.$$

’t Hooft coupling

Let us perform perturbative expansions.

$$Z_{matrix} = \int [d\Phi] \exp[-S_0(\Phi)] \cdot \sum_{n=0}^{\infty} \left(-S_1(\Phi)\right)^n ,$$

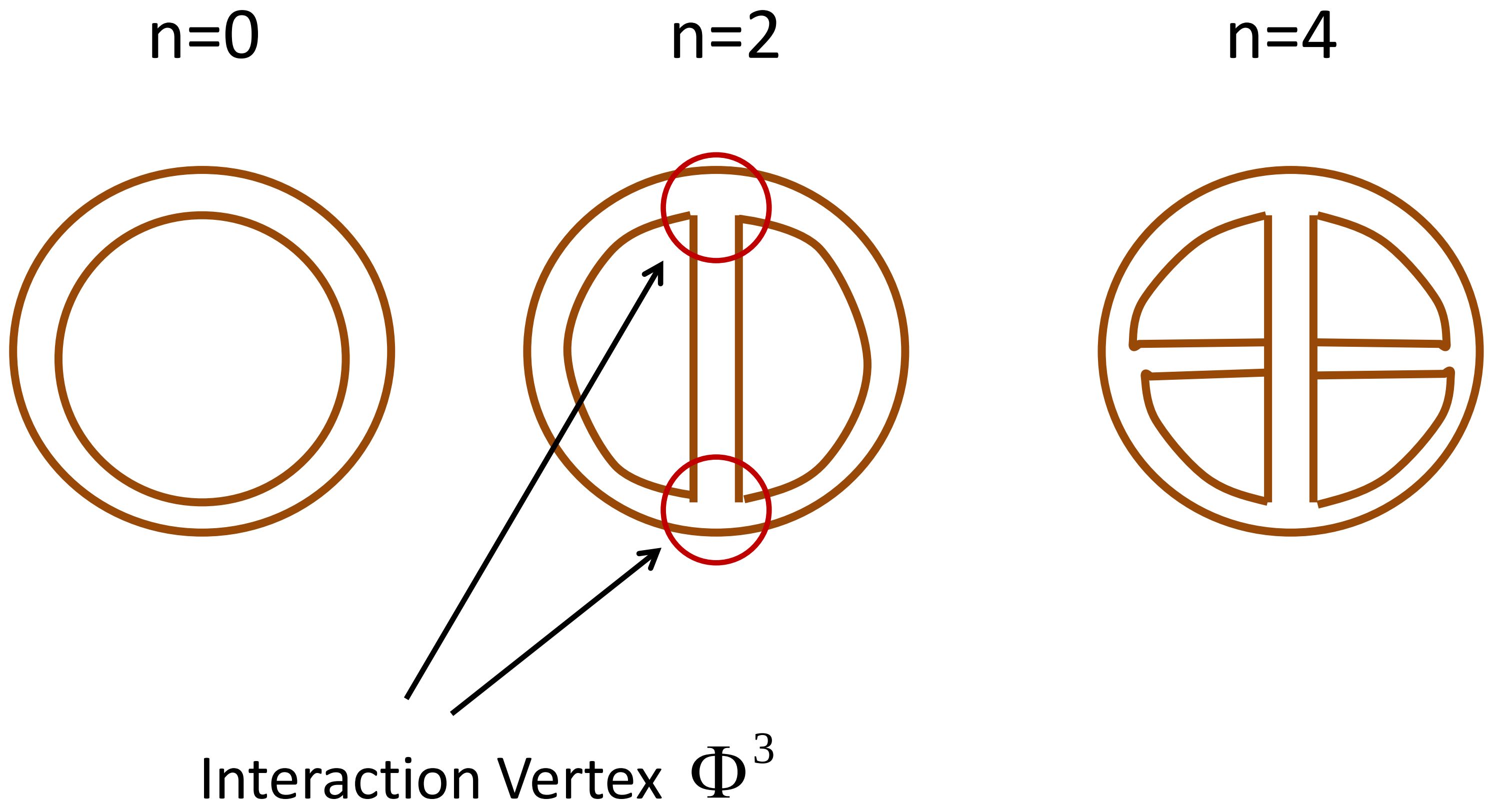
$$S_0(\Phi) \equiv \frac{N}{\lambda} \mathrm{Tr}[\Phi^2], \qquad S_1(\Phi) \equiv \frac{N}{\lambda} \mathrm{Tr}[\Phi^3].$$

The propagator is given by

$$\left\langle \Phi_{ab} \Phi_{cd} \right\rangle = \int [d\Phi]\, \Phi_{ab} \Phi_{cd} \exp[-S_0(\Phi)] \sim \frac{\lambda}{N} \delta_{ad} \delta_{bc} \quad .$$

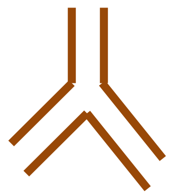
$$c.f. \qquad \frac{\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} \cdot x^2}{\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2}} = \frac{1}{2a}$$

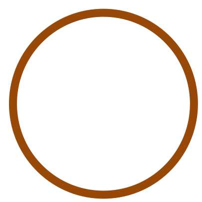
Perturbative expansions are expressed by using the Feynman diagrams:



Rules

Propagator  → $\left(\frac{\lambda}{N}\right)^{I}$ for I propagators

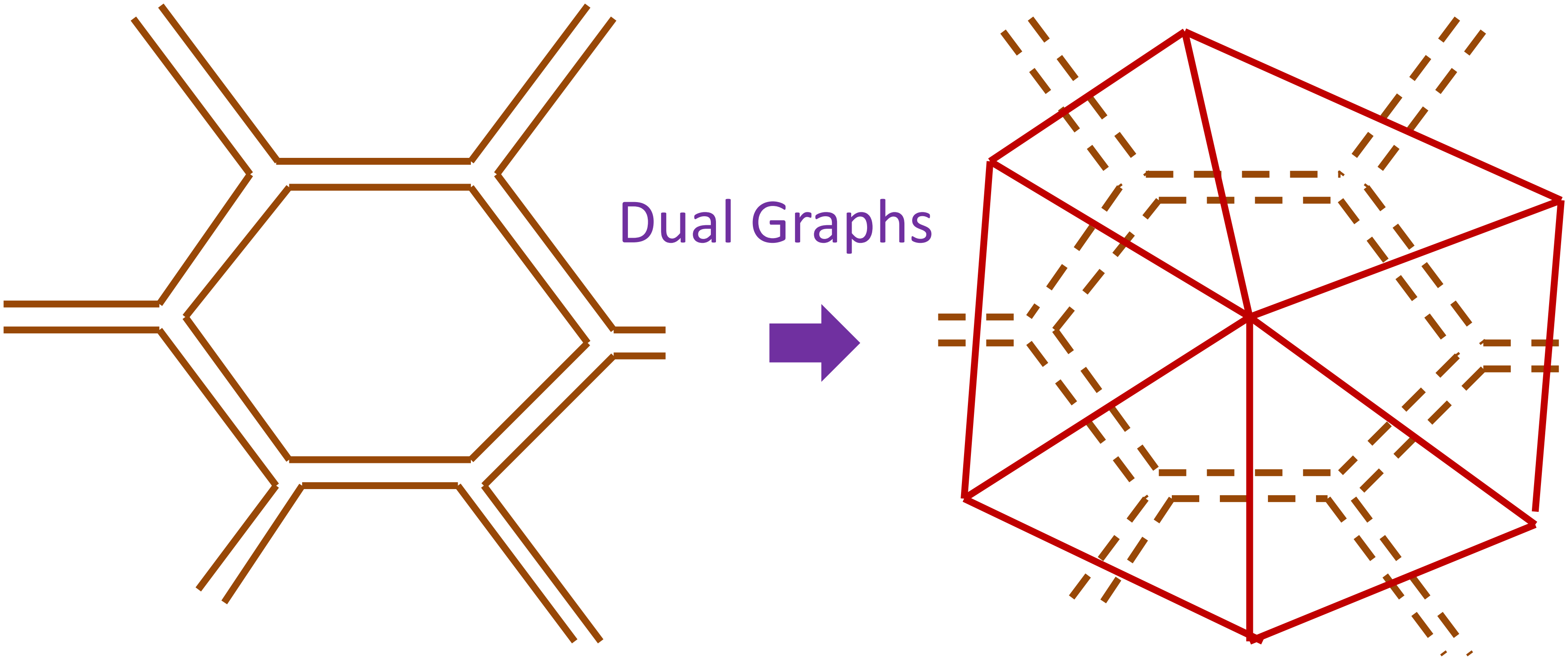
Vertex  → $\left(\frac{N}{\lambda}\right)^{V}$ for V vertices

Loop  → N^{L} for L loops

Thus we find

$$Z_{matrix} \sim \sum_{\text{all graphs}} \left(\frac{\lambda}{N}\right)^{I} \cdot N^{L} \cdot \left(\frac{N}{\lambda}\right)^{V} = \sum_{\text{all graphs}} N^{L+V-I} \cdot \lambda^{I-V}.$$

Triangulation and Feynman Diagrams



$$\text{Loop} \quad (L) \quad \Leftrightarrow \quad \text{point} \quad (0\text{-cycle})$$

$$\text{Propagator} \, (I) \quad \Leftrightarrow \quad \text{line} \quad (1\text{-cycle})$$

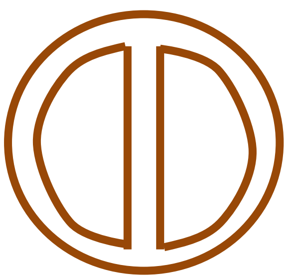
$$\text{Vertex} \quad (V) \quad \Leftrightarrow \quad \text{Face} \quad (2\text{-cycle})$$

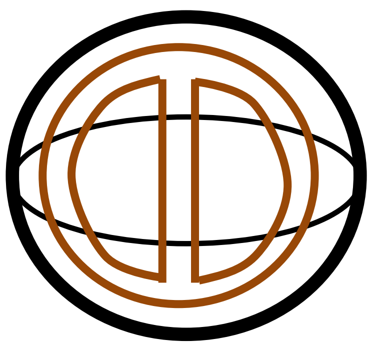
Therefore,

$$L - I + V = \#(\text{points}) - \#(\text{lines}) + \#(\text{faces}) = \chi \quad (\text{Euler number}).$$

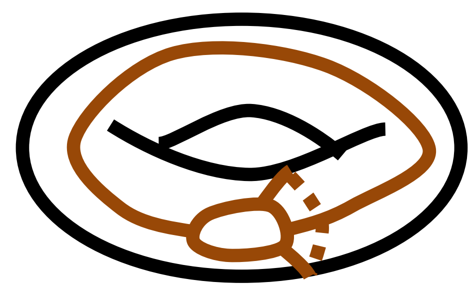
In this way, $$Z_{matrix} \sim \sum_{\text{all graphs}} N^{\chi} \cdot \lambda^{I-V}.$$

Planar ex. S^2





Non-planar ex. T^2



$L = 3, \quad I = 3, \quad V = 2$

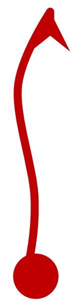
$\Rightarrow \chi = 2 \quad$ (Sphere)

$L = 2, \quad I = 6, \quad V = 4$

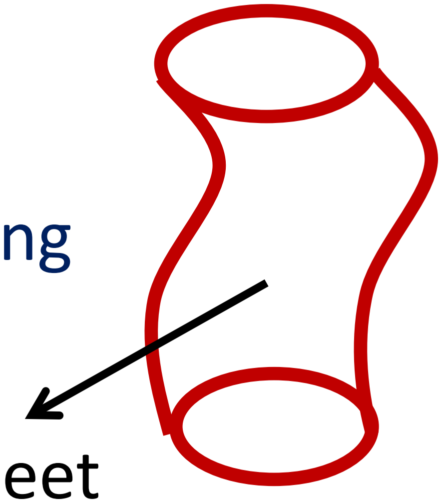
$\Rightarrow \chi = 0 \quad$ (Torus)

String Theory

Particle

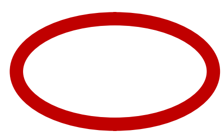


String



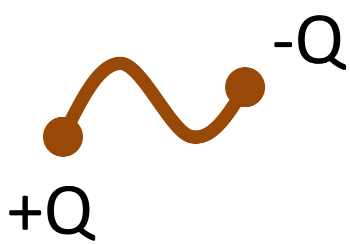
World-sheet
(Riemann surface)

Closed string



⇒ (Quantum) Gravity

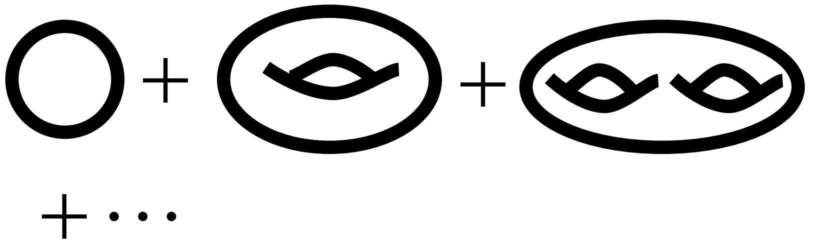
Open string



⇒ Electromagnetism

Actually, the same topological expansion appears in string theory:

$$Z_{String} = \sum_{\substack{\text{Riemann} \\ \text{Surfaces}}} g_s^{-\chi} \cdot f_g(R/l_s)$$



world-sheets

g_s : string coupling constant

l_s : string length

Thus we expect the correspondence: $N \Leftrightarrow g_s^{-1}$

Therefore, the large N expansion $\Leftrightarrow$ g_s expansion

$$\left(\text{we can also find } \lambda \Leftrightarrow \frac{R}{l_s} \text{ in AdS/CFT.} \right)$$

(4-2) Matrix Model and 2D String Theory

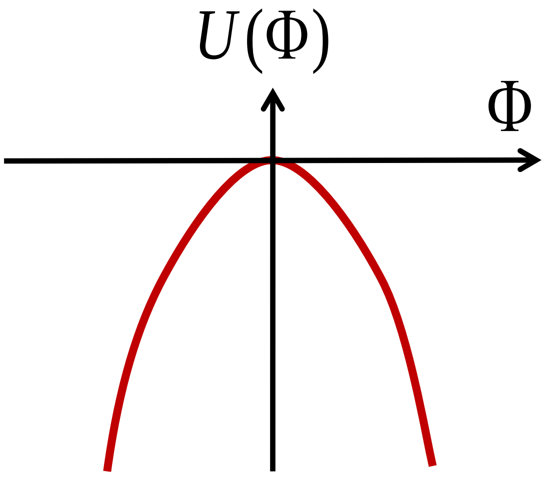
Before we go back to AdS/CFT, let us briefly introduce the simplest example of holography: 2D string theory.

Consider a matrix quantum mechanics:

$$L_{MQM} = \mathrm{Tr}[(\partial_t \Phi)^2 - U(\Phi)] \ ,$$

$$U(\Phi) = -l_s^{-2} \cdot \Phi^2 \quad \text{(potential energy)},$$

$$\Phi \to g \cdot \Phi \cdot g^{-1} \quad \text{(gauge sym.)}.$$



Using the U(N) gauge sym. , we diagonalize the matrix

$$g \cdot \Phi \cdot g^{-1} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix}$$

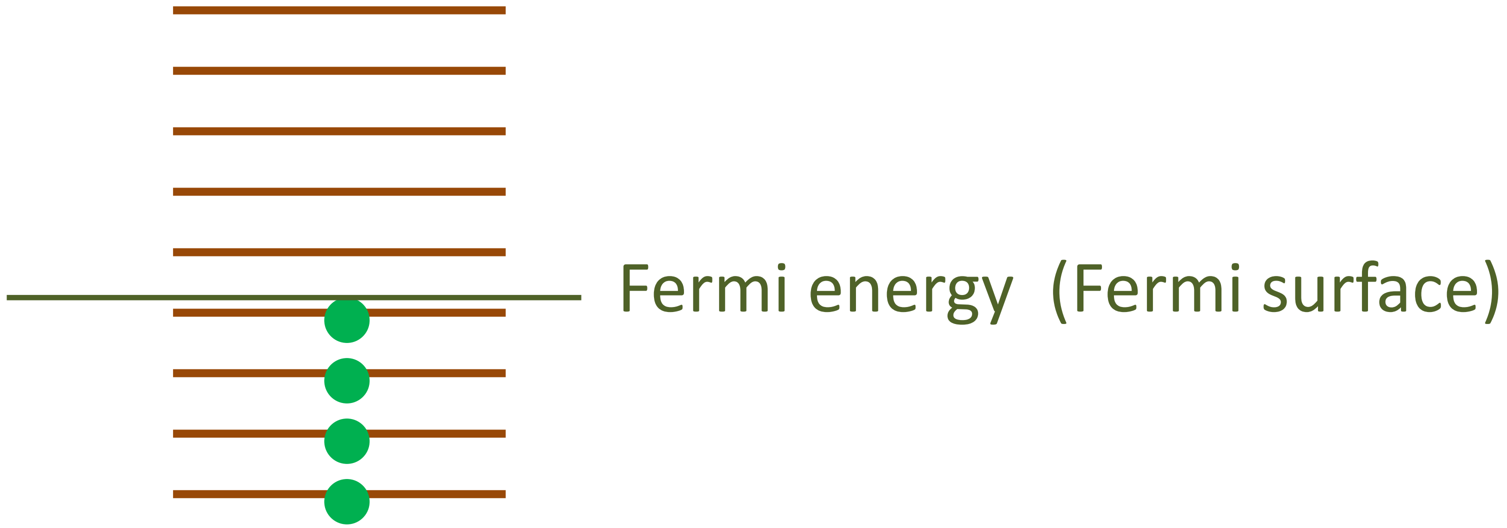
$$[d\Phi] = \prod_{i=1}^{N} d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2$$

This shows that the system is described by *N free fermions*

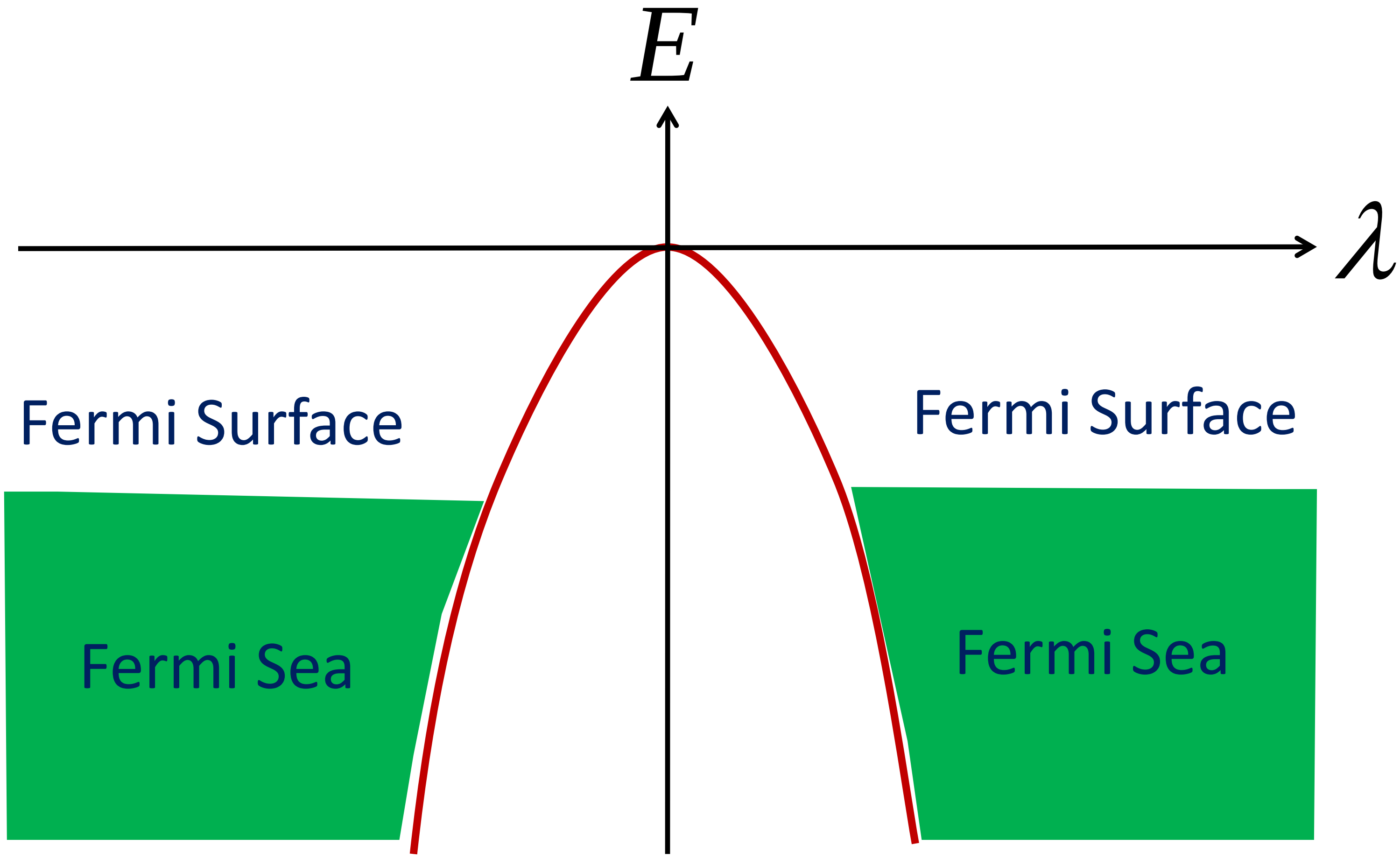
$$H = \sum_{i=1}^{N} \left[(\partial_t \lambda)^2 + U(\lambda_i) \right] \quad ,$$

$$\left(\because \qquad \Psi(\lambda_1, \lambda_2, \ldots\ldots, \lambda_N) \propto \prod_{i<j} (\lambda_i - \lambda_j) \right)$$

Pauli's exclusion principle requires that there is only one fermion on each energy level.



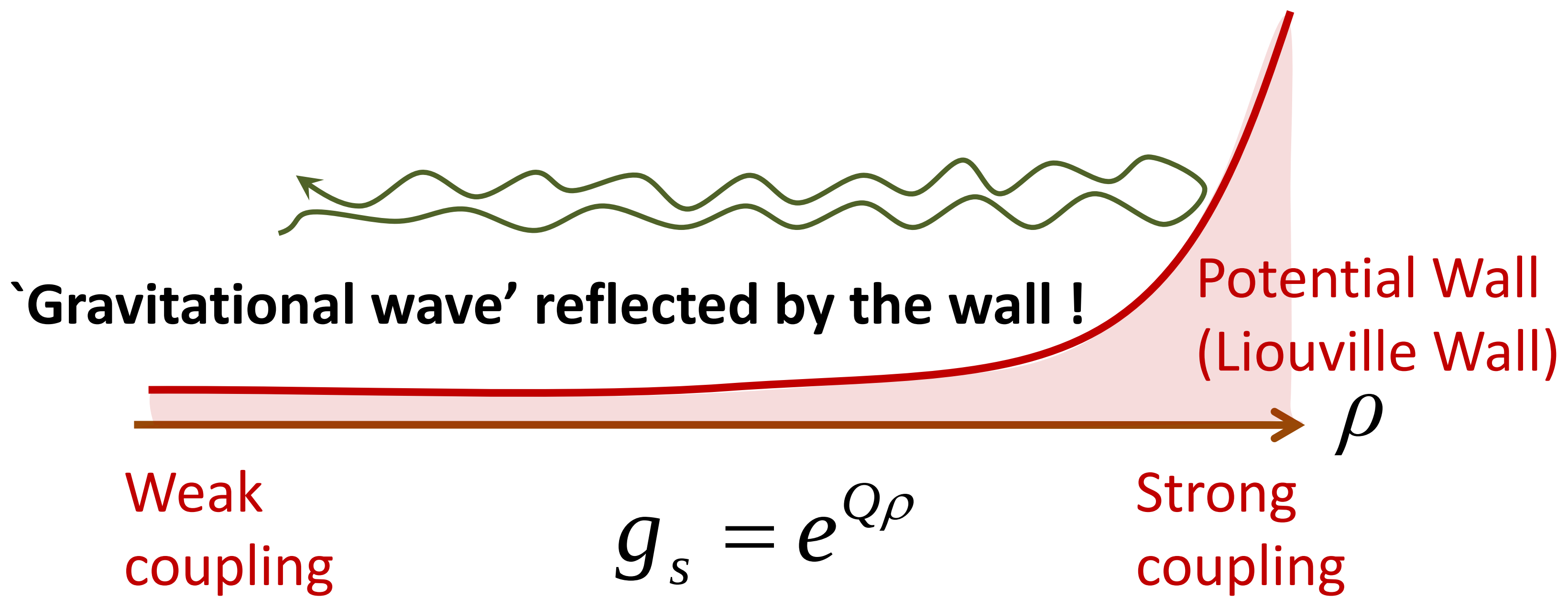
In our matrix model, we find the following structure:



This matrix model is known to be equivalent to the two dim. string theory.

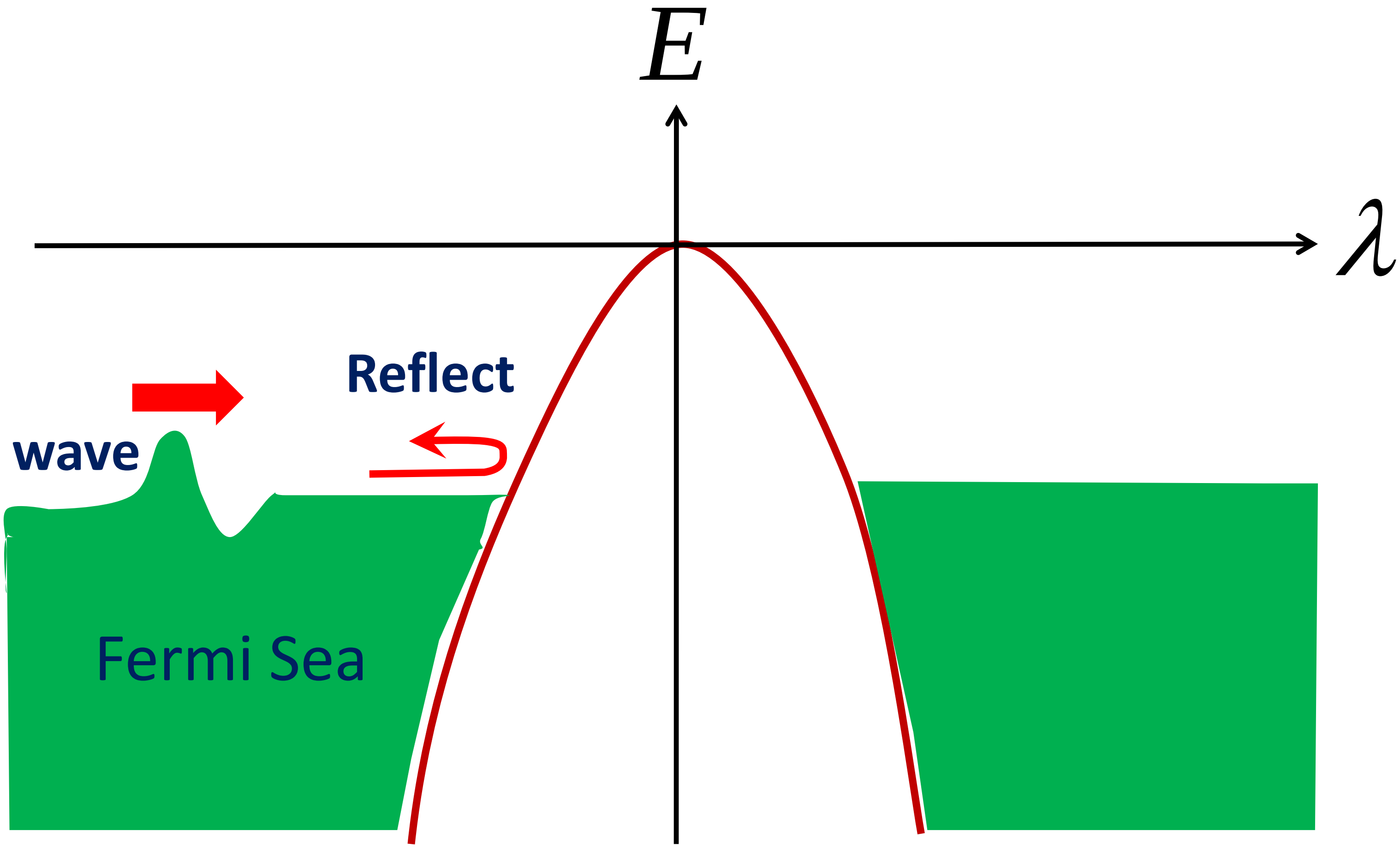
$$\text{spacetime}: \ (t, \rho) \in R^{1,1}$$

Note: Usually, string theory is defined only in 10 dim. To obtain 2D string, we turn on non-trivial dilaton field (described by the Liouville CFT).



$$g_s = e^{Q\rho}$$

The waves on the fermi surface = `Gravitational waves’



In this way, we find the following simple holography:

Matrix Quantum Mechanics **=** 2D (super) string theory

($c=1$ *or* $\hat{c}=1$ matrix model) (2D quantum gravity)

0+1 dim. 1+1 dim.

In this example, the space dimension in gravity can be regarded as the direction along the Fermi surface.

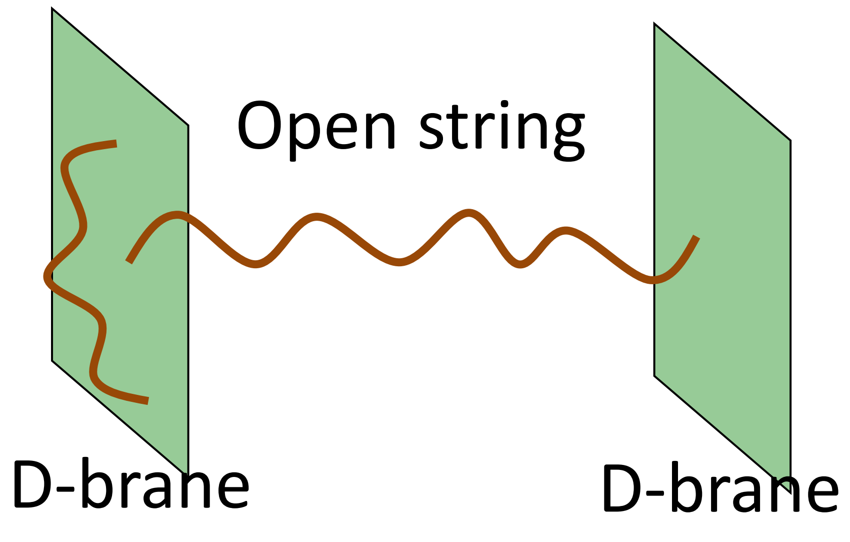
$\Rightarrow$ A mechanism of emergent space dimension !

Bosonic string [Polyakov, ….Gross-Miljkovic, Berezin-Kazakov-Zamolodchikov, Ginsparg-ZinnJustin 90], D-brane interpretation [Mcgreevy-Verlinde 03]
Especially, superstring (type 0 string) version becomes non-perturbatively stable. [Toumbas-TT 03, Douglas-Klebanov-Kutasov-Maldacena-Martinec-Seiberg 03]

(4-3) AdS/CFT from String Theory

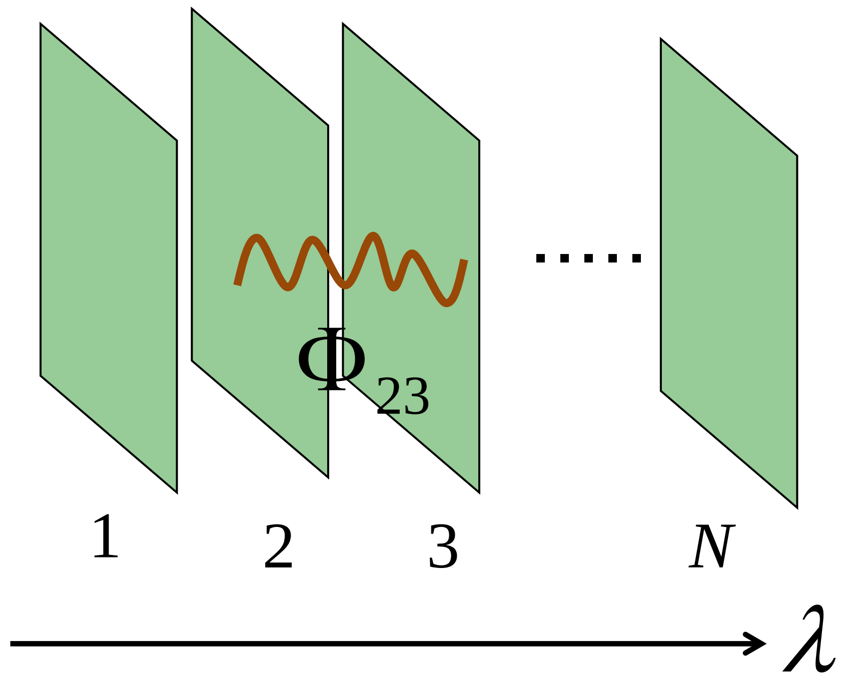
Now let us go back to the AdS/CFT correspondence. Originally, the AdS/CFT has been found in string theory systems with so called ***D-branes***.

Dp-brane ⇒ (p+1) dim. charged object in string theory



Open strings can end on D-branes.

Consider N Dp-branes:



$\Leftrightarrow$

N × N matrix

$$\Phi_{ab}$$

$$\Phi \approx \begin{bmatrix} \lambda^{(1)} & & & \\ & \lambda^{(2)} & & \\ & & \ddots & \\ & & & \lambda^{(N)} \end{bmatrix}$$

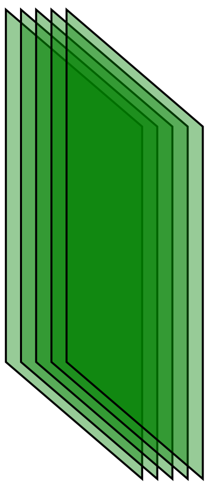
N Dp-branes ⇒ a p+1 dim. U(N) gauge theory $\Phi_{ab}(x)$

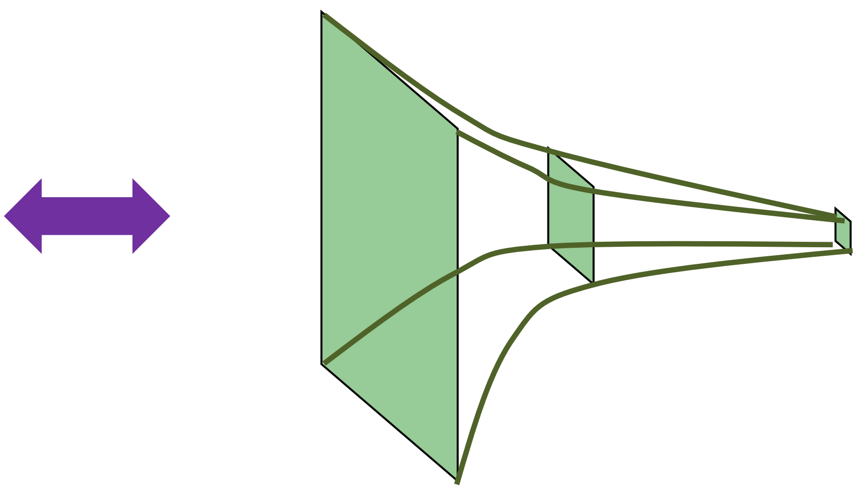
(theory of `matrix values functions')

A basic example of AdS/CFT correspondence is obtained by looking at the low energy limit of D3-branes.

Open string viewpoint = Closed string viewpoint

N D3-branes (very heavy) 5 dim. AdS space

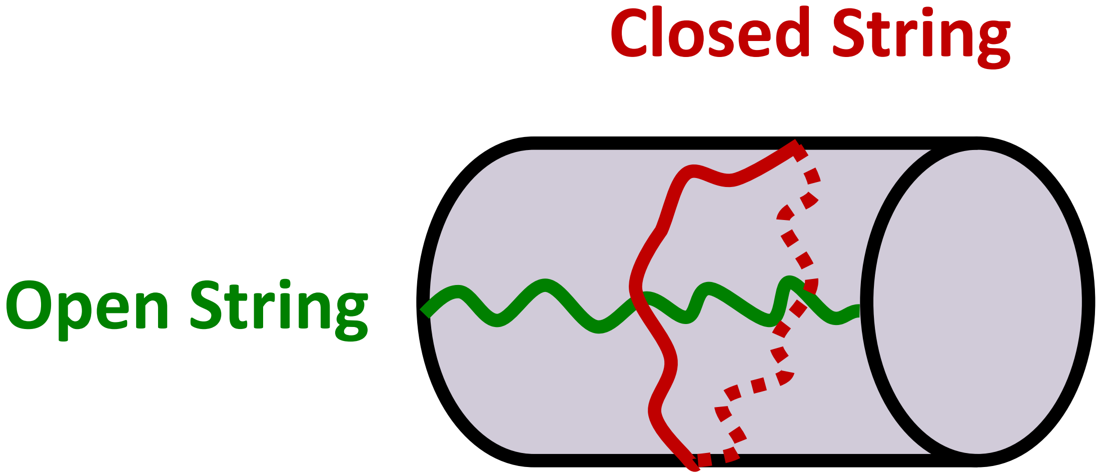




3+1dim. $N=4$ SU(N) Super Yang-Mills (CFT)

Type IIB String on $AdS_5 \times S^5$ (Gravity on AdS)

Note: Open-Closed Duality



Basic Modifications

$$\text{Type IIB String on } AdS_5 \times S^5$$

Einstein Manifolds N_5
with negative curvature
(AdS BH, AdS Soliton etc.)
$\Rightarrow$ Breaks SO(2,4)
conformal sym.

Einstein Manifolds M_5
with positive curvature
$\Rightarrow$ Breaks SO(6) R-sym.
of N=4 Supersym.
$\exists$ N=1 Susy if Sasaki-Einstein.

M_5 is Sasaki-Einstein

$\Leftrightarrow \quad \mathrm{C}(M)_6 : \quad ds^2 = dr^2 + r^2 ds^2_{M5}$

has a SU(3) holonomy (i.e. Calabi-Yau).

⑤ Developments of AdS/CFT

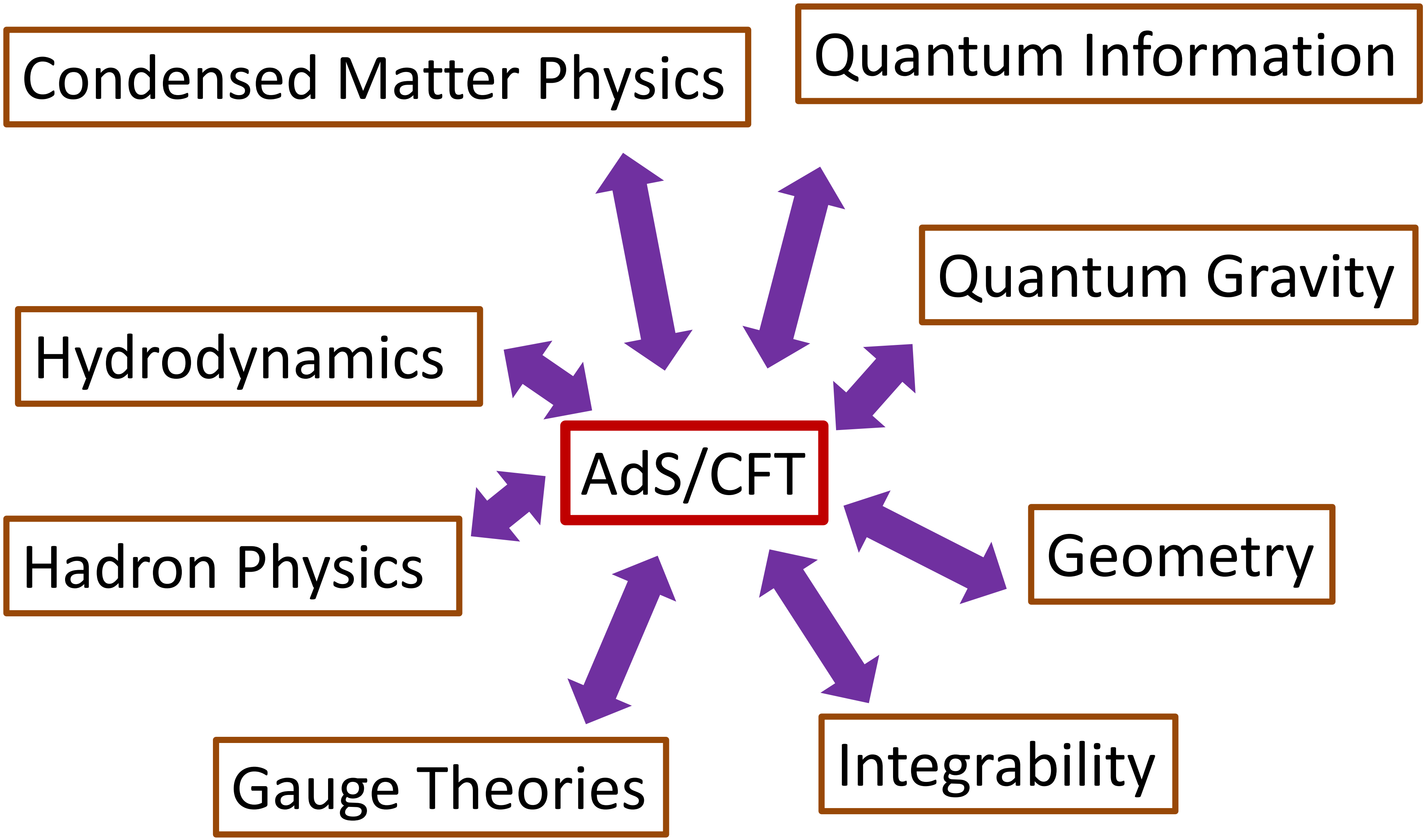
In summary, the AdS/CFT argues an equivalence between theories with gravity and those without gravity.

Gravity on AdS = CFT

\`Geometry' \`Quantum (algebra?)'

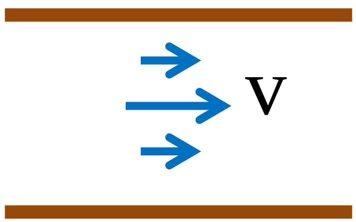
Interestingly, it often happens that when either side is difficult to analyze, the other side gets more tractable.

This enables us to apply the AdS/CFT to many subjects !


Condensed Matter Physics
Quantum Information
Quantum Gravity
Hydrodynamics
AdS/CFT
Hadron Physics
Geometry
Gauge Theories
Integrability

Example 1.   Viscosity in Strongly Coupled Systems

The viscosity is a fundamental property for fluids.



$$\eta \sim 300 \cdot \hbar \cdot n \quad \text{(for water)}$$

In the classical gravity (= large N strongly coupled CFT), the AdS/CFT predicts the universal result

$$\boxed{\frac{\eta}{s} = \frac{\hbar}{4\pi k_B}.}$$

[Kovtun-Son-Starinets 05]

$$(s \sim n \cdot k_B : \text{ entropy density})$$

$$\Rightarrow \eta \sim 0.1 \cdot \hbar \cdot n \quad \text{(very small)}$$

Interestingly, the same order of viscosity has been observed in *two different* strongly coupled systems:

(1) Quark-gluon plasma (QGP)

$$\frac{\eta}{s} \sim 0.1 \cdot \frac{\hbar}{k_B} \quad \text{at} \quad T \sim 10^{12} K.$$

(2) Cold Atoms (Fermi ^{6}Li Gas)

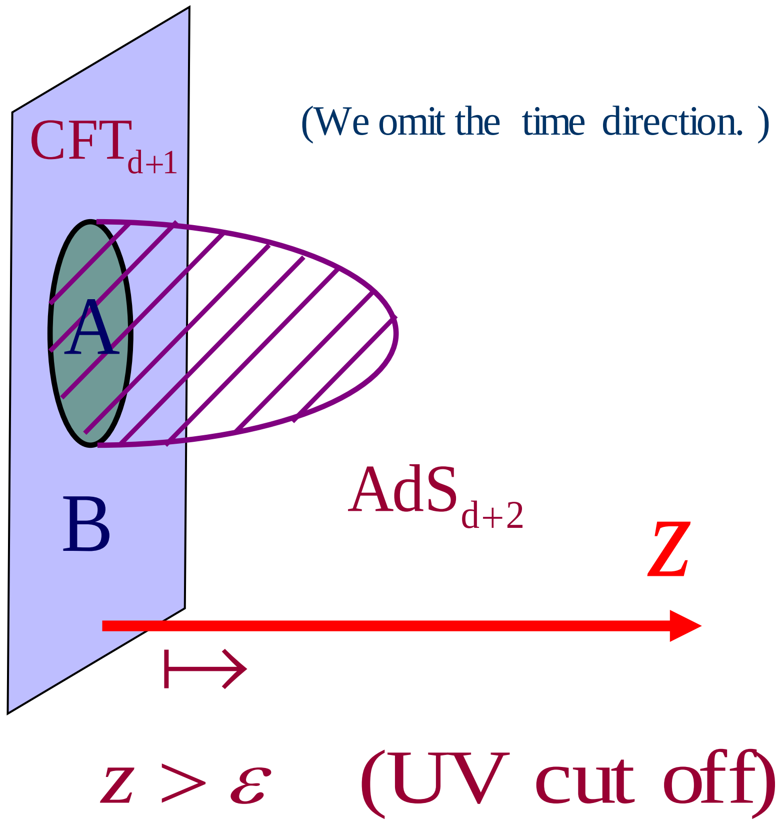
$$\frac{\eta}{s} \sim 0.2 \cdot \frac{\hbar}{k_B} \quad \text{at} \quad T \sim 10^{-6} K.$$

Example 2. Entanglement Entropy

$S_A = -\mathrm{Tr}[\rho_A \log \rho_A]$ ~ Hidden information in A

$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N}$$

[Ryu-TT 06]



γ_A is the **minimal area surface** (codim.=2) such that

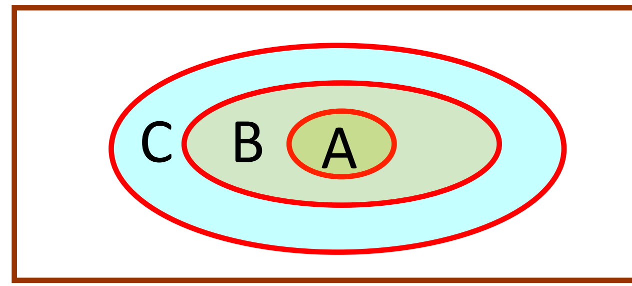
$\partial A = \partial \gamma_A$ and $A \sim \gamma_A$.

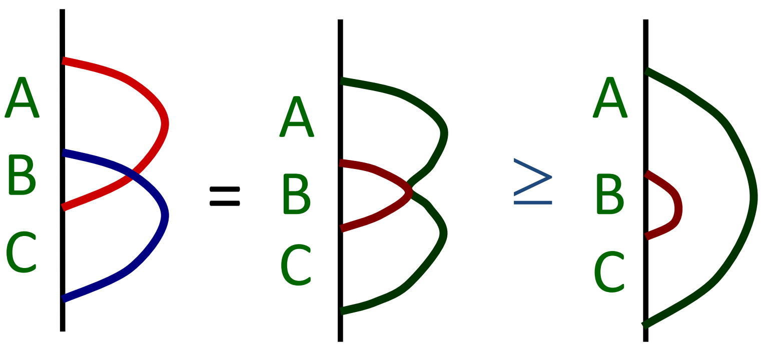
homologous

Holographic Proof of Strong Subadditivity [Headrick-TT 07]

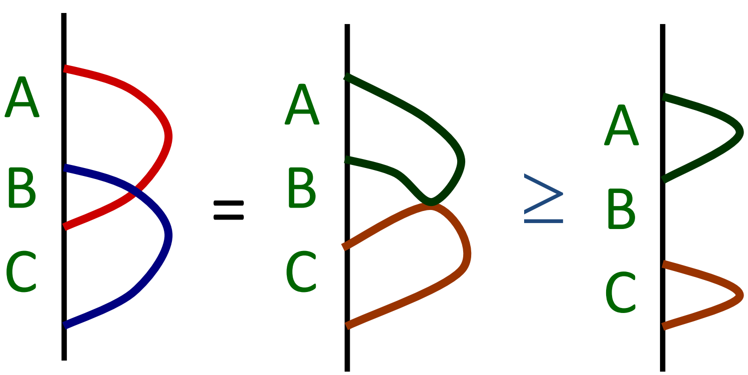
We can easily derive the ***strong subadditivity***, which is known as the most important inequality satisfied by EE.

[Lieb-Ruskai 73]





$$\Rightarrow \quad S_{A\cup B} + S_{B\cup C} \geq S_{A\cup B\cup C} + S_{B}$$

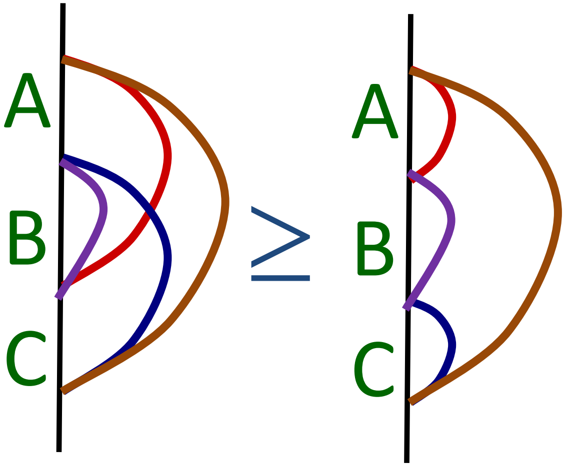


$$\Rightarrow \quad S_{A\cup B} + S_{B\cup C} \geq S_{A} + S_{C}$$

Tripartite Information [Hayden-Headrick-Maloney 11]

Recently, the holographic entanglement entropy is shown to have a special property called ***monogamy***.

$$S_{AB}+S_{BC}+S_{AC}\geq S_A+S_B+S_C+S_{ABC}$$
$$\Leftrightarrow I(A:B)+I(A:C)\leq I(A:BC)$$



AdS/CFT predicts that **large N gauge theories are monogamous**.

Future Problems

- Proof of AdS/CFT
- Holography for Cosmological spacetimes ?

  (e.g. de Sitter space)
- Search of New States of Matter ?
- More universal predictions ?

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