LECTURES ON CORRELATION FUNCTIONS IN INTGRABLE MODELS OF QUANTUM FIELD THEORY.

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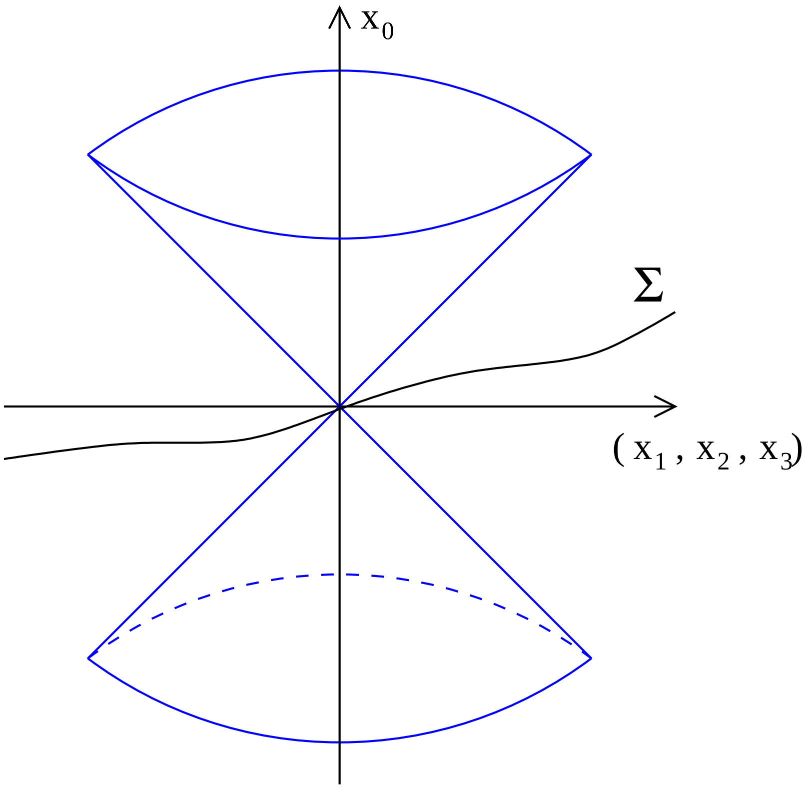
1. Lehman-Symanzik-Zimmermann axiomatics for QFT.



$$x = (x_0, x_1, x_2, x_3), \quad x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 \,.$$

Poincaré group: $\mathbb{R}^{1,3} \rtimes O(1,3)$.

1. The space of states.

The space of states is the Fock space of particles. Every particle carries the momentum $p = (p_0, p_1, p_2, p_3)$ satisfying

$$p^2 = p_0^2 - p_1^2 - p_2^2 - p_3^2 = m^2 \,.$$

It may have external degrees of freedom counted by $\epsilon = 1, \cdots, N$. They are Lorenz and isotopic. For simplicity I consider only Lorenz scalars. There is unique state called vacuum: $|\mathrm{vac}\rangle$. The creation-annihilation operators $a^*_{\mathrm{in},\epsilon}(k)$, $a^\epsilon_{\mathrm{in}}(k)$, $k = (p_1, p_2, p_3)$ satisfy

$$[a^\epsilon_{\mathrm{in}}(k), a^*_{\mathrm{in},\epsilon'}(k')] = \delta^\epsilon_{\epsilon'} \delta^{(3)}(k - k') \,.$$

Annihilation operators kill the vacuum

$$a^\epsilon_{\mathrm{in}}(k)|\mathrm{vac}\rangle = 0 \,,$$

creation operators create n-particle states.

There is another set of operators $a^*_{\mathrm{out},\epsilon}(k)$, $a^\epsilon_{\mathrm{out}}(k)$. The two sets are related by unitary operators called S-matrix:

$$a^*_{\mathrm{out},\epsilon}(k) = S a^*_{\mathrm{in},\epsilon}(k) S^* \,, \qquad S\,|\mathrm{vac}\rangle = |\mathrm{vac}\rangle \,.$$

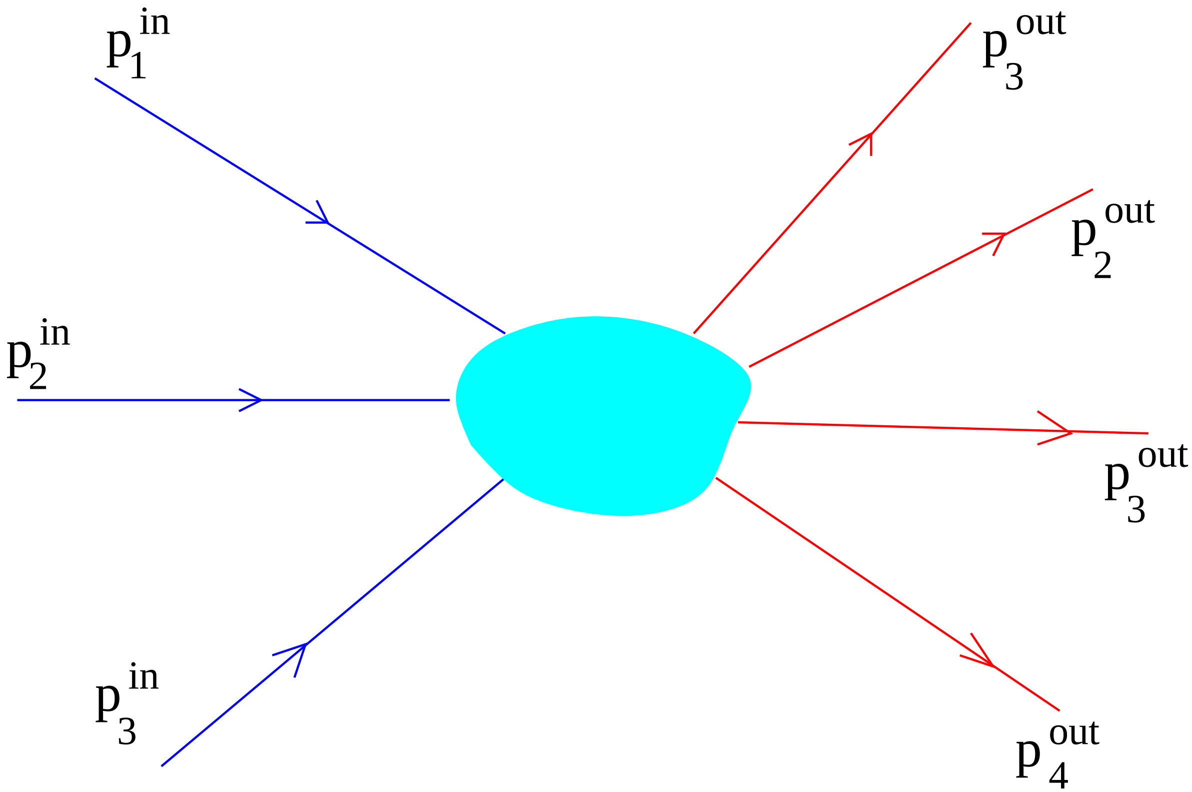
More precisely

$$S = I + \sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\int d^3k_1 \cdots \int d^3k_m \int d^3k'_1 \cdots \int d^3k'_n$$

$$\cdot\,\delta^{(3)}\big(\sum k_j - \sum k'_j\big)\ \ S(k_1,\cdots,k_m|k'_1,\cdots,k'_n)^{\epsilon_1,\cdots,\epsilon_m}_{\epsilon'_1,\cdots,\epsilon'_n}$$

$$\cdot\, a^*_{\mathrm{in},\epsilon_1}(k_1)\cdots a^*_{\mathrm{in},\epsilon_m}(k_n) a^{\epsilon'_1}_{\mathrm{in}}(k'_1)\cdots a^{\epsilon'_m}_{\mathrm{in}}(k'_m)$$

Every matrix element corresponds to scattering process



2. Locality. There are local operators $O(x)$. Locality means

$$[O_1(x), O_2(0)] = 0, \quad x^2 < 0\,.$$

Among these operators there is the energy-momentum tensor $T_{\mu,\nu}(x)$ such that

$$T_{\mu,\nu}(x) = T_{\nu,\mu}(x), \quad \partial_\mu T_{\mu,\nu}(x) = 0, \quad P_\mu = \int_\Sigma T_{\mu,0}(x)\,,$$

and

$$[P_\mu\ ,\ a^*_{\substack{\mathrm{in}\\ \mathrm{out}}}(k)] = p_\mu\ a^*_{\substack{\mathrm{in}\\ \mathrm{out}}}(k), \qquad P_\mu|\mathrm{vac}\rangle = 0\,.$$

Self-consistency:

$$\mathcal{O}(x) = e^{iP_\mu x_\mu} O(0) e^{-iP_\mu x_\mu}\,.$$

Interpolating field $\varphi_\epsilon(x)$:

$$\text{w}-\lim_{x_0\to\mp\infty}\varphi_\epsilon(x) = \varphi_{\substack{\text{in}\\ \text{out}},\epsilon}(x)$$
$$= \int \left(e^{ip_\mu x_\mu} a^*_{\substack{\text{in}\\ \text{out}},\epsilon}(k) + e^{-ip_\mu x_\mu} a^\epsilon_{\substack{\text{in}\\ \text{out}}}(k) \right) \frac{d^3k}{(2\pi)^3\sqrt{k^2+m^2}} .$$

Comments.

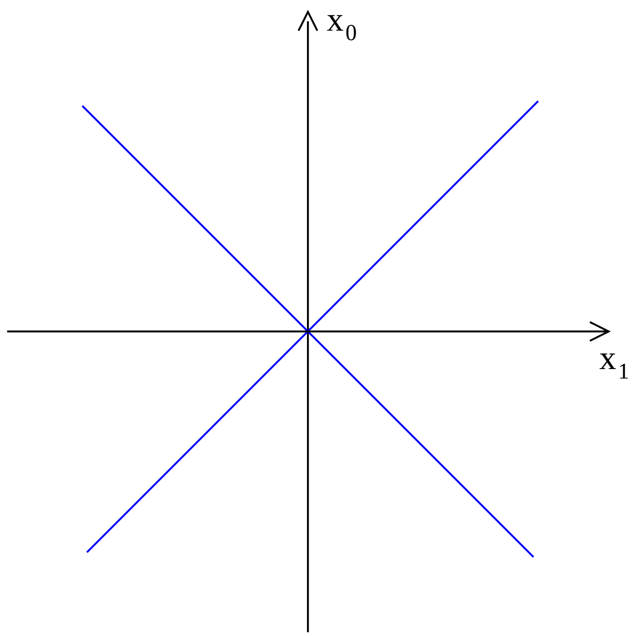
1. The theory is called free if $S = I$. The goal is to find a theory with non-trivial S-matrix.

2. Analyticity. The locality implies rather rich and complicated analytical properties of the S-matrix.

3. Mathematically the difference between the scattering theory in Quantum Mechanics and Quantum Field Theory is due to the difference between strong and weak limits. Hence the problem with the perturbation theory in QFT.

2. Integrable models in two dimensions.

Consider the two-dimensional space-time:



Parametrization of the energy-momentum of particles:

$$p_0^2 - p_1^2 = m^2, \quad p_0 = m \cosh \beta, \quad p_1 = m \sinh \beta \, .$$

Integrability. We had the conservation law $\partial_\mu T_{\mu,\nu}(x) = 0$. The light-cone components of the energy momentum tensor $P_\pm = P_0 \pm P_1$ have the eigenvalues on the asymptotical states:

$$P_\pm a^*_{\mathrm{in},\epsilon_1}(\beta_1)\cdots a^*_{\mathrm{in},\epsilon_n}(\beta_n)|\mathrm{vac}\rangle = m\sum e^{\pm\beta_j}\ a^*_{\mathrm{in},\epsilon_1}(\beta_1)\cdots a^*_{\mathrm{in},\epsilon_n}(\beta_n)|\mathrm{vac}\rangle\,.$$

Integrability implies existence of local operators $T^{(s)}_{\mu,\pm}(x)$ satisfying the conservation law $\partial_\mu T^{(s)}_{\mu,\pm}(x) = 0$. Such that

$$[I^{(s)}_\pm, I^{(s')}_\pm] = 0, \quad I^{(s)}_\pm = \int_\Sigma T^{(s)}_{0,\pm}(x)\,, \quad I^{(1)}_\pm = P_\pm\,.$$

Further

$$I^{(s)}_\pm a^*_{\mathrm{in},\epsilon_1}(\beta_1)\cdots a^*_{\mathrm{in},\epsilon_n}(\beta_n)|\mathrm{vac}\rangle = m^{(s)}\sum e^{\pm s\beta_j}\ a^*_{\mathrm{in},\epsilon_1}(\beta_1)\cdots a^*_{\mathrm{in},\epsilon_n}(\beta_n)|\mathrm{vac}\rangle\,.$$

Implications for scattering. S-matrix commutes with all $I_{\pm}^{(s)}$. It is impossible to satisfy infinite number of equations

$$\sum_{j=1}^{m} e^{\pm s\beta_j} = \sum_{j=1}^{n} e^{\pm s\beta'_j} \ ,$$

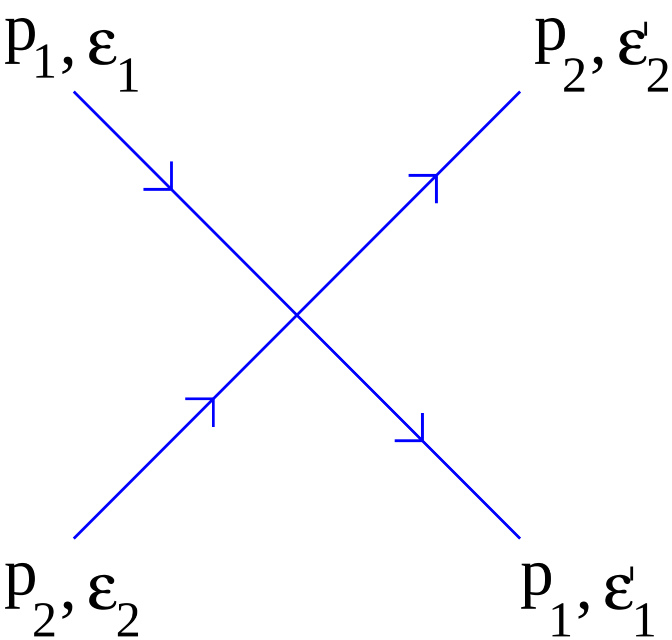
except the trivial solution $m = n$, $\{\beta_j\} = \{\beta'_j\}$.

Hence the first conclusion. The scattering is purely elastic:

$$S = I + \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \int \beta_1 \cdots \int d\beta_m \ S(\beta_1, \cdots, \beta_m)_{\epsilon'_1,\cdots,\epsilon'_n}^{\epsilon_1,\cdots,\epsilon_m}$$

$$\cdot\, a^*_{\mathrm{in},\epsilon_1}(\beta_1) \cdots a^*_{\mathrm{in},\epsilon_m}(\beta_m) a_{\mathrm{in}}^{\epsilon'_1}(\beta_1) \cdots a_{\mathrm{in}}^{\epsilon'_m}(\beta_m)$$

We start with the two-particle S-matrix. Graphically it is represented as

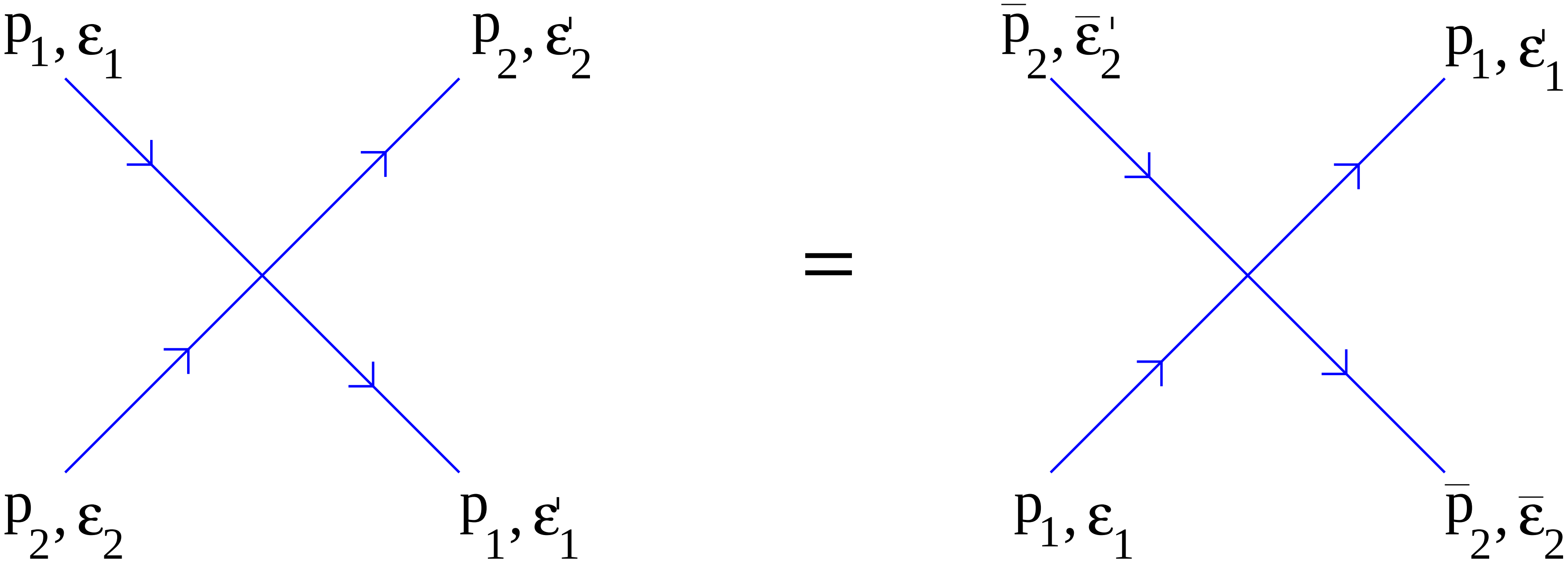


$$= \quad S^{\varepsilon'_1 \varepsilon'_2}_{\varepsilon_1 \varepsilon_2}(\beta_1 - \beta_2)$$

1. Analyticity. $S^{\epsilon'_1,\epsilon'_2}_{\epsilon_1,\epsilon_2}(\beta)$ is meromorphic function of β regular at $0 \leq \mathrm{Im}(\beta) \leq \pi$.
2. Unitarity. Unitarity implies two relations

$$\overline{S^{\epsilon'_1,\epsilon'_2}_{\epsilon_1,\epsilon_2}(\beta)} = S^{\epsilon'_2,\epsilon'_1}_{\epsilon_2,\epsilon_1}(-\beta),\ \beta \in \mathbb{R}; \quad S^{\epsilon'_1,\epsilon'_2}_{\epsilon''_1,\epsilon''_2}(\beta) S^{\epsilon''_2,\epsilon''_1}_{\epsilon'_2,\epsilon'_1}(-\beta) = \delta^{\epsilon_1}_{\epsilon'_1}\delta^{\epsilon_2}_{\epsilon'_2}\ .$$

3.Crossing symmetry.



This is written as

$$S_{\epsilon_2,\epsilon_1}^{\epsilon_2',\epsilon_1'}(\pi i-\beta)=c_{\epsilon_2,\epsilon_2''}\;S_{\epsilon_1,\epsilon_2'''}^{\epsilon_1',\epsilon_2''}(\beta)\;c^{\epsilon_2''',\epsilon_2'}\;.$$

4. Factorizability of scattering. The most important restriction on the two-particle S-matrix comes from consideration of multi-particle scattering.

Multi-particle scattering reduces to sequence of two-particle ones.

I. Ya. Aref'eva, V.E.Korepin. *S-matrix for Sin-Gordon theory.* Pisma JETF, **20** (1974)

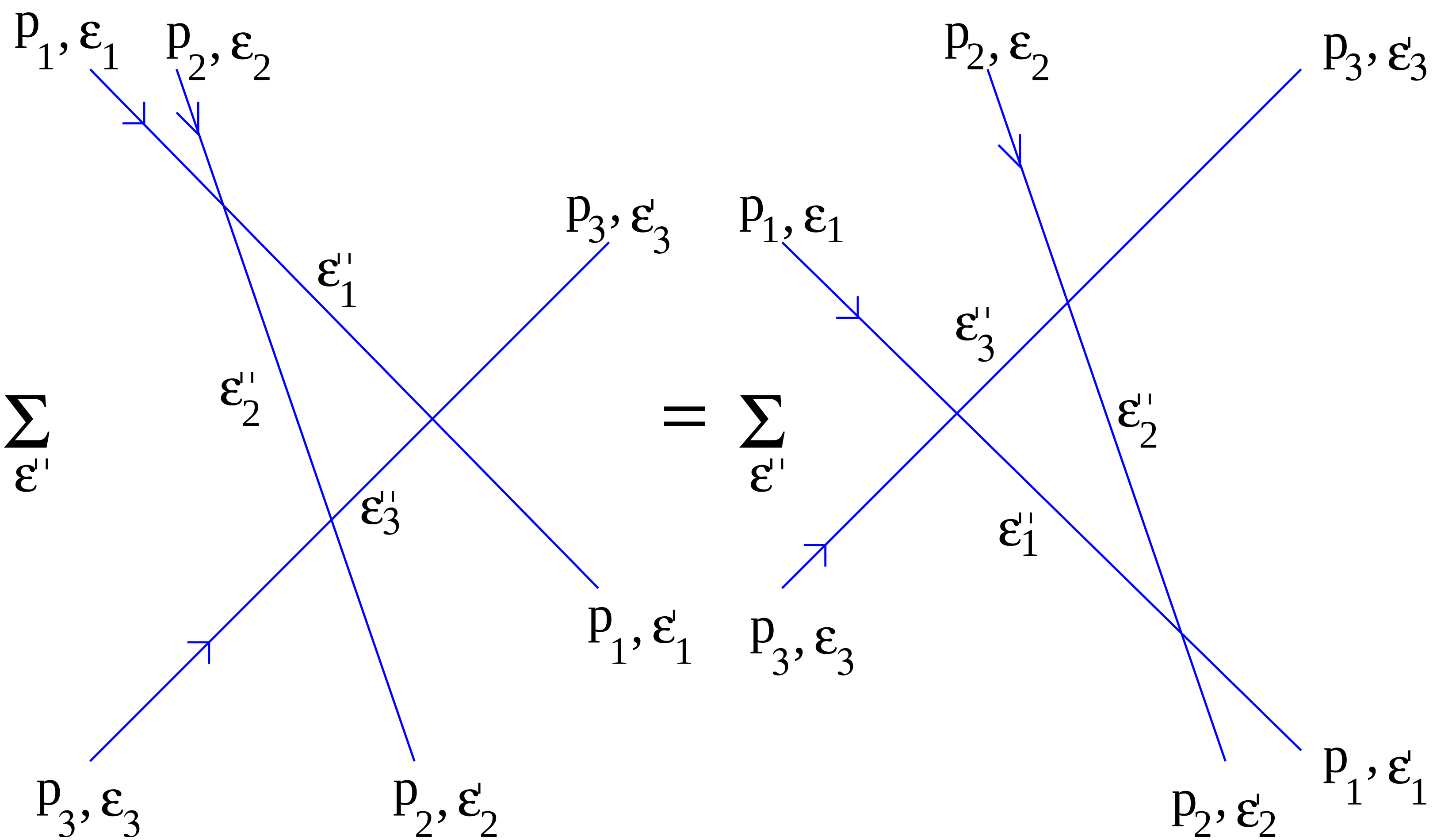
D. Iagolnitzer, *Factorization of the multiparticle S matrix in two-dimensional space-time models.* Phys. Rev. D **18** (1978)

A.B. Zamolodchikov, Al.B Zamolodchikov *factorized scattering in two dimensions of certain relativistic quantum field theory models.* Annals of Physics, **120** (1979)

Consistency relation (Yang-Baxter equation).

$$S^{\epsilon_1,\epsilon_2}_{\epsilon_1'',\epsilon_2''}(\beta_1-\beta_2)S^{\epsilon_1'',\epsilon_3}_{\epsilon_1',\epsilon_3''}(\beta_1-\beta_3)S^{\epsilon_2'',\epsilon_3''}_{\epsilon_2',\epsilon_3'}(\beta_2-\beta_3)$$
$$=S^{\epsilon_2,\epsilon_3}_{\epsilon_2'',\epsilon_3''}(\beta_2-\beta_3)S^{\epsilon_1,\epsilon_3''}_{\epsilon_1'',\epsilon_3'}(\beta_1-\beta_3)S^{\epsilon_1'',\epsilon_2''}_{\epsilon_1',\epsilon_2'}(\beta_1-\beta_2)\,.$$

Graphical illustration to Yang-Baxter equation.



3. Form factor bootstrap.

Consider a local operator $O(x)$. It is completely defined by its matrix elements

$$\langle\text{vac}|a_{\text{in}}^{\epsilon_1}(\beta_1)\cdots a_{\text{in}}^{\epsilon_m}(\beta_m)\ O(x)\ a^*_{\text{in},\epsilon'_1}(\beta'_1)\cdots a^*_{\text{in},\epsilon'_k}(\beta'_k)|\text{vac}\rangle$$
$$= e^{ix_\mu\left(\sum p_\mu(\beta_j)-\sum p_\mu(\beta'_j)\right)}\langle\text{vac}|a_{\text{in}}^{\epsilon_1}(\beta_1)\cdots a_{\text{in}}^{\epsilon_m}(\beta_m)\ O(x)\ a^*_{\text{in},\epsilon'_1}(\beta'_1)\cdots a^*_{\text{in},\epsilon'_k}(\beta'_k)|\text{vac}\rangle\,.$$

It is sufficient to find the form factors

$$f(\beta_1,\cdots,\beta_n)^{\epsilon_1,\cdots,\epsilon_n}\ =\ \langle\text{vac}|\ a_{\text{in}}^{\epsilon_1}(\beta_1)\cdots a_{\text{in}}^{\epsilon_n}(\beta_n)\ O(0)\ |\text{vac}\rangle\,.$$

We set $\beta_1<\cdots<\beta_n$ and then continue analytically. The result is assumed to be meromorphic. General matrix elements are obtained by analytical continuation $\beta_j\to\beta_j+\pi i$ for last k rapidities ($n=m+k$), and by lowering indices by the matrix c.

Form factor axioms.

F.A. Smirnov
Form factors in completely integrable models of quantum field theory.
World Scientific (1992) 208 p.

0. Analyticity. $f(\beta_1,\cdots,\beta_n)^{\epsilon_1,\cdots,\epsilon_n}$ is meromorphic function of all its arguments. As function of β_n is has in the strip $0\le \mathrm{Im}(\beta_n)\le 2\pi$ only simple poles at $\beta_n=\beta_j+\pi i$, $j=1,\cdots,n-1$.

1. Symmetry.

$$S^{\epsilon_j,\epsilon_{j+1}}_{\epsilon'_j,\epsilon'_{j+1}}(\beta_j-\beta_{j+1})f(\beta_1,\cdots,\beta_j,\beta_{j+1},\cdots,\beta_n)^{\epsilon_1,\cdots,\epsilon'_j,\epsilon'_{j+1},\cdots,\epsilon_n}$$
$$=f(\beta_1,\cdots,\beta_{j+1},\beta_j,\cdots,\beta_n)_{\epsilon_1,\cdots,\epsilon_{j+1},\epsilon_j,\cdots,\epsilon_n}\,.$$

2. Riemann-Hilbert problem.

$$f(\beta_1,\cdots,\beta_{n-1},\beta_n+2\pi i)^{\epsilon_1,\cdots,\epsilon_{n-1},\epsilon_n}=f(\beta_n,\beta_1,\cdots,\beta_{n-1})^{\epsilon_n,\epsilon_1,\cdots,\epsilon_{n-1}}\,.$$

3. Annihilation pole.

$$2\pi i \ \mathrm{res}_{\beta_n=\beta_{n-1}+\pi i} f(\beta_1,\cdots,\beta_{n-2},\beta_{n-1},\beta_n)^{\epsilon_1,\cdots,\epsilon_{n-2},\epsilon_{n-1},\epsilon_n}$$
$$= f(\beta_1,\cdots,\beta_{n-2})^{\epsilon_1,\cdots,\epsilon_{n-2}} c^{\epsilon_{n-1},\epsilon_n}$$
$$- S^{\epsilon_{n-1},\epsilon_1}_{\epsilon'_{n-1},\epsilon'_1}(\beta_{n-1}-\beta_1)\cdots S^{\epsilon_{n-1},\epsilon_{n-2}}_{\epsilon'_{n-1},\epsilon'_{n-2}}(\beta_{n-1}-\beta_{n-2}) f(\beta_1,\cdots,\beta_{n-2})^{\epsilon'_1,\cdots,\epsilon'_{n-2}} c^{\epsilon'_{n-1},\epsilon_n}$$

This is the origin of simple poles for general matrix elements. The way of understanding these poles can be explained.

Theorems.

1. Local commutativity theorem. *Suppose the form factors of two operators* $\mathcal{O}_{1,2}(x)$ *obey the axioms. Then the operators are local*

$$[O_1(x)\ ,\ O_2(0)] = 0 \quad \text{for} \quad x_\mu^2 < 0\,.$$

2. Asymptotical theorem. *Suppose the operator $\varphi_\epsilon(x)$ satisfies the axioms and has non-vanishing one-particle form factor $f(\beta)_\epsilon \neq 0$ .Then*

$$\mathrm{w} - \lim_{x_0 \to \mp\infty} \varphi_\epsilon(x) = \varphi_{\substack{\mathrm{in} \\ \mathrm{out}},\epsilon(x)} \,.$$

3. Energy-momentum theorem. *Suppose we have the operators $T_{\mu,\nu}(x)$ such that their form factors $f_{\mu,\nu}$ are of the form*

$$f_{\mu,\nu}(\beta_1, \cdots, \beta_n)^{\epsilon_1, \cdots, \epsilon_n}$$
$$= m^2 \sum (e^{\beta_j} - (-)^\mu e^{-\beta_j}) \sum (e^{\beta_j} - (-)^\nu e^{-\beta_j}) g(\beta_1, \cdots, \beta_n)^{\epsilon_1, \cdots, \epsilon_n} \,,$$

and $g(\beta_1, \cdots, \beta_n)$ satisfy all the axioms except for additional simple pole at two-particle form factor

$$2\pi i \operatorname{res}_{\beta_2 = \beta_1 + \pi i} \, g(\beta_1, \beta_2)^{\epsilon_1, \epsilon_2} = c^{\epsilon_1, \epsilon_2} \,.$$

Then $T_{\mu,\nu}(x)$ can be taken for the energy-momentum tensor.

Correlation functions.

Let $x^2 = -r^2 < 0$, then considering for simplicity Lorenz scalar operators we have

$$G(r) = \langle \mathrm{vac}|O_1(x)O_2(0)|\mathrm{vac}\rangle$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int d\beta_1 \cdots d\beta_n f_1(\beta_n, \cdots, \beta_1)_{\epsilon_1, \cdots, \epsilon_n} f_2(\beta_1, \cdots, \beta_n)^{\epsilon_n, \cdots, \epsilon_1} e^{-mr \sum \cosh \beta_j} .$$

We begin with $e^{im(x_0 \sum \cosh \beta_j - x_1 \sum \sinh \beta_j)}$, then we make the Lorenz transformation $\beta_j \to \beta_j - \mathrm{arcsinh}(x_1/r)$ arriving at $e^{-imr \sum \sinh \beta_j}$, then we shift $\beta_j \to \beta_j - \pi i/2$. The latter is possible since

$$e^{-imr \sinh(\beta - i\theta)} = e^{-imr \sinh \beta \cos \theta - mr \cosh \beta \sin \theta} .$$

Two-point space-like correlation function is the same as two-point Euclidean correlation function.

The difficulty with describing the short-distance behaviour is obvious.

Sine-Gordon model.

$$\mathcal{A}^{\mathrm{sG}} = \int \Big[\frac{1}{16\pi}(\partial_\mu \boldsymbol{\varphi}(x))^2 + \frac{\boldsymbol{\mu}^2}{\sin \pi\beta^2} 2\cos(\beta\boldsymbol{\varphi}(x))\Big] d^2x \,.$$

I shall use the parameter

$$\nu = 1 - \beta^2 \,, \qquad 1 > \nu > 0 \,.$$

Semi-classical domain $\nu \to 1$.
The spectrum of the model consists of soliton-antisoliton with mass M and for $1/2 < \nu < 1$ of $\left[\frac{\nu}{1-\nu}\right] - 1$ bound states (breathers) with masses $2M \sin\left(\pi \frac{1-\nu}{2\nu} j\right)$, $j = 1, \cdots \left[\frac{\nu}{1-\nu}\right] - 1$.

Free fermion point $\nu = \frac{1}{2}$.

Two soliton S-matrix.

I use the notations

$$\mathfrak{b}_j = e^{\frac{2\nu}{1-\nu}\beta_j}, \quad \mathfrak{q} = e^{\pi i \frac{1}{1-\nu}} .$$

$$S_{i,j}(\beta_i - \beta_j) = S_0(\beta_i - \beta_j)\widetilde{S}_{i,j}(\mathfrak{b}_i/\mathfrak{b}_j) ,$$

$$S_0(\beta) = \exp\left(-i\int\limits_0^\infty \frac{\sin(2k\nu\beta)\sinh((2\nu-1)\pi k)}{k\cosh(\pi\nu k)\sinh(\pi(1-\nu)k)}dk\right) ,$$

and

$$\widetilde{S}_{i,j}(\mathfrak{b}_i/\mathfrak{b}_j) = \tfrac{1}{2}(I_i\otimes I_j + \sigma_i^3\otimes\sigma_j^3) + \frac{\mathfrak{b}_i - \mathfrak{b}_j}{\mathfrak{b}_i\mathfrak{q}^{-1} - \mathfrak{b}_j\mathfrak{q}}\cdot\tfrac{1}{2}(I_i\otimes I_j - \sigma_i^3\otimes\sigma_j^3)$$
$$+ \sqrt{\mathfrak{b}_i\mathfrak{b}_j}\frac{\mathfrak{q}^{-1} - \mathfrak{q}}{\mathfrak{b}_i\mathfrak{q}^{-1} - \mathfrak{b}_j\mathfrak{q}}\cdot(\sigma_i^+\otimes\sigma_j^- + \sigma_i^-\otimes\sigma_j^+) .$$

Local fields.

We shall consider the "primary fields" $\Phi_\alpha(x) = e^{i\alpha \frac{\nu}{2\sqrt{1-\nu}} \varphi(x)}$, and their relatives ("descendants"). The form factor axioms are in this case:

Symmetry axiom.

$$S_{j,j+1}(\beta_j - \beta_{j+1}) f_{\mathcal{O}_\alpha}(\beta_1, \cdots, \beta_j, \beta_{j+1}, \cdots, \beta_{2n}) = f_{\mathcal{O}_\alpha}(\beta_1, \cdots, \beta_{j+1}, \beta_j, \cdots, \beta_{2n}),$$

Riemann-Hilbert problem axiom.

$$f_{\mathcal{O}_\alpha}(\beta_1, \cdots, \beta_{2n-1}, \beta_{2n} + 2\pi i) = e^{-\frac{\pi i \nu}{1-\nu}\alpha\sigma^3_{2n}} f_{\mathcal{O}_\alpha}(\beta_{2n}, \beta_1, \cdots, \cdots, \beta_{2n-1}).$$

Residue axiom.

$$2\pi i \ \mathrm{res}_{\beta_{2n}=\beta_{2n-1}+\pi i} f_{\mathcal{O}_\alpha}(\beta_1, \cdots, \beta_{2n-2}, \beta_{2n-1}, \beta_{2n}) =$$
$$\Big(1 - e^{-\frac{\pi i \nu}{1-\nu}\alpha\sigma^3_{2n}} S_{2n-1,1}(\beta_{2n-1} - \beta_1) \cdots S_{2n-1,2n-2}(\beta_{2n-1} - \beta_{2n-2})\Big)$$
$$\times f_{\mathcal{O}_\alpha}(\beta_1, \cdots, \beta_{2n-2}) \otimes s_{2n-1,2n},$$

where $s_{i,j} = e_i^+ \otimes e_j^- + e_i^- \otimes e_j^+$.

2. Example: free fermions and Painlevé.

For $\nu = 1/2$ the S-matrix trivialises:

$$S_{1,2}(\theta) = -I\,,$$

there are now breathers, and the form factors for solitons are simple:

$$f_\alpha(\theta_1,\cdots,\theta_n,\theta_{n+1},\cdots\theta_{2n})_{\frac{1}{2},\cdots\frac{1}{2},-\frac{1}{2},\cdots-\frac{1}{2}}$$

$$= \left(\frac{2\sin\frac{\pi\alpha}{2}}{\pi}\right)^n e^{\frac{1}{2}\alpha\sum\limits_{j=1}^{n}(\theta_j-\theta_{n+j})} \frac{\prod\limits_{i<j}^{2n}\sinh\frac{1}{2}(\theta_i-\theta_j)}{\prod\limits_{i=1}^{n}\prod\limits_{j=n+1}^{2n}\sinh(\theta_i-\theta_i)}\,.$$

The form factor series turn into the Fredholm determinant which is the tau-functioon for Painleve III. To cut the long story short

$$\frac{\langle\Phi_{\alpha_1}(\mathbf{x})\Phi_{\alpha_2}(0)\rangle^{\mathrm{sG}}}{\langle\Phi_{\alpha}(0)\rangle^{\mathrm{sG}}}=\frac{\langle\Phi_{\alpha_1}(0)\rangle^{\mathrm{sG}}\langle\Phi_{\alpha_2}(0)\rangle^{\mathrm{sG}}}{\langle\Phi_{\alpha}(0)\rangle^{\mathrm{sG}}}\cdot\tau((\tfrac{1}{2}Mr)^2),\quad \mathrm{recall}\quad -\mathbf{x}^2=r^2\,.$$

We denote $\alpha=\alpha_1+\alpha_2\ ,\ \theta=\alpha_1-\alpha_2$. One-point function is

$$\langle\Phi_{\alpha}(0)\rangle^{\mathrm{sG}}=\Bigl(\frac{M}{2}\Bigr)^{\frac{\alpha^2}{4}}\exp\Bigl(\int\limits_0^{\infty}\Bigl(\frac{\sinh^2(\frac{\alpha t}{2})}{\sinh^2 t}-\frac{\alpha^2}{4}e^{-2t}\Bigr)\frac{dt}{t}\Bigr)\,.$$

Set

$$\zeta(t)=t\frac{d}{dt}\log\tau(t)\,.$$

we have

$$\left(t\frac{d^2\zeta}{dt^2}\right)^2=4\frac{d\zeta}{dt}\left(\frac{d\zeta}{dt}-1\right)\left(\zeta-t\frac{d\zeta}{dt}\right)+\frac{\theta^2}{4}\left(\frac{d\zeta}{dt}\right)^2\,.$$

Asymptotics for $0 < \alpha < 2$:

$$\frac{\langle \Phi_{\alpha_1}(\mathbf{x})\Phi_{\alpha_2}(0)\rangle^{\mathrm{sG}}}{\langle \Phi_{\alpha}(0)\rangle^{\mathrm{sG}}}$$

$$\simeq r^{\frac{\alpha_1\alpha_2}{2}}\left(1 + \frac{\alpha_1\alpha_2 M^2}{4\alpha^2}\left[r^2 - \frac{4M^\alpha}{(2+\alpha)^2}sr^{2+\alpha} - \frac{4M^{-\alpha}}{(2-\alpha)^2}s^{-1}r^{2-\alpha}\right] + \cdots\right),$$

where

$$s = 2^\alpha \frac{\Gamma(1-\frac{\alpha}{2})^2}{\Gamma(1+\frac{\alpha}{2})^2}\,\frac{\Gamma\left(1+\frac{\alpha_1}{2}\right)}{\Gamma\left(1-\frac{\alpha_1}{2}\right)}\,\frac{\Gamma\left(1+\frac{\alpha_2}{2}\right)}{\Gamma\left(1-\frac{\alpha_2}{2}\right)}\,.$$

Euclidean case.

In Euclidean case we take seriously the functional integral. For example the two-pont function is

$$\langle \mathcal{O}_1(z,\bar{z})\mathcal{O}_2(0)\rangle = \frac{\int e^{-\frac{1}{\hbar}\mathcal{A}[\varphi]}\mathcal{O}_1(z,\bar{z})\mathcal{O}_2(0)\prod_{w,\bar{w}}\mathcal{D}\phi(w,\bar{w})}{\int e^{-\frac{1}{\hbar}\mathcal{A}[\varphi]}\prod_{w,\bar{w}}\mathcal{D}\phi(w,\bar{w})}\,.$$

I shall talk more on this later.

Conformal field theory.

Energy-momentum tensor describe variations with respect to external metric

$$\frac{\delta}{\delta g^{a,b}(z,\bar{z})} Z \Big|_{g^{a,b}\ \mathrm{Euclidean}} = \int T_{a,b}(z,\bar{z}) e^{-\frac{1}{\hbar}\mathcal{A}[\varphi]} \prod_{w,\bar{w}} \mathcal{D}\phi(w,\bar{w})\,.$$

There are three components of T for which we use the complex notations:

$$T(z,\bar{z}) = T_{z,z}(z,\bar{z}),\ \ \bar{T}(z,\bar{z}) = T_{\bar{z},\bar{z}}(z,\bar{z}),\ \ \Theta(z,\bar{z}) = T_{z,\bar{z}}(z,\bar{z})\,.$$

They satisfy the conservation

$$\partial_{\bar{z}} T(z,\bar{z}) = \Theta(z,\bar{z}),\quad \partial_z \bar{T}(z,\bar{z}) = \Theta(z,\bar{z})\,.$$

By definition for CFT $\Theta(z,\bar{z}) = 0$. Hence $T(z,\bar{z}) = T(z)$, $\bar{T}(z,\bar{z}) = \bar{T}(\bar{z})$.

Operator product expansion. Every observable $\mathcal{O}$ is characterized by scaling dimensions $\Delta, \bar{\Delta}$, in particular

$$\mathcal{O}(az, a\bar{z}) = a^{-\Delta-\bar{\Delta}}\mathcal{O}(z,\bar{z})\,.$$

Suppose we have complete set of local observables, then we must have OPE

$$\mathcal{O}_i(z,\bar{z})\mathcal{O}_j(0) = \sum_k z^{-\Delta_i-\Delta_j+\Delta_k}\,\bar{z}^{-\bar{\Delta}_i-\bar{\Delta}_j+\bar{\Delta}_k} C_{i,j}^k \mathcal{O}_k(0)\,.$$

The set of constants $C_{i,j}^k$ completely characterizes the CFT.
We have action of Virasoro algebra

$$\mathbf{l}_k O(0) = \int_\gamma z^{k+1} T(y) O(0) \frac{dz}{2\pi i}\,,$$

primary fields and their descendants.

Perturbed CFT.

Al. B. Zamolodchikov, Two-point correlation function in scaling Lee-Yang model (1991)

$$\mathcal{A}^{\mathrm{PCFT}} = \mathcal{A}^{\mathrm{CFT}} + g \int \phi(z,\bar{z}) d^2 z, \quad d^2 z = \frac{i dz \wedge d\bar{z}}{2},$$

where $\phi(z,\bar{z})$ has scaling dimension (Δ,Δ), being relevant $\Delta < 1$.

$$g = [\mathrm{Length}]^{2\Delta - 2}.$$

Naïve attempt of computing the short distance asymptotics of the two-point function for PCFT:

$$\int \mathcal{O}_1(z,\bar{z})\mathcal{O}_2(0) e^{-\frac{1}{\hbar}\mathcal{A}^{\mathrm{PCFT}}} \prod_{w,\bar{w}} \mathcal{D}\phi(w,\bar{w})$$

$$= \sum_{n=0}^{\infty} \frac{g^n}{n!} \int d^2 z_1 \cdots \int d^2 z_n \langle \mathcal{O}_1(z,\bar{z})\mathcal{O}_2(0)\phi(z_1,\bar{z}_1)\cdots\phi(z_n,\bar{z}_n)\rangle_{\mathrm{CFT}}.$$

This is wrong because the integrals are IR divergent (they are UV convergent for $\Delta < 1/2$). But this is rather good than bad because this series contradict even to

$$\frac{\langle\Phi_{\alpha_1}(\mathbf{x})\Phi_{\alpha_2}(0)\rangle^{\mathrm{sG}}}{\langle\Phi_{\alpha}(0)\rangle^{\mathrm{sG}}}$$
$$\simeq r^{\frac{\alpha_1\alpha_2}{2}}\Big(1+\frac{\alpha_1\alpha_2 M^2}{4\alpha^2}\Big[r^2-\frac{4M^{\alpha}}{(2+\alpha)^2}sr^{2+\alpha}-\frac{4M^{-\alpha}}{(2-\alpha)^2}s^{-1}r^{2-\alpha}\Big]+\cdots\Big)\,,$$

in this case $g = M/2$.

The main conclusion is that the perturbation theory must be used rather for OPE than for the correlation functions. For irrational dimensions local operators can be identified, and we have

$$\mathcal{O}_i(z,\bar z)\mathcal{O}_j(0)=\sum_k z^{-\Delta_i-\Delta_j+\Delta_k}\,\bar z^{-\bar\Delta_i-\bar\Delta_j+\bar\Delta_k}C_{i,j}^k(g(z\bar z)^{1-\Delta})\mathcal{O}_k(0)\,,$$

where $C_{i,j}^k(x)=C_{i,j}^k+C_{i,j}^{k,(1)}x+C_{i,j}^{k,(2)}x^2+\cdots$.

The structural functions can be computed from PCFT, but the one-point functions

$$\langle \mathcal{O}_k(0) \rangle^{\rm PCFT} = g^{-\frac{\Delta_k + \bar{\Delta}_k}{2-2\Delta}} G_k$$

can not being non-analytical in g.

Generalization. We can impose some geometric environment, them OPE remain the same, and only G_k depend on the geometry. For example, for the cylinder of radius R we have

$$\langle \mathcal{O}_k(0) \rangle^{\rm PCFT}_R = g^{-\frac{\Delta_k + \bar{\Delta}_k}{2-2\Delta}} G_k(gR^{2-2\Delta})\,.$$

Returning to sine-Gordon model.

Writing the action as

$$\mathcal{A}^{\mathrm{sG}} = \int \Big\{ \Big[\frac{1}{4\pi} \partial_z \boldsymbol{\varphi}(z,\bar{z}) \partial_{\bar{z}} \boldsymbol{\varphi}(z,\bar{z}) - \frac{\boldsymbol{\mu}^2}{\sin \pi\beta^2} e^{-i\beta\boldsymbol{\varphi}(z,\bar{z})} \Big] - \frac{\boldsymbol{\mu}^2}{\sin \pi\beta^2} e^{i\beta\boldsymbol{\varphi}(z,\bar{z})} \Big\} \frac{idz \wedge d\bar{z}}{2} \,.$$

we consider sG model as perturbation of the complex Liouville model with the central charge

$$c = 1 - 6\frac{\nu^2}{1-\nu} \,,$$

by the primary field

$$e^{i\beta\boldsymbol{\varphi}(z,\bar{z})} \,.$$

OPE:

$$\Phi_{\alpha_1}(z,\bar z)\Phi_{\alpha_2}(0) = \sum_{m=-\infty}^{\infty}\sum_{N,\bar N}\left(\boldsymbol{\mu}^2 r^{2\nu}\right)^{|m|} C^{m,N,\bar N}_{\alpha_1,\alpha_2}\left(\boldsymbol{\mu}^4 r^{4\nu}\right)$$
$$\times\, r^{\frac{\nu^2}{1-\nu}\alpha_1\alpha_2+2m^2(1-\nu)+2\alpha m\nu}\; z^{|N|}\bar z^{|\bar N|}\,\mathbf{l}_{-N}\bar{\mathbf{l}}_{-\bar N}\Phi_{\alpha+2m\frac{1-\nu}{\nu}}(0)\,,$$

where $\alpha = \alpha_1 + \alpha_2$.
Coefficients of functions

$$C^{m,N,\bar N}_{\alpha_1,\alpha_2}(x) = C^{m,N,\bar N,(0)}_{\alpha_1,\alpha_2} + C^{m,N,\bar N(1)}_{\alpha_1,\alpha_2}x + C^{m,N,\bar N(2)}_{\alpha_1,\alpha_2}x^2 + \cdots$$

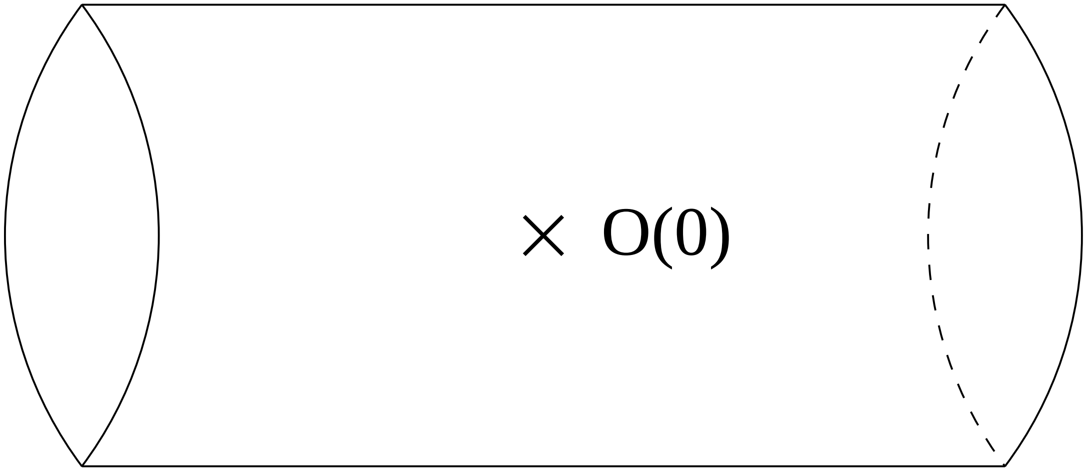
are given by Coulomb gas type integrals.

We generalise the problem considering the correlation functions on a cylinder

So, our goal is to compute

$$\frac{\langle \mathbf{l}_{-N}\bar{\mathbf{l}}_{-\bar{N}}\Phi_{\alpha+2m\frac{1-\nu}{\nu}}(0)\rangle_R^{\mathrm{sG}}}{\langle \Phi_\alpha(0)\rangle_R^{\mathrm{sG}}}\,.$$

This is the ratio of two functional integrals on



Fermionic description of the space of local fields.

Local integrals of motion I_{2j-1}, $\bar{I}_{2j-1}$ act on the local operators by commutators. Corresponding action is denoted by $\mathbf{i}_{2j-1}$, $\bar{\mathbf{i}}_{2j-1}$ and should be factored out.
We claim that the quotient space of

$$\bigotimes_{m=-\infty}^{\infty} \mathcal{V}_{\alpha+2m\frac{1-\nu}{\nu}} \otimes \overline{\mathcal{V}}_{\alpha+2m\frac{1-\nu}{\nu}} \,,$$

by action of $\mathbf{i}_{2j-1}$, $\bar{\mathbf{i}}_{2j-1}$ can be described by action on $\Phi_\alpha(0)$ of four sets of fermionic creation operators:

$$\boldsymbol{\beta}^*_{2j-1},\ \boldsymbol{\gamma}^*_{2j-1},\ \bar{\boldsymbol{\beta}}^*_{2j-1},\ \bar{\boldsymbol{\gamma}}^*_{2j-1}\,.$$

What do we know about descendants of these fermions to Virasoro descendants?

Introduce the notations

$$I^+ = \{2i_1^+ - 1, \cdots, 2i_p^+ - 1\}, \quad \boldsymbol{\beta}^*_{I^+} = \boldsymbol{\beta}^*_{2i_1^+-1} \cdots \boldsymbol{\beta}^*_{2i_p^+-1}, \quad etc\,.$$

Then if $\#(I^+) = \#(I^-)$,

$$\boldsymbol{\beta}^*_{I^+} \boldsymbol{\gamma}^*_{I^-} \Phi_\alpha(0) = \prod_{2j-1 \in I^+} D_{2j-1}(\alpha) \prod_{2j-1 \in I^-} D_{2j-1}(2-\alpha)$$
$$\times \left[P^{\text{even}}_{I^+,I^-}(\{\mathbf{l}_{-2k}\}|\Delta_\alpha, c) + d_\alpha P^{\text{odd}}_{I^+,I^-}(\{\mathbf{l}_{-2k}\}|\Delta_\alpha, c) \right] \Phi_\alpha(0)\,,$$

where

$$D_{2j-1}(\alpha) = -\sqrt{\frac{i}{\nu}}\,\Gamma(\nu)^{-\frac{2j-1}{\nu}} (1-\nu)^{\frac{2j-1}{2}} \frac{\Gamma\left(\frac{\alpha}{2} + \frac{1}{2\nu}(2j-1)\right)}{(j-1)!\Gamma\left(\frac{\alpha}{2} + \frac{1-\nu}{2\nu}(2j-1)\right)}\,,$$
$$d_\alpha = \tfrac{1}{6}\sqrt{(25-c)(24\Delta_\alpha + 1 - c)}\,.$$

Similarly for other chirality changing $\alpha \to 2 - \alpha$.

For $\boldsymbol{\beta}^*_{I^+}\bar{\boldsymbol{\beta}}^*_{\bar{I}^+}\bar{\boldsymbol{\gamma}}^*_{\bar{I}^-}\boldsymbol{\gamma}^*_{I^-}\Phi_\alpha(0)$ the only real requirement is $\#(I^+)+\#(\bar{I}^+)=\#(I^-)+\#(\bar{I}^-)$. We have

$$\begin{aligned}&\boldsymbol{\beta}^*_{I^+}\bar{\boldsymbol{\beta}}^*_{\bar{I}^+}\bar{\boldsymbol{\gamma}}^*_{\bar{I}^-}\boldsymbol{\gamma}^*_{I^-}\Phi_{\alpha+2m\frac{1-\nu}{\nu}}(0)\\ &\cong C_m(\alpha)\boldsymbol{\beta}^*_{I^++2m}\bar{\boldsymbol{\beta}}^*_{\bar{I}^+-2m}\bar{\boldsymbol{\gamma}}^*_{\bar{I}^-+2m}\boldsymbol{\gamma}^*_{I^--2m}\boldsymbol{\beta}^*_{I_{\mathrm{odd}}(m)}\bar{\boldsymbol{\gamma}}^*_{I_{\mathrm{odd}}(m)}\Phi_\alpha(0)\,,\end{aligned}$$

where $I_{\mathrm{odd}}(m)=\{1,3,\cdots 2m-1\}$,

$$\boldsymbol{\gamma}^*_{-a}=\tfrac{i}{\nu}\cot\tfrac{\pi}{2\nu}(\nu\alpha+a)\boldsymbol{\beta}_a,\quad \bar{\boldsymbol{\beta}}^*_{-a}=\tfrac{i}{\nu}\cot\tfrac{\pi}{2\nu}(\nu\alpha+a)\bar{\boldsymbol{\gamma}}_a\,,$$

$$C_m(\alpha)=\boldsymbol{\mu}^{2m\alpha-2m^2}\prod_{j=0}^{m-1}U(\alpha+2j\tfrac{1-\nu}{\nu})\,.$$

$$U(\alpha)=-\nu\Gamma(\nu)^{4x}\,\frac{\Gamma(-2\nu x)}{\Gamma(2\nu x)}\cdot\frac{\Gamma(x)}{\Gamma(x+1/2)}\cdot\frac{\Gamma(-x+1/2)}{\Gamma(-x)}i\cot\pi x\,,\ x=\tfrac{\alpha}{2}+\tfrac{1-\nu}{2\nu}\,.$$

Our main theorem.

Consider

$$\boldsymbol{\beta}^*(\zeta) = \sum_{j=1}^{\infty} \boldsymbol{\beta}^*_{2j-1}(\zeta \cdot \boldsymbol{\mu})^{-\frac{2j-1}{\nu}} + \sum_{j=1}^{\infty} \bar{\boldsymbol{\beta}}^*_{2j-1}(\zeta / \boldsymbol{\mu})^{\frac{2j-1}{\nu}} \quad etc\,.$$

Then

$$\frac{\langle \boldsymbol{\beta}^*(\zeta_1)\cdots\boldsymbol{\beta}^*(\zeta_m)\boldsymbol{\gamma}^*(\xi_m)\cdots\boldsymbol{\gamma}^*(\xi_1)\Phi_\alpha(0)\rangle_R^{\mathrm{sG}}}{\langle \Phi_\alpha(0)\rangle_R^{\mathrm{sG}}} = \det \omega_R^{\mathrm{sG}}(\zeta_i, \xi_j|\alpha)\,,$$

and the function $\omega_R^{\mathrm{sG}}(\zeta, \xi|\alpha)$ can be described through the TBA data.

Desrti-DeVega equations.

Free energy on the cylinder is defined by the maximal eigenvalue of Matrubara transfer-matrix. DDV equation

$$\frac{1}{i}\log \mathfrak{a}(\zeta) = \pi M R(\zeta^{1/\nu} - \zeta^{-1/\nu}) - 2\mathrm{Im}\int\limits_0^\infty R(\zeta/\xi)\log(1+\mathfrak{a}(\xi e^{+i0}))\frac{d\xi^2}{\xi^2}\,,$$

where $R(\zeta) = R(\zeta, 0)$,

$$R(\zeta,\alpha) = \int\limits_{-\infty}^{\infty} \zeta^{2ik}\widehat{R}(k,\alpha)\frac{dk}{2\pi}, \quad \widehat{R}(k,\alpha) = \frac{\sinh\pi((2\nu-1)k - i\alpha/2)}{2\sinh\pi((1-\nu)k + i\alpha/2)\cosh(\pi\nu k)}\,,$$

Mass of soliton is related to $\boldsymbol{\mu}$ by

$$(\boldsymbol{\mu}\Gamma(\nu))^{\frac{1}{\nu}} = M\frac{\sqrt{\pi}\;\Gamma(\frac{1}{2\nu})}{2\Gamma(\frac{1-\nu}{2\nu})}\,.$$

Function $\omega_R^{\mathrm{sG}}(\zeta,\xi)$

It is convenient to use the Mellin transform

$$\omega_R^{\mathrm{sG}}(\zeta,\xi|\alpha) = -\frac{\pi i}{2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{dl}{2\pi}\frac{dm}{2\pi}\zeta^{2il}\xi^{2im}\frac{e^{-\pi\nu l}}{\cosh(\pi\nu l)}\Theta_R^{\mathrm{sG}}(l,m|\alpha)\frac{e^{-\pi\nu m}}{\cosh(\pi\nu m)}$$

$$+\,\omega_0(\zeta/\xi,\alpha)\,.$$

where

$$\omega_0(\zeta,\alpha)$$

$$=\frac{i}{\nu}\Bigl(\sum_{j=1}^{\infty}\zeta^{-\frac{2j-1}{\nu}}\cot\frac{\pi}{2\nu}(\nu\alpha+(2j-1))-\sum_{j=1}^{\infty}\zeta^{\frac{2j-1}{\nu}}\cot\frac{\pi}{2\nu}(\nu\alpha-(2j-1))\Bigr),$$

The function $\Theta_R^{\rm sG}(l,m|\alpha)$ is defined from the equation

$$\Theta_R^{\rm sG}(l,m|\alpha)+G(l+m)+\int\limits_{-\infty}^{\infty} G(l-k)\widehat{R}(k,\alpha)\Theta_R^{\rm sG}(k,m|\alpha)\frac{dk}{2\pi}=0\,,$$

where

$$G(k)=\int_0^\infty \zeta^{-2ik}dm(\zeta)\,,\quad dm(\zeta)=2{\rm Re}\left(\frac{1}{1+\mathfrak{a}(\zeta e^{-i0})}\right)\frac{d\zeta^2}{\zeta^2}\,,$$

QFT and second order phase transitions.

We considered

$$\langle \mathcal{O}_1(z,\bar{z})\mathcal{O}_2(0)\rangle = \frac{\int e^{-\frac{1}{\hbar}\mathcal{A}[\varphi]}\mathcal{O}_1(z,\bar{z})\mathcal{O}_2(0)\prod_{w,\bar{w}}\mathcal{D}\phi(w,\bar{w})}{\int e^{-\frac{1}{\hbar}\mathcal{A}[\varphi]}\prod_{w,\bar{w}}\mathcal{D}\phi(w,\bar{w})}\,.$$

There is an important analogy between D-dimensional Euclidean QFT and D-dimensional classical statistical mechanics. Partition function:

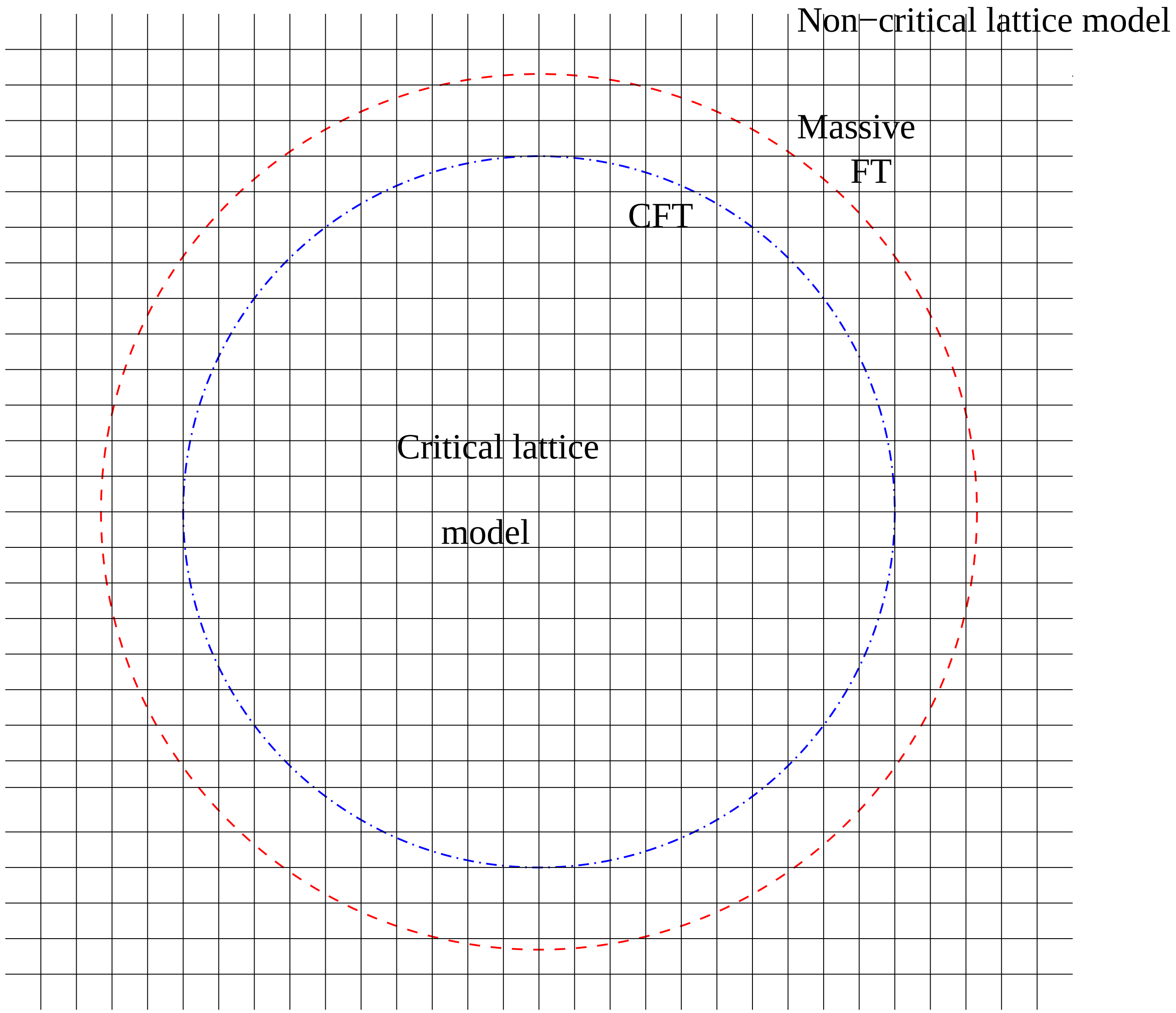
$$Z = \sum_{\text{configurations}} \left(e^{-\frac{1}{T}\mathcal{H}}\right).$$

The rule is $\mathcal{A} \leftrightarrow \mathcal{H}$. If we try the lattice regularization. Typically

$$\langle \sigma(r)\sigma(0)\rangle = \frac{1}{Z}\sum_{\text{configurations}} \left(e^{-\frac{1}{T}\mathcal{H}}\sigma(r)\sigma(0)\right) \simeq_{r\to\infty} e^{-\frac{r}{\xi(T)}}\,.$$

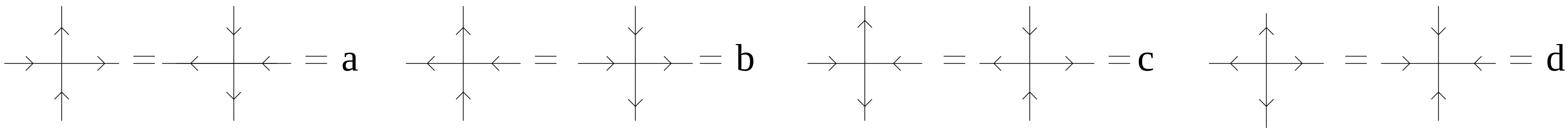
Scaling limit at T_c where $\xi(T) \simeq (T-T_c)^{-\nu}$ (another ν).

Lattice model near the point of the second order phase transition.



Example.

Eight-vertex model



We consider homogeneous case when the Boltzmann weights are parametrized by two parameters: ν and k (which parametrizes the temperature).

$$a : b : c : d = \mathrm{sn}(\nu/2) : \mathrm{sn}(\nu/2) : \mathrm{sn}(\nu) : k(\mathrm{sn}(\nu/2))^2\mathrm{sn}(\nu)\,.$$

Critical temperature $k = 0$. In this case $d = 0$,

$$a : b : c = \sin(\nu/2) : \sin(\nu/2) : \sin(\nu)\,.$$

This is homogeneous six-vertex model. Scaling theory is Euclidean sG:

$$\mathcal{A}^{\mathrm{sG}} = \int \left\{ \frac{1}{4\pi}\partial_z \boldsymbol{\varphi}(z,\bar{z})\partial_{\bar{z}}\boldsymbol{\varphi}(z,\bar{z}) - \frac{\boldsymbol{\mu}^2}{\sin \pi\beta^2} 2\cos(\beta\boldsymbol{\varphi}(z,\bar{z})) \right\} \frac{idz \wedge d\bar{z}}{2}\,.$$

Expectation values for six vertex model.

Consider the partition function with defect:

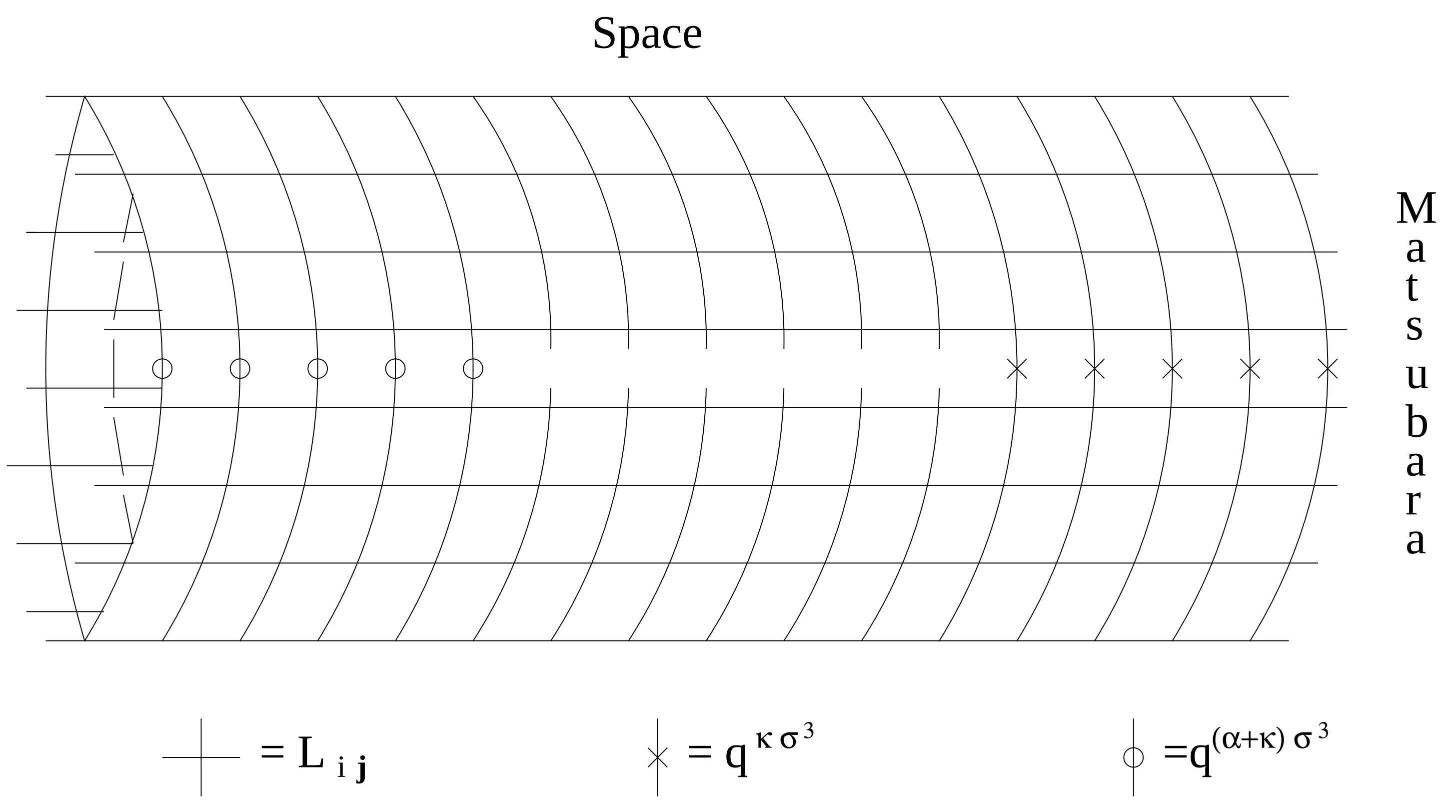


fig. 1

Exact definition. R-matrix

$$R_{1,2}(\zeta)=q^{\frac{1}{2}(\sigma_1^3\sigma_2^3+1)}\zeta-q^{-\frac{1}{2}(\sigma_1^3\sigma_2^3+1)}\zeta^{-1}+(q-q^{-1})(\sigma_1^+\sigma_2^-+\sigma_1^-\sigma_2^+)\,.$$

This R-matrix satisfies: Yang-Baxter equations:

$$R_{1,2}(\zeta_1/\zeta_2)R_{1,3}(\zeta_1/\zeta_3)R_{2,3}(\zeta_2/\zeta_3)=R_{2,3}(\zeta_2/\zeta_3)R_{1,3}(\zeta_1/\zeta_3)R_{1,2}(\zeta_1/\zeta_2)\,.$$

One more important property

$$R_{1,2}(1)=(q-q^{-1})P_{1,2}\,.$$

we shall formally use the space $\mathfrak{H}_{\mathrm{S}}=\bigotimes_{j=-\infty}^{\infty}\mathbb{C}^2$. Let us consider also the space $\mathfrak{H}_{\mathbf{M}}=\bigotimes_{\mathbf{j=1}}^{\mathbf{n}}\mathbb{C}^2$, where $\mathbf{M}$ stands for Matsubara.

Introduce

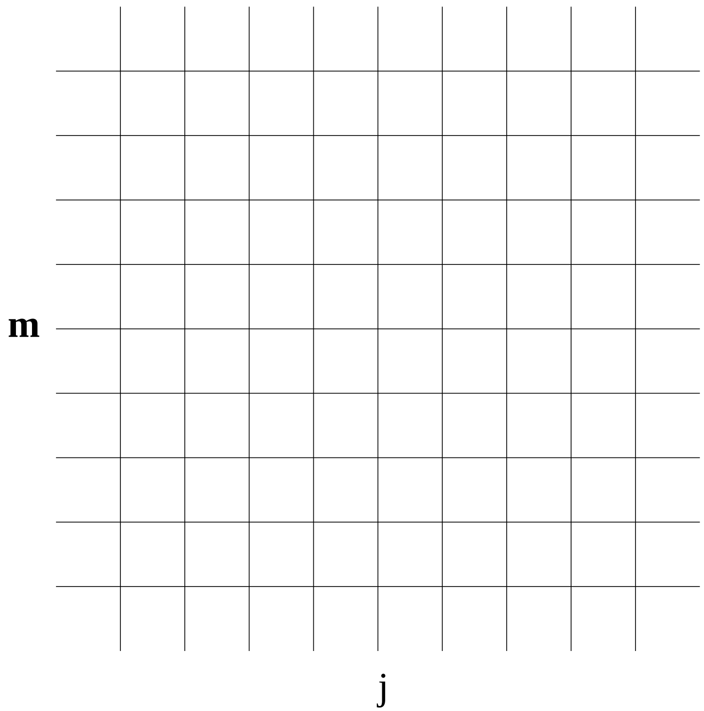
$$T_{j,\mathbf{M}}(\zeta) = R_{j,\mathbf{n}}(\zeta q^{-1/2}) \cdots R_{j,\mathbf{1}}(\zeta q^{-1/2}) \, ,$$

and

$$T_{j,\mathbf{M}} = T_{j,\mathbf{M}}(1) \, .$$

Further,

$$T_{\mathrm{S},\mathbf{M}} = \lim_{N \to \infty} T_{-N+1,\mathbf{M}} \cdots T_{N,\mathbf{M}} \, .$$



Our main object, the partition function with defect can be presented as

$$Z^{\kappa}_{\mathbf{n}}\left\{q^{2\alpha S(0)}\mathcal{O}\right\}=\frac{\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathbf{M}}\left(T_{\mathrm{S},\mathbf{M}}q^{2\kappa S+2\alpha S(0)}\mathcal{O}\right)}{\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathbf{M}}\left(T_{\mathrm{S},\mathbf{M}}q^{2\kappa S+2\alpha S(0)}\right)}\,.$$

Notice the importance of maximal eigenvalues. Non-degeneracy condition $\langle\kappa+\alpha|\kappa\rangle\neq 0$.

Our results.

Consider the space

$$\mathcal{W}^{(\alpha)}=\bigoplus_{s=-\infty}^{\infty}\mathcal{W}_{\alpha-s,s}\,,\quad \mathcal{W}_{\alpha-s,s}\ni q^{(\alpha-s)\sum\limits_{j=-\infty}^{0}\sigma^3_j}\cdot\mathcal{O}^{(s)}\,.$$

On this space we defined the creation operators $\mathbf{t}^*(\zeta)$, $\mathbf{b}^*(\zeta)$, $\mathbf{c}^*(\zeta)$ and annihilation operators $\mathbf{b}(\zeta)$, $\mathbf{c}(\zeta)$.

These are one-parameter families of operators of the form

$$\mathbf{t}^*(\zeta) = \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \mathbf{t}_p^*\,,$$

$$\mathbf{b}^*(\zeta) = \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \mathbf{b}_p^*\,, \quad \mathbf{c}^*(\zeta) = \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \mathbf{c}_p^*\,,$$

$$\mathbf{b}(\zeta) = \sum_{p=1}^{\infty} (\zeta^2 - 1)^{-p} \mathbf{b}_p\,, \quad \mathbf{c}(\zeta) = \sum_{p=1}^{\infty} (\zeta^2 - 1)^{-p} \mathbf{c}_p\,.$$

The operator $\mathbf{t}^*(\zeta)$ is in the center of our algebra of creation-annihilation operators,

$$[\mathbf{t}^*(\zeta_1), \mathbf{t}^*(\zeta_2)] = [\mathbf{t}^*(\zeta_1), \mathbf{c}^*(\zeta_2)] = [\mathbf{t}^*(\zeta_1), \mathbf{b}^*(\zeta_2)] = 0,$$

$$[\mathbf{t}^*(\zeta_1), \mathbf{c}(\zeta_2)] = [\mathbf{t}^*(\zeta_1), \mathbf{b}(\zeta_2)] = 0\,.$$

The rest of the operators $\mathbf{b}$, $\mathbf{c}$, $\mathbf{b}^*$, $\mathbf{c}^*$ are fermionic. The only non-vanishing anti-commutators are

$$[\mathbf{b}(\zeta_1), \mathbf{b}^*(\zeta_2)]_+ = -\psi(\zeta_2/\zeta_1, \alpha)\,, \quad [\mathbf{c}(\zeta_1), \mathbf{c}^*(\zeta_2)]_+ = \psi(\zeta_1/\zeta_2, \alpha)\,,$$

where

$$\psi(\zeta, \alpha) = \frac{\zeta^\alpha}{(\zeta^2 - 1)}\,.$$

Each Fourier mode has the block structure

$$\mathbf{t}_p^* \;:\; \mathcal{W}_{\alpha-s,s} \;\to\; \mathcal{W}_{\alpha-s,s}$$

$$\mathbf{b}_p^*, \mathbf{c}_p \;:\; \mathcal{W}_{\alpha-s+1,s-1} \;\to\; \mathcal{W}_{\alpha-s,s}\,, \quad \mathbf{c}_p^*, \mathbf{b}_p \;:\; \mathcal{W}_{\alpha-s-1,s+1} \;\to\; \mathcal{W}_{\alpha-s,s}\,.$$

Further

$$\mathbf{x}_p(X) = 0, \quad p > \mathrm{length}\big(X\big), \qquad \mathbf{x} = \mathbf{b}, \mathbf{c}\,,$$

$$\mathrm{length}\big(\mathbf{x}_p^*\big(X\big)\big) \le \mathrm{length}\big(X\big) + p\,, \qquad \mathbf{x} = \mathbf{b}, \mathbf{c}, \mathbf{t}\,.$$

Among them, $\boldsymbol{\tau} = \mathbf{t}_1^*/2$ plays a special role. It is the right shift by one site along the chain. Consider the set of operators

$$\boldsymbol{\tau}^m \mathbf{t}^*_{p_1} \cdots \mathbf{t}^*_{p_j} \mathbf{b}^*_{q_1} \cdots \mathbf{b}^*_{q_k} \mathbf{c}^*_{r_1} \cdots \mathbf{c}^*_{r_k} \left(q^{2\alpha S(0)} \right),$$

where $m \in \mathbb{Z}$, $j, k \in \mathbb{Z}_{\geq 0}$, $p_1 \geq \cdots \geq p_j \geq 2$, $q_1 > \cdots > q_k \geq 1$ and $r_1 > \cdots > r_k \geq 1$. constitutes a basis of $\mathcal{W}_{\alpha,0}$.

Main theorem relating Space and Matsubara

$$Z^\kappa \big\{ \mathbf{t}^*(\zeta)(X) \big\} = 2\rho(\zeta) Z^\kappa \{X\} \,,$$

$$Z^\kappa \big\{ \mathbf{b}^*(\zeta)(X) \big\} = \frac{1}{2\pi i} \oint\limits_\Gamma \omega(\zeta, \xi) Z^\kappa \big\{ \mathbf{c}(\xi)(X) \big\} \frac{d\xi^2}{\xi^2} \,,$$

$$Z^\kappa \big\{ \mathbf{c}^*(\zeta)(X) \big\} = -\frac{1}{2\pi i} \oint\limits_\Gamma \omega(\xi, \zeta) Z^\kappa \big\{ \mathbf{b}(\xi)(X) \big\} \frac{d\xi^2}{\xi^2} \,,$$

the functions ρ and ω are defined by Matsubara.

Since

$$\mathbf{c}(\zeta)(q^{2\alpha S(0)}) = 0, \quad \mathbf{b}(\zeta)(q^{2\alpha S(0)}) = 0\,,$$

we obtain

$$Z^\kappa\Big\{\mathbf{t}^*(\zeta_1^0)\cdots\mathbf{t}^*(\zeta_k^0)\mathbf{b}^*(\zeta_1^+)\cdots\mathbf{b}^*(\zeta_l^+)\mathbf{c}^*(\zeta_l^-)\cdots\mathbf{c}^*(\zeta_1^-)\big(q^{2\alpha S(0)}\big)\Big\}$$
$$= \prod_{p=1}^{k} 2\rho(\zeta_p^0) \times \det\big(\omega(\zeta_i^+,\zeta_j^-)\big)_{i,j=1,\cdots,l}\,.$$

Taking the Taylor coefficients in $(\zeta_i^\epsilon)^2 - 1$ in both sides, one obtains the value of Z^κ on an arbitrary element of the fermionic basis.

The analogy with the CFT becomes transparent at this point.

Quantum loop algebra $U_q'(\widehat{\mathfrak{sl}}_2)$.

Quantum groups. Multiplication (with unit 1):

$$m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A},$$

and comultiplication

$$\Delta: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A},$$

with the requirement that Δ is a homomorphism:

$$\Delta(xy) = \Delta(x)\Delta(y).$$

Antipode is an anti-homomorphism $s: \mathcal{A} \to \mathcal{A}$, it is a deformation of inverse for Lie algebra. Counit is a homomorphism $\epsilon: \mathcal{A} \to \mathcal{A}$

$$m \circ (s \otimes id) \circ \Delta(x) = m \circ (id \otimes s) \circ \Delta(x) = \epsilon(x).$$

Let σ be the permutation of two copies of $\mathcal{A}$ in the tensor product:

$$\sigma: \ \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}\,, \quad \sigma(x \otimes y) = y \otimes x\,,$$

and

$$\Delta' = \sigma \circ \Delta\,.$$

The quasi-triangularity requires the existence of a universal R-matrix, $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ which intertwines two comultiplications:

$$\Delta' = \mathcal{R} \Delta \mathcal{R}^{-1}\,.$$

The universal R-matrix satisfies the Yang-Baxter equation:

$$\mathcal{R}_{1,2}\mathcal{R}_{1,3}\mathcal{R}_{2,3} = \mathcal{R}_{2,3}\mathcal{R}_{1,3}\mathcal{R}_{1,2}\,,$$

another important property is

$$(id \otimes s)\mathcal{R} = \mathcal{R}^{-1}\,.$$

$U'_q(\widehat{\mathfrak{sl}}_2)$ is generated by e_i, f_i, h_i ($i = 0, 1$). We consider the case of central charge equal to zero: $h_1 = -h_0 \equiv h$. Two Borel subalgebras $U_q(\mathfrak{b}^+)$ and $U_q(\mathfrak{b}^-)$ are generated respectively by e_i, h and f_i, h. We have the commutation relations:

$$[e_i, f_j] = \delta_{i,j} \frac{t_i - t_i^{-1}}{q - q^{-1}} \,,$$

where $t_i = q^{h_i}$. The deformed Serre relations are

$$e_i^3 e_j + (q^2 + q^{-2} + 1)(e_i^2 e_j e_i - e_i e_j e_i^2) - e_j e_i^3 = 0 \,,$$
$$f_i^3 f_j + (q^2 + q^{-2} + 1)(f_i^2 f_j f_i - f_i f_j f_i^2) - f_j f_i^3 = 0$$

The comultiplication and antipode are given by

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \quad \Delta(t_i) = t_i \otimes t_i \,,$$
$$s(e_i) = -t_i^{-1} e_i, \quad \Delta(f_i) = f_i t_i, \quad s(t_i) = t_i^{-1} \,.$$

The comultiplication looks quite simple, but the universal R-matrix intertwining Δ and Δ' is complicated. It can be written as follows:

$$\mathcal{R} = \overline{\mathcal{R}} q^{-\frac{h \otimes h}{2}} \,,$$

$$\overline{\mathcal{R}} = 1 - (q - q^{-1}) \sum_{i=0}^{1} e_i \otimes f_j + \cdots \; \in \; U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{b}^-) \,,$$

where the $\cdots$ stands for terms of higher degree in generators.

Representations. Let E, F, H be generators of $U_q(\mathfrak{sl}_2)$. The evaluation representation:

$$ev_\zeta(e_0) = \zeta F, \quad ev_\zeta(e_1) = \zeta E, \quad ev_\zeta(f_0) = \zeta^{-1} E, \quad ev_\zeta(f_1) = \zeta^{-1} F,$$
$$ev_\zeta(h) = H \,.$$

Choosing finite-dimensional representation of dimension $2s + 1$ we obtain $\pi_\zeta^{(2s)}$.

We have $(ev_{\zeta_1} \otimes \pi^{(1)}_{\zeta_2})(\mathcal{R}) = \tau(\zeta) L(\zeta)\,, \quad \zeta = \zeta_1/\zeta_2\,,$

$$L(\zeta) = \begin{pmatrix} 1 - \zeta^2 q^{H+1} & -(q-q^{-1})\zeta F \\ -(q-q^{-1})\zeta E & 1-\zeta^2 q^{-H+1} \end{pmatrix} t_0^{\sigma^3/2}\,,$$

This will be used for Finite-dimensional of dimension $2s+1$:

$$Fv_j = v_{j+1}, \quad Hv_j = (-2s+2j)v_j, \quad t_0 = q^{-H}\,,$$

$$Ev_j = (q^j - q^{-j})(q^{2(s-2s-1)} - q^{-2(j-2s-1)})v_{j-1}, \quad j = 0, \cdots, 2s\,.$$

Important generalization.

Bazhanov, Lukyanov, Zamolodchikov (1996).

Recall that $\overline{\mathcal{R}} \in U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{b}^-)$. Suppose we are given two algebras $A^{\pm}$ and homomorphisms $U_q(\mathfrak{b}^+) \to A^+$, $U_q(\mathfrak{b}^-) \to A^-$. I shall use the term L-operator for the image of the universal R-matrix under these maps. The q-oscillator algebra Osc is an associative algebra with generators $\mathbf{a}, \mathbf{a}^*, q^D$, and defining relations

$$q^D \mathbf{a}\, q^{-D} = q^{-1}\mathbf{a}, \quad q^D\, \mathbf{a}^* q^{-D} = q\, \mathbf{a}^*,$$
$$\mathbf{a}\, \mathbf{a}^* = 1 - q^{2D+2}, \quad \mathbf{a}^*\mathbf{a} = 1 - q^{2D}\,.$$

Representations of Osc relevant to us are $\rho^{\pm} : Osc \to \mathrm{End}(W^{\pm})$ defined by

$$W^+ = \oplus_{k\geq 0}\mathbb{C}|k\rangle, \quad W^- = \oplus_{k<0}\mathbb{C}|k\rangle,$$
$$q^D|k\rangle = q^k|k\rangle,\ \mathbf{a}|k\rangle = (1-q^{2k})|k-1\rangle,\ \mathbf{a}^*|k\rangle = (1-\delta_{k,-1})|k+1\rangle\,.$$

In what follows I shall consider ρ^+ only.

Trace

$$\mathrm{Tr}(q^{2\alpha D}XY) = \mathrm{Tr}(q^{2\alpha D}q^{2\alpha d(X)}YX) \quad (X, Y \in Osc, q^D X q^{-D} = q^{d(X)}X),$$
$$\mathrm{Tr}(q^{2\alpha D}q^{mD}) = \frac{1}{1-q^{2\alpha+m}} \quad (m \in \mathbb{Z}).$$

There is a homomorphism of algebras $o_\zeta : U_q\mathfrak{b}^+ \to Osc$ given by

$$o_\zeta(e_0) = \frac{\zeta}{q-q^{-1}}\mathbf{a}, \quad o_\zeta(e_1) = \frac{\zeta}{q-q^{-1}}\mathbf{a}^*, \quad o_\zeta(t_0) = q^{-2D}, \quad o_\zeta(t_1) = q^{2D}.$$

We define representation $o_\zeta^+ : U_q\mathfrak{b}^+ \to \mathrm{End}(W^\pm)$ by $o_\zeta^+ = \rho^+ \circ o_\zeta$.

We define

$$(o_\zeta^+ \otimes \pi_\xi)\mathcal{R} = \sigma(\zeta/\xi) \cdot L_{Aj}(\zeta/\xi)\,,$$

Then by self-consistency one finds:

$$L_{A,j}(\zeta) := \begin{pmatrix} 1 - \zeta^2 q^{2D_A+2} & -\zeta \mathbf{a}_A \\ -\zeta \mathbf{a}_A^* & 1 \end{pmatrix}_j \begin{pmatrix} q^{-D_A} & 0 \\ 0 & q^{D_A} \end{pmatrix}_j, \; .$$

Notice the indices j, A, a!

R-matrices. Obvious:

$$L_{a,b}(\zeta_2/\zeta_1)L_{a,\mathbf{j}}(\zeta_2)L_{b,\mathbf{j}}(\zeta_1) = L_{b,\mathbf{j}}(\zeta_1)L_{a,\mathbf{j}}(\zeta_2)L_{a,b}(\zeta_2/\zeta_1)\,,$$

$$L_{A,a}(\zeta_2/\zeta_1)L_{A,\mathbf{j}}(\zeta_2)L_{a,\mathbf{j}}(\zeta_1) = L_{a,\mathbf{j}}(\zeta_1)L_{A,\mathbf{j}}(\zeta_2)L_{A,a}(\zeta_2/\zeta_1)\,.$$

Less obvious. $R_{A,B}(\zeta_1/\zeta_2)$ satisfying

$$R_{A,B}(\zeta_1/\zeta_2)L_{A,\mathbf{j}}(\zeta_1)L_{B,\mathbf{j}}(\zeta_2) = L_{B,\mathbf{j}}(\zeta_2)L_{A,\mathbf{j}}(\zeta_1)R_{A,B}(\zeta_1/\zeta_2)\,.$$

does exist. It is given by

$$R_{A,B}(\zeta) = P_{A,B}h(\zeta, u_{A,B})\zeta^{D_A+D_B},$$

where $u_{A,B} = \mathbf{a}_A^* q^{-2D_A}\mathbf{a}_B$, and $h(\zeta, u)$ is the unique formal power series in u satisfying

$$(1+\zeta u)h(\zeta,u) = (1+\zeta^{-1}u)h(\zeta,q^2u),$$
$$h(\zeta,u) = (1+\zeta^{-1}u)(1+q^{-2}\zeta u)h(q^{-2}\zeta,u)$$

and $h(\zeta, 0) = 1$.

One more important property.

$$L_{\{a,A\},j}(\zeta)(F_{a,A})^{-1}L_{a,j}(\zeta)L_{A,j}(\zeta)F_{a,A}$$

$$= \begin{pmatrix} 1 & 0 \\ \frac{q-q^{-1}}{\zeta-\zeta^{-1}}\,\sigma_j^+ & 1 \end{pmatrix}_a \begin{pmatrix} (\zeta^2-1)L_{A,j}(q\zeta)q^{-\sigma_j^3/2} & 0 \\ 0 & (\zeta^2q^2-1)L_{A,j}(q^{-1}\zeta)q^{\sigma_j^3/2} \end{pmatrix}_a ,$$

where $F_{a,A}=1-\mathbf{a}_A\sigma_a^+$.

Construction of annihilation operators.

Consider the operator $X_{[k,l]}\in \mathrm{End}\left(\mathbb{C}^{\otimes(l-k+1)}\right)$. Define

$$T_{a,[k,l]}(\zeta)=L_{a,l}(\zeta)\cdots L_{a,k}(\zeta)\,,$$

and the adjoint monodromy matrix

$$\mathbb{T}_a(\zeta,\alpha)(X_{[k,l]})=T_{a,[k,l]}(\zeta)q^{\alpha\sigma_a^3}X_{[k,l]}T_{a,[k,l]}(\zeta)^{-1}\,,\ .$$

Define further

$$\mathbb{S}(X_{[k,l]}) := [S_{[k,l]}, X_{[k,l]}], \quad S_{[k,l]} := \tfrac{1}{2}\sum_{j\in[k,l]} \sigma^3_j .$$

Then

$$(F_{a,A})^{-1}\left(\mathbb{T}_a(\zeta,\alpha)\mathbb{T}_A(\zeta,\alpha)(X_{[k,l]})\right) F_{a,A} = \begin{pmatrix} \mathbb{A}_A(\zeta,\alpha)(X_{[k,l]}) & 0 \\ \mathbb{C}_A(\zeta,\alpha)(X_{[k,l]}) & \mathbb{D}_A(\zeta,\alpha)(X_{[k,l]}) \end{pmatrix}_a ,$$

$$\mathbb{A}_A(\zeta,\alpha) = \mathbb{T}_A(\zeta q,\alpha)q^{-\mathbb{S}}, \quad \mathbb{D}_A(\zeta,\alpha) = \mathbb{T}_A(\zeta q^{-1},\alpha)q^{\mathbb{S}} .$$

Define

$$\mathbf{k}(\zeta,\alpha)(X_{[k,l]}) := \mathrm{Tr}_A\left\{\mathbb{C}_A(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}}\left(q^{-2S_{[k,l]}}\, X_{[k,l]}\right)\right\} .$$

$\mathbf{k}(\zeta,\alpha)(X_{[k,l]})$ has poles of hight order at $\zeta^2 = 1, q^{\pm 2}$.

Define the operation

$$\Delta_\zeta f(\zeta) = f(\zeta q) - f(\zeta q^{-1})\,.$$

Definition. Exact q-one form is an expression of the form $\Delta_\zeta f(\zeta)$ with $f(\zeta)$ having poles at $\zeta^2 = 1$.

Using our algebra it can be shown.

$$\mathbf{k}(\zeta_1, \alpha)\mathbf{k}(\zeta_2, \alpha+1) + \mathbf{k}(\zeta_2, \alpha)\mathbf{k}(\zeta_1, \alpha+1)$$
$$= \Delta_{\zeta_1}\mathbf{m}(\zeta_1, \zeta_2, \alpha) + \Delta_{\zeta_2}\mathbf{m}(\zeta_2, \zeta_1, \alpha),$$

In RHS we have exact q-two forms.

Consider

$$\bar{\mathbf{c}}(\zeta,\alpha)(X_{[k,l]}) := \frac{1}{2\pi i}\int\limits_{\Gamma}\psi(\zeta/\xi,\alpha+\mathbb{S})\mathbf{k}(\xi,\alpha)(X_{[k,l]})\frac{d\xi^2}{\xi^2}\,,$$

$$\mathbf{c}(\zeta,\alpha)(X_{[k,l]}) := \frac{1}{4\pi i}\int\limits_{\Gamma}\psi(\zeta/\xi,\alpha+\mathbb{S})\left\{\mathbf{k}(q\xi,\alpha)+\mathbf{k}(q^{-1}\xi,\alpha)\right\}(X_{[k,l]})\frac{d\xi^2}{\xi^2}\,,$$

where Γ goes around $\zeta^2=1$.

Then

$$\mathbf{c}(\zeta_1,\alpha)\mathbf{c}(\zeta_2,\alpha+1)+\mathbf{c}(\zeta_2,\alpha)\mathbf{c}(\zeta_1,\alpha+1)=0\,,$$

$$\bar{\mathbf{c}}(\zeta_1,\alpha)\bar{\mathbf{c}}(\zeta_2,\alpha+1)+\bar{\mathbf{c}}(\zeta_2,\alpha)\bar{\mathbf{c}}(\zeta_1,\alpha+1)=0\,,$$

$$\bar{\mathbf{c}}(\zeta_1,\alpha)\mathbf{c}(\zeta_2,\alpha+1)+\mathbf{c}(\zeta_2,\alpha)\bar{\mathbf{c}}(\zeta_1,\alpha+1)=0\,.$$