On confluent A-hypergeometric functions

(joint work with A. Esterov)

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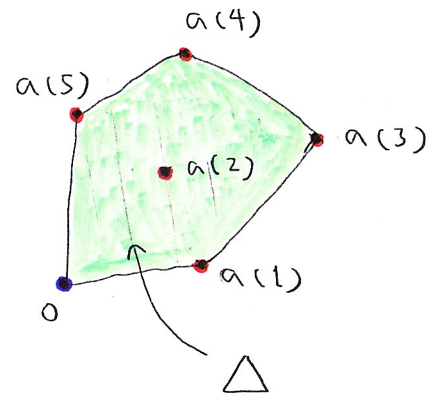
§ Adolphson's results (Duke math. J, 1994)

$A = \{ a(1), a(2), \cdots\cdots, a(N) \} \subset \mathbb{Z}^n$: a finite subset.

Assume that $\sum_{j=1}^{N} \mathbb{Z}\, a(j) = \mathbb{Z}^n$. Set

$\mathbb{R}^n$



$$\Delta \underset{\text{def}}{:=} \operatorname{conv}(\{0\} \cup A) \subset \mathbb{R}^n \quad \text{and}$$

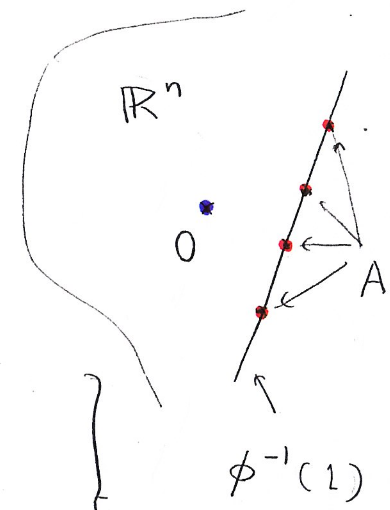
$$A \underset{\text{def}}{:=} (a_{ij}) = n\Big\updownarrow \underbrace{\left({}^t a(1), \cdots\cdots, {}^t a(N) \right)}_{N} \in M(n, N; \mathbb{C}).$$

Then Adolphson's confluent A-HG system for $c \in \mathbb{C}^n$ is

$$\begin{cases} \left(\sum_{j=1}^{N} a_{ij} z_j \dfrac{\partial}{\partial z_j} + c_i \right) u(z) = 0 & (1 \leqq i \leqq n), \\[2ex] \left[\prod_{\mu_j > 0} \left(\dfrac{\partial}{\partial z_j} \right)^{\mu_j} - \prod_{\mu_j < 0} \left(\dfrac{\partial}{\partial z_j} \right)^{-\mu_j} \right] u(z) = 0 & \left(\begin{array}{l} \mu = (\mu_1, \cdots, \mu_N) \\ \in \operatorname{Ker} A \cap \mathbb{Z}^N \end{array} \right). \end{cases}$$

(2)

Rem If $\exists$ a linear map $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ s.t. $\phi(\mathbb{Z}^n) \subset \mathbb{Z}$ and $A \subset \phi^{-1}(1)$, then this system on $X := \mathbb{C}^N_z$ was introduced by G-K-Z.



Set

$$\Omega \underset{\text{def}}{:=} \left\{ z \in X := \mathbb{C}^N \;\middle|\; \begin{array}{l} \text{the Laurent polynomial} \\ \quad h_z(x) = \sum_{j=1}^{N} z_j x^{a(j)} \\ \text{on } T = (\mathbb{C}^*)^n_x \text{ is non-deg.} \end{array} \right\} \underset{\text{open dense}}{\subset} X.$$

Thm (Adolphson) Assume that the parameter $c \in \mathbb{C}^n$ is semi-nonresonant. Then the corresponding (holonomic) $\mathcal{D}_X$-module

$$\mathcal{M}_{A,c} := \mathcal{D}_X \Big/ \sum_{1 \leq i \leq n} \mathcal{D}_X Z_{i,c} + \sum_{\mu \in \operatorname{Ker} A \cap \mathbb{Z}^N} \mathcal{D}_X \square_\mu$$

is an integrable connection of rank $\underbrace{\mathrm{Vol}_{\mathbb{Z}}(\Delta)}_{\parallel \; n!\,\mathrm{Vol}(\Delta)} \in \mathbb{Z}$ on Ω.
(see S-S-T etc.)

$$\Rightarrow \mathcal{H}om_{\mathcal{D}_{X^{an}}}\left((\mathcal{M}_{A,c})^{an}, \mathcal{O}_{X^{an}}\right) = \left\{ u \in \mathcal{O}_{X^{an}} \;\middle|\; \begin{array}{l} Z_{i,c}\, u = 0 \\ \Box_\mu\, u = 0 \end{array} \right\} \subset \mathcal{O}_{X^{an}}$$

the sheaf of confluent A-HG functions

is a local system of rank $\mathrm{Vol}_{\mathbb{Z}}(\Delta)$ on $\Omega^{an} \subset X^{an}$.

$$\mathcal{H}(A,c) \underset{\text{def}}{:=} \mathcal{H}om_{\mathcal{D}_{X^{an}}}\left((\mathcal{M}_{A,c})^{an}, \mathcal{O}_{X^{an}}\right)\Big|_{\Omega^{an}} \subset \mathcal{O}_{\Omega^{an}}.$$

<u>Problem</u> (i) $\exists$ an integral rep for $\mathcal{H}(A,c)$?

(ii) monodromy of $\mathcal{H}(A,c) = ?$ (iii) asymptotic expansions ?

In the classical case of G-K-Z, $\mathcal{M}_{A,c}$ is regular holonomic (Hotta's thm) and we have ($\gamma \subset (\mathbb{C}^*)^{n-1}$):

$$u(z) = \int_\gamma P_z(y_1, \cdots, y_{n-1})^\alpha\, y_1^{\beta_1} \cdots y_{n-1}^{\beta_{n-1}}\, dy_1 \wedge \cdots \wedge dy_{n-1}$$

(G-K-Z, Adv. Math, 1990).

§ Hien's rapid decay homologies

$\mathcal{O}_U^{\oplus m}$ $\underset{\text{Zariski locally}}{\cong}$

U : smooth quasi-projective alg var $/\mathbb{C}$, $(\mathcal{E}, \nabla)$: flat (integrable) connection on U. For $p \in \mathbb{Z}$ its p-th alg de Rham cohomology is defined by

$$H^p_{dR}(U, \mathcal{E}, \nabla)$$

$$\underset{\text{def}}{:=} H^p\left[\cdots 0 \longrightarrow \Gamma(U; \mathcal{E}) \xrightarrow{\nabla} \Gamma(U; \mathcal{E} \otimes_{\mathcal{O}_U} \Omega^1_U) \xrightarrow{\nabla} \Gamma(U; \mathcal{E} \otimes_{\mathcal{O}_U} \Omega^2_U) \xrightarrow{\nabla} \cdots\cdots \right].$$

$(\mathcal{E}^*, \nabla^*)$: the dual connection of $(\mathcal{E}, \nabla)$.

We define a local system $\mathcal{L}$ on U^{an} by

$$\mathcal{L} \underset{\text{def}}{:=} \mathrm{Ker}\left[(\mathcal{E}^*)^{an} \xrightarrow{(\nabla^*)^{an}} (\mathcal{E}^*)^{an} \otimes_{\mathcal{O}_{U^{an}}} \Omega^1_{U^{an}} \right].$$

<u>Thm</u> (Grothendieck - Deligne) If $(\mathcal{E}, \nabla)$ is regular (at ∞) there exists a perfect pairing :

$$H^p_{dR}(U, \mathcal{E}, \nabla) \times H_p(U^{an}; \mathcal{L}) \longrightarrow \mathbb{C} \quad \text{for } {}^\forall p \in \mathbb{Z}.$$

<u>Thm</u> (Hien, Invent Math, 2009) (Even if $(\mathcal{E}, \nabla)$ is irregular)

for the rapid decay homologies $H_p^{rd}(U, \mathcal{E}^*, \nabla^*)$ $(p \in \mathbb{Z})$ we have perfect pairings :

$$H_{dR}^p(U, \mathcal{E}, \nabla) \times H_p^{rd}(U, \mathcal{E}^*, \nabla^*) \longrightarrow \mathbb{C}.$$

For the definition of $H_p^{rd}(\cdots\cdots)$ we need some deep results of Sabbah and Mochizuki.

<u>Example</u> $U = T = (\mathbb{C}^*)^n_x$. For a <u>fixed</u> $z \in \Omega \subset X = \mathbb{C}^N$ consider the (non-deg) Laurent polynomial

$$h_z(x) \underset{def}{:=} \sum_{j=1}^{N} z_j x^{a(j)}$$

on $U = T$ and set

$$g(x) = \exp(h_z(x)) \cdot x_1^{c_1 - 1} \cdots\cdots x_n^{c_n - 1}$$

(a multi-valued holo ft on $U^{an} = T^{an}$).

On $\mathcal{E} := \mathcal{O}_U\, g \simeq \mathcal{O}_U$ we define a connection ∇ by

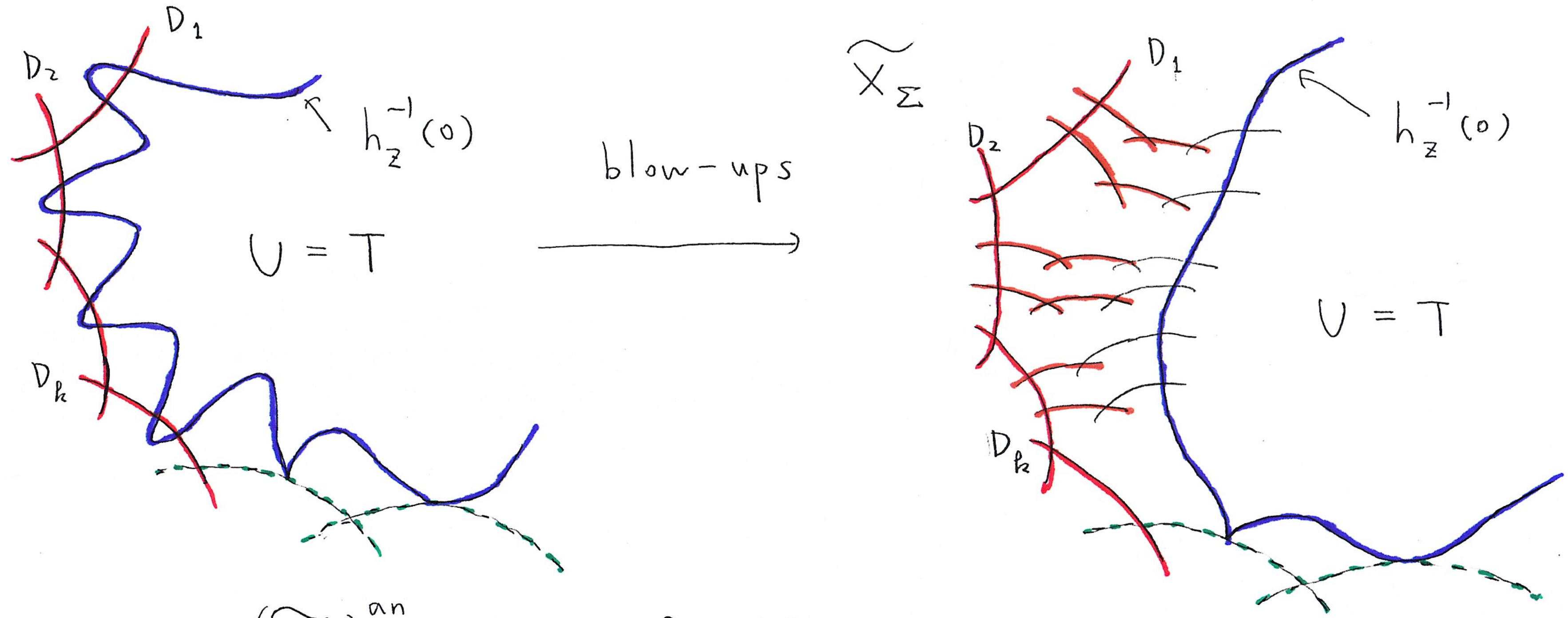
symbol

$$\nabla : \mathcal{E} = \mathcal{O}_U\, g \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_U} \Omega^1_U$$

$$v \cdot g \longmapsto dv \cdot g + v \cdot dg = \Big(dv + \underbrace{\tfrac{dg}{g}}_{\text{algebraic!}} \Big) \cdot g$$

So that we have $\operatorname{Ker} \nabla^{an} \simeq \mathbb{C}_{U^{an}} \left(\frac{1}{g}\right)$, $\mathcal{L} = \operatorname{Ker}(\nabla^*)^{an} \simeq \mathbb{C}_{U^{an}}\, g$.

X_Σ ↗ a nice toric compactif. of $U = T$



U^{an}

$\pi : \widetilde{Z} \twoheadrightarrow Z := (\widetilde{X_\Sigma})^{an}$: the real blow-up of Z along $D^{an} = (\widetilde{X}_\Sigma \setminus U)^{an}$,

[**Def** (Hien) $c \otimes g \in \mathcal{C}^p_{\widetilde{Z}, \pi^{-1}(D^{an})} \otimes_{\mathbb{C}_{\widetilde{Z}}} c_* \mathcal{L}$ is rapid decay for g if ....,

§ An integral rep of confluent A-HG functions

Let $\mathcal{H}_n^{rd}$ be the local system on $\Omega^{an} \subset X^{an} = \mathbb{C}_z^N$ s.t. for $\forall z \in \Omega^{an}$ we have

$$\left(\mathcal{H}_n^{rd}\right)_z \simeq H_n^{rd}\left(T, \mathcal{E}^*_{(z)}, \nabla^*_{(z)}\right).$$

$\uparrow$ stalk at z

Thm (Esterov − T)

Assume that $c \in \mathbb{C}^n$ is non-resonant. Then on Ω^{an} we have an isomorphism:

$$\mathcal{H}_n^{rd} \xrightarrow[\sim]{\Phi} \mathcal{H}(A, c) \subset \mathcal{O}_{\Omega^{an}}$$

$$\gamma = \{\gamma_z\} \longmapsto u(z) = \int_{\gamma_z} \underbrace{\exp\left(\sum_{j=1}^N z_j x^{\alpha(j)}\right)}_{h_z(x)} x_1^{c_1 - 1} \cdots\cdots x_n^{c_n - 1}\, dx_1 \wedge \cdots\cdots \wedge dx_n.$$

γ: a family of rapid decay n-cycles in $T = (\mathbb{C}^*)^n_x$.

proof

(A generalization of Hien − Roucairol's thm) + (Katz − Laumon's thm) etc. ▨

<u>Rem</u> (i) Bessel function: $$v(t) = \frac{1}{2\pi i}\int_\gamma \exp\left(\frac{t}{2}x - \frac{t}{2}x^{-1}\right) x^{-\nu-1}\,dx,$$

Airy function: $$Ai(t) = \frac{1}{2\pi i}\int_\gamma \exp\left(\frac{1}{3}x^3 - tx\right)dx \quad \text{etc.}$$

(ii) For confluent HG functions related to <u>hyperplane arrangements</u>, see Kimura-Haraoka-Takano's works.

<u>Cor</u> (M. Saito, Compositio Math, 2011 and Schulze-Walther)

$\mathcal{M}_{A,c}\big|_{\Omega}$ is an <u>irreducible</u> integrable connection.

§ A construction of the basis of $\mathcal{H}_n^{rd}$

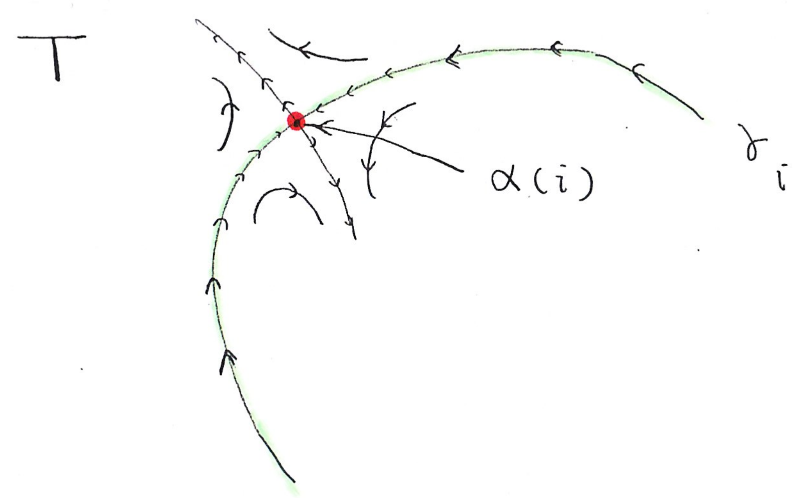
$$\Omega_0 \underset{\text{def}}{:=} \left\{ z \in \Omega \;\middle|\; \begin{array}{l} h_z : T = (\mathbb{C}^*)^n \longrightarrow \mathbb{C} \text{ has exactly } \mathrm{Vol}_{\mathbb{Z}}(\Delta) \\ \text{non-degenerate (Morse) critical points in } T \end{array} \right\}.$$

<u>Lemma</u> (E-T) $\Omega_0 \subset X = \mathbb{C}^N_z$ is <u>open dense</u> and stable by the $(\mathbb{C}^*)$-action if $0 \in \mathrm{Int}(\Delta)$.

<u>Thm (E-T)</u> Assume that <u>$0 \in \mathrm{Int}(\Delta)$</u>. For $z \in \Omega_0 \subset \Omega$ let $\alpha(1), \alpha(2), \cdots\cdots, \alpha(\mathrm{Vol}_{\mathbb{Z}}(\Delta)) \in T = (\mathbb{C}^*)^n$ be the non-deg. (Morse) critical points of $h_z : T \longrightarrow \mathbb{C}$ ($\Longleftrightarrow$ critical pts of $\mathrm{Re}(h_z) : T \longrightarrow \mathbb{R}$). Then for each $1 \leqq i \leqq \mathrm{Vol}_{\mathbb{Z}}(\Delta)$ there exists a <u>rapid decay</u> n-cycle $\gamma_i \subset T$ for

$$\exp(h_z(x))\, x_1^{c_1-1} \cdots\cdots x_n^{c_n-1}$$

which coincides with the <u>stable manifold</u> of $\overrightarrow{\mathrm{grad}}(\mathrm{Re}(h_z))$ near $\alpha(i) \in T$ and $[\gamma_1], \cdots\cdots, [\gamma_{\mathrm{Vol}_{\mathbb{Z}}(\Delta)}] \in (\mathcal{M}_n^{rd})_z \simeq$
$\simeq H_n^{rd}(T, \mathcal{E}_z^*, \nabla_z^*)$ is a basis of $(\mathcal{M}_n^{rd})_z$.



<u>proof</u> by "twisted" Morse theory for $\mathrm{Re}(h_z) : T \longrightarrow \mathbb{R}$ ∎

By applying the "higher-dimensional" saddle point (steepest descent) method we obtain:

<u>Thm (E-T)</u> In the situation as above ($0 \in \mathrm{Int}(\Delta)$ etc.), for each $1 \leqq i \leqq \mathrm{Vol}_{\mathbb{Z}}(\Delta)$ we have an asymptotic expansion:

$$u_i(\underset{\substack{\vee \\ 0}}{r} z) := \int_{\gamma_i} \exp\left(r \sum_{j=1}^{N} z_j x^{a(j)} \right) x_1^{c_1 - 1} \cdots\cdots x_n^{c_n - 1} \, dx_1 \wedge \cdots \wedge dx_n$$

$$\sim (\sqrt{-1})^n \alpha(i)_1^{c_1-1} \cdots\cdots \alpha(i)_n^{c_n-1} \times \exp\left(r \underbrace{h_z(\alpha(i))}_{} \right) \cdot \left\{ \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{H(\alpha(i))}} \cdot \frac{1}{r^{\frac{n}{2}}} + \frac{{}^{\exists}b_1}{r^{\frac{n}{2}+1}} + \frac{{}^{\exists}b_2}{r^{\frac{n}{2}+2}} + \cdots\cdots + \cdots\cdots \right\}$$

the critical value of h_z at $\alpha(i) \in T$.

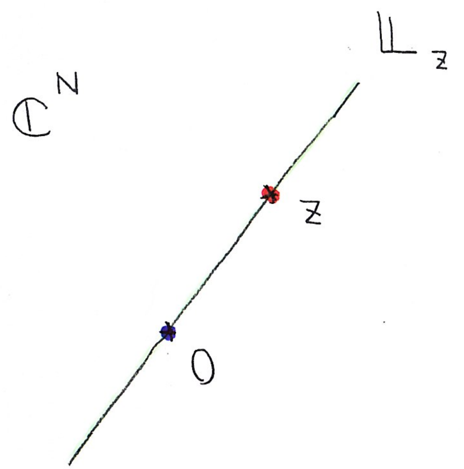
we can calculate them!

as $\underset{\substack{\vee \\ 0}}{r} \longrightarrow +\infty$, where we set $H(\alpha) = \det\left(\dfrac{\partial^2 h_z}{\partial x_i \partial x_j} \right) \Big|_{x=\alpha}$.

Stokes phenomena Fix $z \in \Omega_0 \subset \Omega \subset X = \mathbb{C}^N$ and consider the complex line through it:

$$\mathbb{L}_z \underset{\text{def}}{:=} \left\{ \varsigma \cdot z \in \mathbb{C}^N \,\middle|\, \varsigma \in \mathbb{C} \right\} \simeq \mathbb{C}_\varsigma$$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ re^{i\theta} \times z & \longleftrightarrow & re^{i\theta}. \end{array}$$

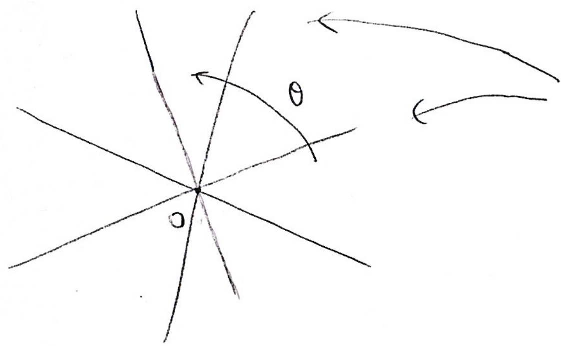


Then the dominance ordering of the functions

$$u_i \Big|_{\mathbb{L}_z} : \mathbb{L}_z \simeq \mathbb{C}_{re^{i\theta}} \longrightarrow \mathbb{C} \quad \left(1 \leq i \leq \mathrm{Vol}_{\mathbb{Z}}(\Delta) \right)$$

changes as θ goes from 0 to 2π.

$\mathbb{L}_z \simeq \mathbb{C}_\varsigma$



Stokes lines $(i \neq j)$:

$$\left\{ \varsigma \,\middle|\, \mathrm{Re}\left\{ \varsigma \cdot \left(h_z(\alpha(i)) - h_z(\alpha(j)) \right) \right\} = 0 \right\}.$$