Long-distance properties of baryons in the Sakai-Sugimoto model

Aleksey Cherman

UNIVERSITY OF CAMBRIDGE

Based on recent work with Takaaki Ishii, arXiv:1109.4665
and older work with T. Cohen and M. Nielsen, PRL 103, (2009) 022001

at IPMU, January 30, 2012

$$\mathcal{L}_{\mathrm{QCD}} = \frac{1}{2g^2}\mathrm{tr}F^{\mu\nu}F_{\mu\nu} + \bar{q}\left(\not{D} + m_q\right)q = ???$$

QCD is the theory of quarks and gluons, but the physical states are baryons and mesons...

39 Year Old Goal: get from quarks and glue to e.g. baryons

Seems to be too hard to do this directly in QCD without brute force numerics

Even the large N limit doesn't help much!

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Popular recent direction:

Change the theory but keep it `QCD-like',
then use gauge-gravity duality to get info.

Then hope lessons learned are applicable to real QCD...

Crucial to know how much like QCD these 'QCD-like' theories are!

But how do we tell, since we can't compute in (large N) QCD?

Phenomenological vs theoretical tests...

Plan of the talk

1) Review large N_c limit of QCD, and discuss an observable for which quantitative large N_c QCD predictions are available

2) Give an overview of the Sakai-Sugimoto model, the most successful holographic model for QCD

3) Present baryon property puzzle in SS model

4) Discuss analysis of model without standard instanton approximation, and resulting possible resolution of puzzle.

Lightning review of large N

'*t Hooft large N_c limit:* $N_c \to \infty$, keeping $\lambda = g_{YM}^2 N_c, N_f$ fixed.

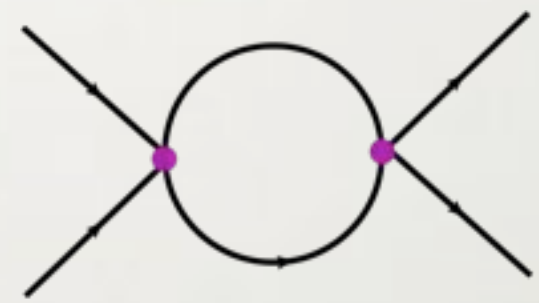
Feynman diagram level:

Non-planar diagrams and quark loops suppressed

Meson level:

$\sim 1/N^{1/2}$

$\sim 1/N$



$\sim \frac{1}{N}\frac{N_f}{N} \sim \frac{1}{N^2}$

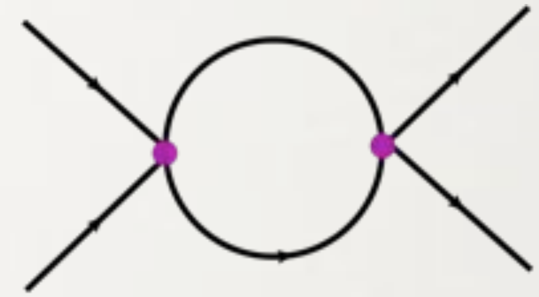
Mesons & glueballs are stable, weakly-interacting for $N_c \gg 1$; meson & glueball loops suppressed.

Large N_c QCD is a classical field theory of (an infinite number of) mesons and glueballs

Good (10-30%) approx. to real world for many observables.

Lightning review of large N_c

't Hooft large N limit: $N \to \infty$, keeping $g^2 N$ fixed, N_f fixed

 $\sim 1/N^{1/2}$  $\sim 1/N$  $\sim \frac{1}{N}\frac{N_f}{N} \sim \frac{1}{N^2}$

Large N_c QCD is a classical field theory of
(an infinite number of) mesons and glueballs

Baryons arise as solitons of meson fields: $M_B \sim 1/g_m \sim N_c$ Witten 1979

Getting baryons with fixed quantum numbers (e.g. isospin, etc)
requires quantizing collective coordinates of the baryon solitons

If we could write down the large N_c `master field theory' of mesons,
we could in principle compute whatever we want for baryons

Unfortunately, we have no idea how to do this for large N_c QCD.

Lightning review of large N_c

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$\sim 1/N^{1/2}$

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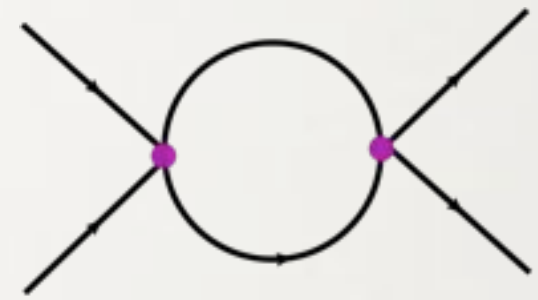

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In theories with gravity duals, we can actually carry out this program, at the cost of moving to a `QCD-like' theory.

But how do we know which lessons carry over to QCD, even qualitatively?

Sometimes, we can calculate in QCD.

If quarks are light, QCD has approximate spontaneously-broken chiral symmetry with powerful implications for low-energy observables

Take $N_f = 2$, consider chiral limit $m_q = 0$.

Low-energy behavior of many observables can be calculated using chiral perturbation theory (ChPT), an effective field theory

Resulting predictions should hold in any theory with same symmetry breaking pattern as QCD!

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$$\mathcal{L}_{\chi PT} = \frac{F_\pi^2}{2} Tr\left[\partial_\mu U \partial^\mu U^\dagger\right] + \ldots, U = \exp\left(i\Pi/F_\pi\right)$$

Higher derivative terms

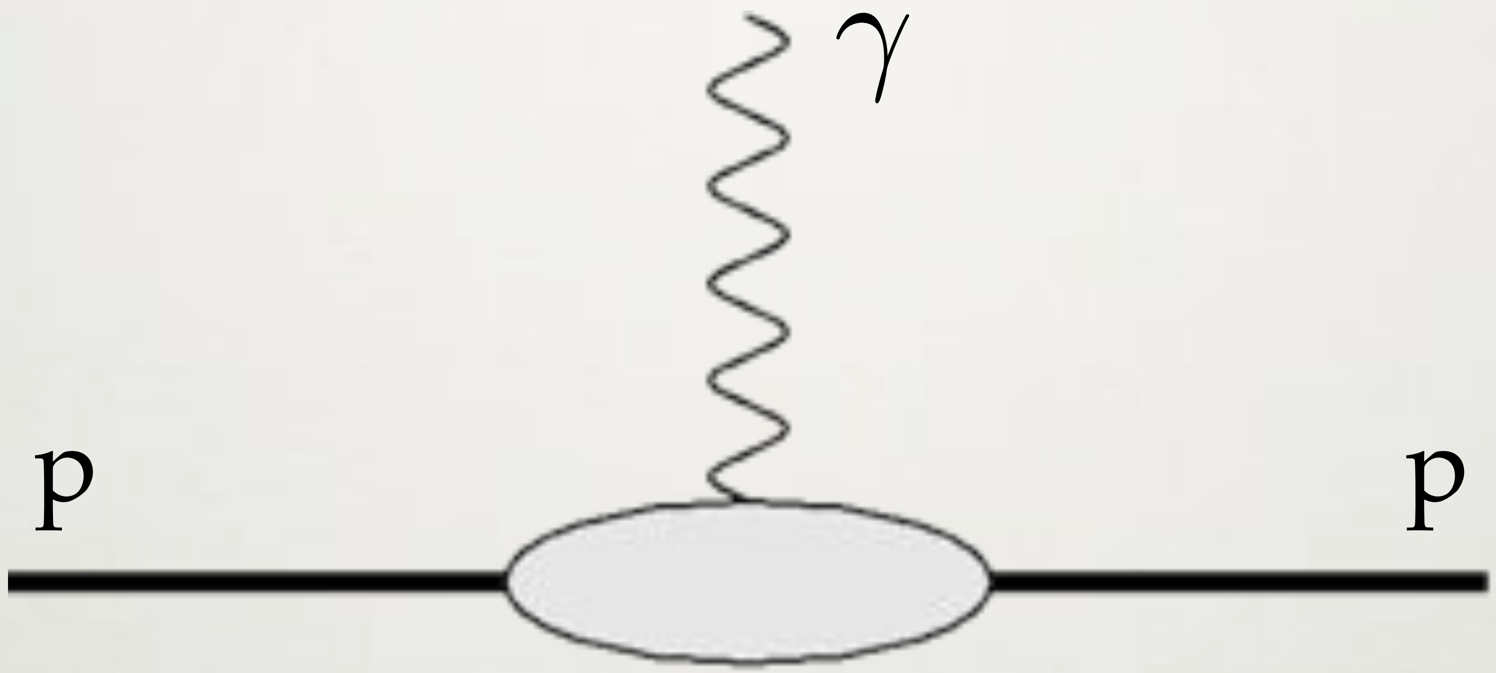
IR properties of QCD dominated by pions, and ChPT gives systematic predictions in terms of a few 'low-energy constants' (LECs) like F_π

All difficulties of `solving' QCD then live inside the LECs

If you want LECs, have to get them from lattice, or AdS/CFT, or ...

Shining a light on baryon properties

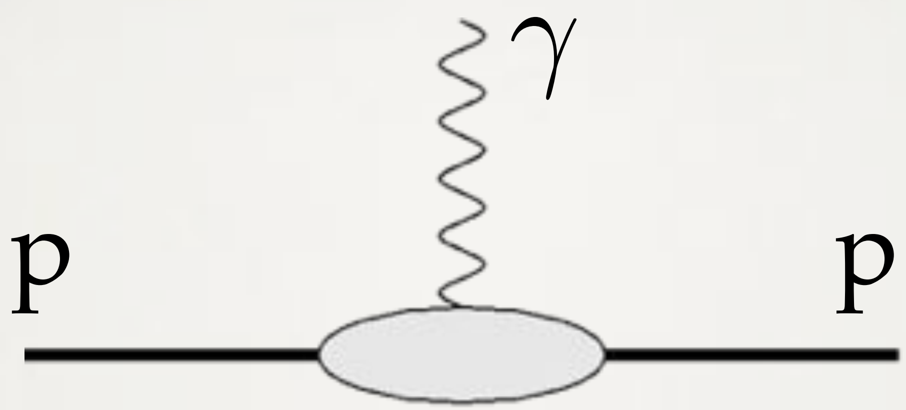
For this talk, consider low-energy EM properties of baryons



Response of proton to EM probes encoded in matrix elements of isoscalar and isovector currents

AC, Cohen, Nielsen 2010

Shining a light on baryon properties



Response of proton to EM probes encoded in matrix elements of isoscalar and isovector currents

$$\tilde{G}_E^{I=0}(r) = \int \frac{d\Omega}{4\pi} \langle p\uparrow | J^0_{I=0} | p\uparrow\rangle$$

$$\tilde{G}_M^{I=0}(r) = \int \frac{d\Omega}{4\pi} \frac{1}{2} \varepsilon_{ij3} \langle p\uparrow | x_i J^j_{I=0} | p\uparrow\rangle$$

$$\tilde{G}_E^{I=1}(r) = \int \frac{d\Omega}{4\pi} \langle p\uparrow | J^{0,a=3}_{I=1} | p\uparrow\rangle$$

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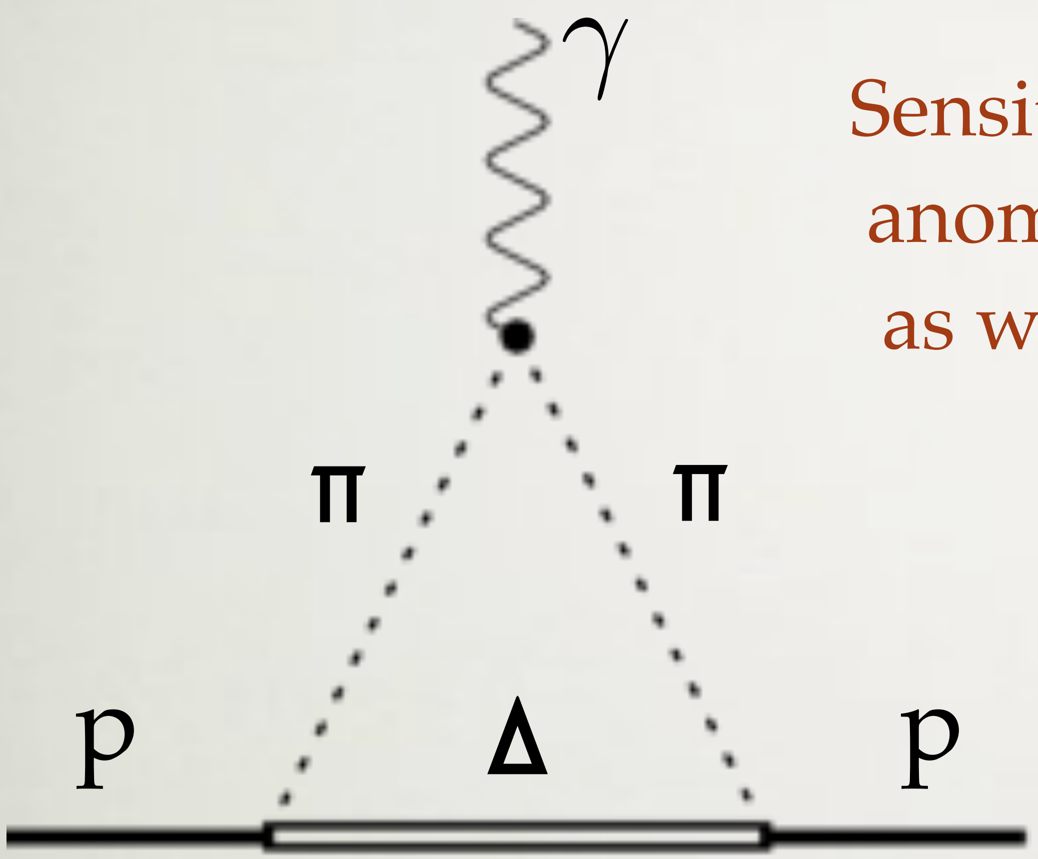
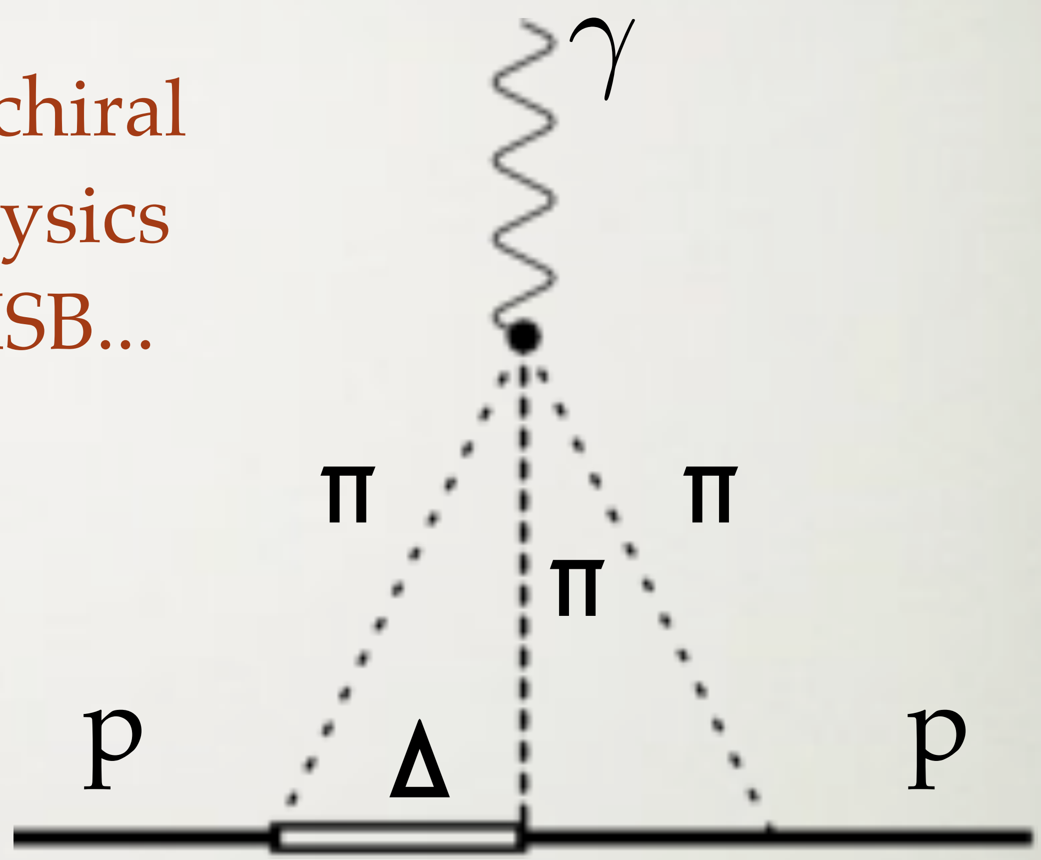
Just Fourier transforms of usual momentum-space form factors

AC, Cohen, Nielsen 2010

Form factors at low energy / large distance



Sensitive to chiral anomaly physics as well as XSB...



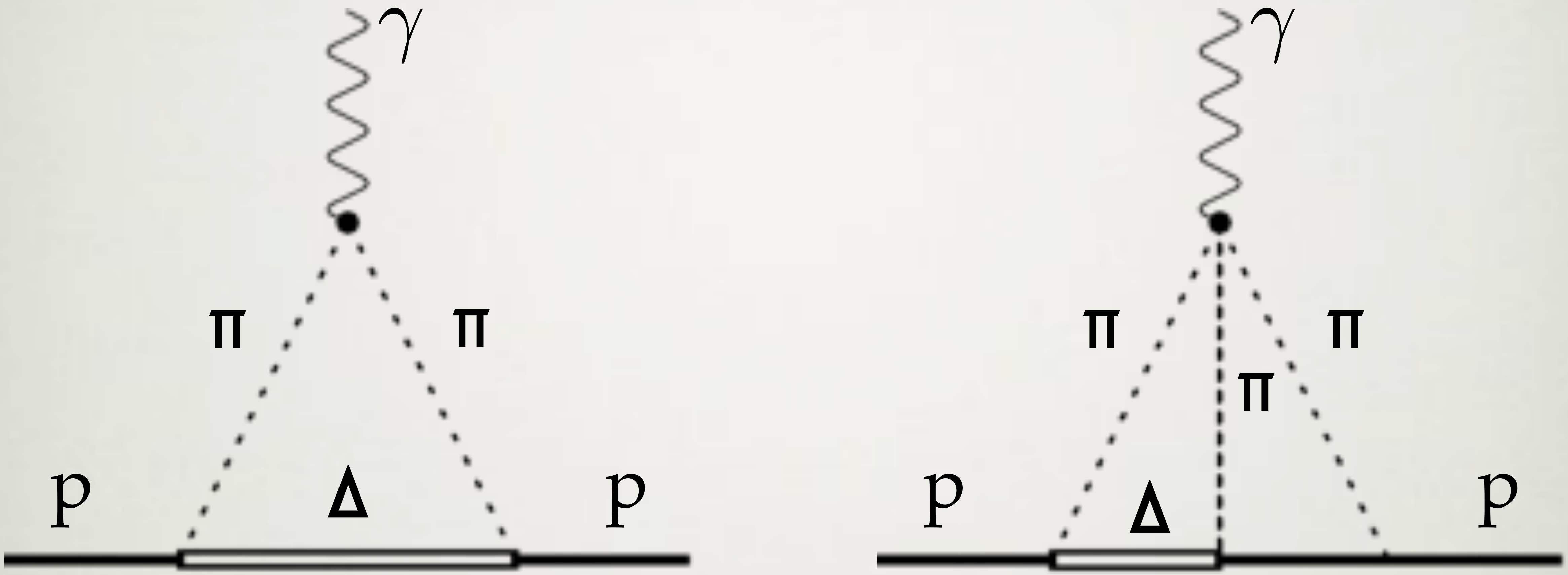
Chiral perturbation theory determines form factors at large r

$$G^E_{I=0} \to \frac{3^3}{2^9\pi^5}\frac{1}{f_\pi^3}\left(\frac{g_A}{f_\pi}\right)^3\frac{1}{r^9} \qquad G^E_{I=1} \to \frac{\Delta}{2^4\pi^2}\left(\frac{g_A}{f_\pi}\right)^2\frac{1}{r^4}$$

$$G^M_{I=0} \to \frac{3\Delta}{2^9\pi^5}\frac{1}{f_\pi^3}\left(\frac{g_A}{f_\pi}\right)^3\frac{1}{r^7} \qquad G^M_{I=1} \to \frac{1}{2^5\pi^2}\left(\frac{g_A}{f_\pi}\right)^2\frac{1}{r^4}$$

The ratio

AC, Cohen, Nielsen 2010
Cohen, Krejcirik 1201.5389



Can put together very simple probe of this physics:

$$\lim_{r\to\infty} r^2 \frac{\tilde{G}_E^{I=0}\tilde{G}_E^{I=1}}{\tilde{G}_M^{I=0}\tilde{G}_M^{I=1}} = 18$$

Should be satisfied by **any** QCD-like theory which is anything like QCD!

The ratio

Cohen, Krejcirik 1201.5389

Ratio is sensitive to the order of limits!

$$\lim_{r\to\infty}\lim_{N_c\to\infty} r^2 \frac{\tilde{G}_E^{I=0}\tilde{G}_E^{I=1}}{\tilde{G}_M^{I=0}\tilde{G}_M^{I=1}} = 18$$

$$\lim_{N_c\to\infty}\lim_{r\to\infty} r^2 \frac{\tilde{G}_E^{I=0}\tilde{G}_E^{I=1}}{\tilde{G}_M^{I=0}\tilde{G}_M^{I=1}} = 9$$

As it happens, all the soliton-based baryons models (that I'm aware of) take large N_c first...

AC, Cohen, Nielsen 2010

The ratio as a probe of baryon models

Form factor ratio is a prediction of low-energy QCD, so should be obeyed by all baryon models that correctly build in chiral physics.

Far-IR is only part of QCD we actually understand analytically
- could take view that models better match at least that much...

AC, Cohen, Nielsen 2010

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Some holographic models also work, e.g.
Pomarol-Wulzer holographic baryon model

AC, Cohen, Nielsen 2010

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What about the most popular QCD-like theory with a gravity dual, the Sakai-Sugimoto model?

Surprisingly, previous calculations suggested a problem:

$$\lim_{r\to\infty} r^2 \frac{\tilde{G}_E^{I=0}\tilde{G}_E^{I=1}}{\tilde{G}_M^{I=0}\tilde{G}_M^{I=1}} = \frac{\lambda\sqrt{40/3}}{\pi\rho_1^2}$$

But improved treatment of model turns out to get ratio right.

Maldacena, Witten, Gubser, Klebanov, Polyakov

Gauge/gravity duality summary

Amazing claim: (some?) gauge theories are dual to string theories

Maldacena, Witten, Gubser,
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- Expansion in $1/N_c$ and $1/\lambda$ map to expansions in g_s and α'. When N_c and λ are large, string theory simplifies to classical gravity on space with 4D boundary + matter fields.
  - Need field theory with a strong-coupling fixed point - not like QCD...

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  - Generating functional of gauge theory identified with exponential of on-shell bulk action.

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- Boundary values of bulk fields act as sources for field theory operators.
  - Generating functional of gauge theory identified with exponential of on-shell bulk action.
- Classical calculations in the gravity theory give information on strongly-coupled quantum physics in the dual field theory.

Sakai-Sugimoto model: field theory side

Start with N=2 $SU(N_c)$ SYM theory in 4+1 D: x_0, x_1,x_2,x_3, x_4

+ N_f flavors of fundamental matter on 3+1D subpace: x_0, x_1,x_2,x_3

Parameters: λ_5, N_c

Compactify x_4 on a circle of size R with anti-periodic BCs for fermions, break SUSY at scale $M_{KK}=1/R$

Everything except gluons and quarks gets mass ~ M_{KK}

$\lambda[M_{KK}] << 1$

$\lambda = \lambda_5 M_{KK}$

$\lambda[M_{KK}] >> 1$

Superpartners decouple. In IR, pure QCD, confining in the usual way at $\Lambda_{QCD} << M_{KK}$

No tractable dual

No decoupling, get glueballs, mesons + lots of exotic states, all at the scale M_{KK}

Gravity dual description

Witten 1998, Sakai+Sugimoto 2004

Sakai-Sugimoto model: gravity side

	x_0	x_1	x_2	x_3	$(x_4 = \tau)$	x_5	x_6	x_7	x_8	x_9
N_c D4	$\times$	$\times$	$\times$	$\times$	$\times$					
N_f D8, $\overline{\text{D8}}$	$\times$	$\times$	$\times$	$\times$		$\times$	$\times$	$\times$	$\times$	$\times$

D4 branes replaced by geometry, giving metric and dilaton...

$$ds^2_{9+1} = \left(\frac{U}{R}\right)^{3/2} \left(\eta_{\mu\nu} dx^\mu dx^\nu + f(U) d\tau^2\right) + \left(\frac{R}{U}\right)^{3/2} \left[\frac{dU^2}{f(U)} + U^2 d\Omega_4^2\right]$$

$$e^\Phi = g_s \left(\frac{U}{R}\right)^{3/4}, \quad f(U) = 1 - \frac{U_{\mathrm{KK}}^3}{U^3}, \quad F_4 = dC_3 = \frac{2\pi N_c}{V_4} \epsilon_4$$

$$U_{\mathrm{KK}} \sim \frac{1}{M_{\mathrm{KK}}}$$

Field theory `lives' on boundary at large U

Witten 1998, Sakai+Sugimoto 2004

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Field theory \`lives' on boundary at large *U*

Probe D8, anti-D8 branes placed at antipodal points on *τ* circle at large *U*, then rest of embedding determined by D8 brane EoM

D8, anti-D8 branes join in the IR

Geometric realization of chiral symmetry breaking!

Sakai-Sugimoto model: the action

Gauge fields on D8 branes source for $U(N_f)_L \times U(N_f)_R$ currents

Action for flavor gauge fields A_M in gravity theory
encodes meson interactions in the field theory

Gives the large N master field theory for mesons!

Turns out only A_μ, A_z couple to mesons with QCD quantum numbers, so set S^4 components of gauge fields to zero

Standard to work with new coordinate: $U^3 = U^3_{\mathrm{KK}} + U_{\mathrm{KK}} z^2$

D8 brane embedding function single-valued in terms of z

Gauge field behavior at large $+z$ and $-z$ sources $U(N_f)_L$ and $U(N_f)_R$.

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Gauge fields behavior at large +z and -z sources $U(N_f)_L$ and $U(N_f)_R$.

$$S = -\kappa \int_{-\infty}^{\infty} d^4x dz \, \mathrm{Tr}\left[\frac{1}{2} h(z) \mathcal{F}_{\mu\nu}^2 + k(z) \mathcal{F}_{\mu z}^2\right] + \frac{N_c}{24\pi^2} \int_{\mathrm{M}_5} \omega_5[\mathcal{A}]$$

$\kappa = \lambda N_c / (216\pi^3)$ $\quad$ $k(z) = 1 + z^2, \; h(z) = k(z)^{-1/3}$ $\quad$ CS 5-form

Action reliable only if F_{MN} varies slowly compared to $1/l_s$

Witten 1998, Sakai+Sugimoto 2004

Sakai-Sugimoto model: baryons

	x_0	x_1	x_2	x_3	(x_4)	x_5	x_6	x_7	x_8	x_9
N_c D4	$\times$	$\times$	$\times$	$\times$	$\times$					
N_f D8, $\overline{\text{D8}}$	$\times$	$\times$	$\times$	$\times$		$\times$	$\times$	$\times$	$\times$	$\times$
N_B D4	$\times$						$\times$	$\times$	$\times$	$\times$

Baryons in field theory map to D4 branes wrapping the S^4

In our case, expect baryonic D4 branes to dissolve in the D8 branes, turn into solitons carrying unit instanton number charge:

$$Q = \frac{1}{8\pi^2} \int_{R^3 \times I} \mathrm{tr} F \wedge F$$

Instanton <-> baryon relation, Atiyah+Manton 1989

Because of non-trivial warp factors and CS term, should not expect the soliton to be self-dual in general.

Standard approach to finding soliton solutions

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- The equations of motion for static soliton configuration hard to solve.
- Assume (x_i , z) SO(4)-symmetric flat-space instanton variational ansatz. Two variational parameters: soliton position and size ρ.
  - Warp factors in YM part of action make soliton sit at $z=0$ and drive it to small size.
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  - Competition between CS and YM terms sets $\rho \sim 1/\lambda^{1/2}$
- Once static configuration is found (analytically!), usual collective coordinate quantization performed to identify baryons with specific quantum numbers.
- Some arguments that variational ansatz becomes exact near $r=0$, $z=0$ at large λ.
  - Space becomes almost flat near $z\sim 0$...
- Control over solution tricky: $l_s \sim 1/\lambda^{1/2}$...

AC, Cohen, Nielsen 2010

Results of standard approach

$$\lim_{r\to\infty} r^2 \frac{\tilde{G}_E^{I=0}\tilde{G}_E^{I=1}}{\tilde{G}_M^{I=0}\tilde{G}_M^{I=1}} = \frac{\lambda\sqrt{40/3}}{\pi\rho_1^2}$$

number related to vector meson mass

But wait a minute! SS model builds in anomaly and XSB physics - so how could it *possibly* not get the form factor ratio right?

Answer: Use of EFT predictions assumes that all LECs are of ‘natural’ size, so that derivative expansion works.

For a generic theory, extremely plausible assumption!

But SS model has an extra parameter compared to a generic theory: λ.

AC, Cohen, Nielsen 2010

Results of standard approach

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Flat-space-instanton analysis suggests $g^2_{mBB} \sim N_c/\lambda,\, M_B \sim \lambda N_c$

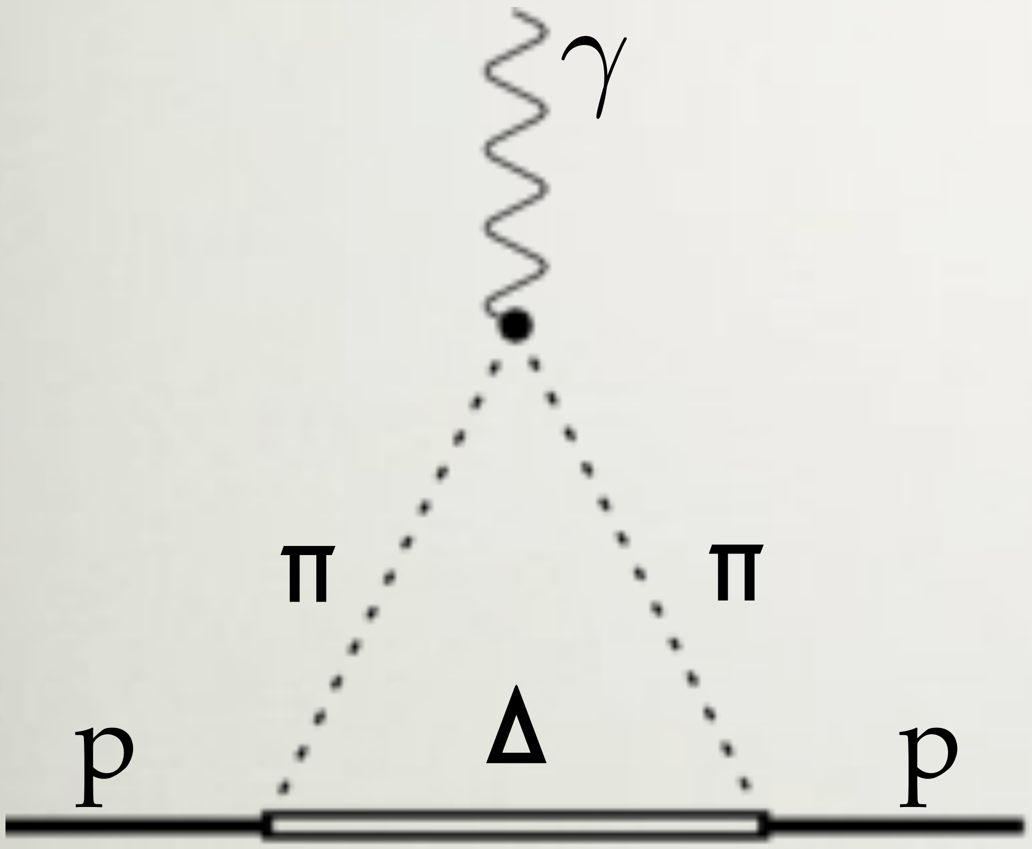
$\sim g^2_{mBB}$ vs. M_B

Apparently meson loops in baryons suppressed in SS model...

Contrast with QCD: $M_B \sim N_c,\, g_{mBB} \sim N_c^{1/2}$

Meson loops in baryons leading order

Large N QCD, large R



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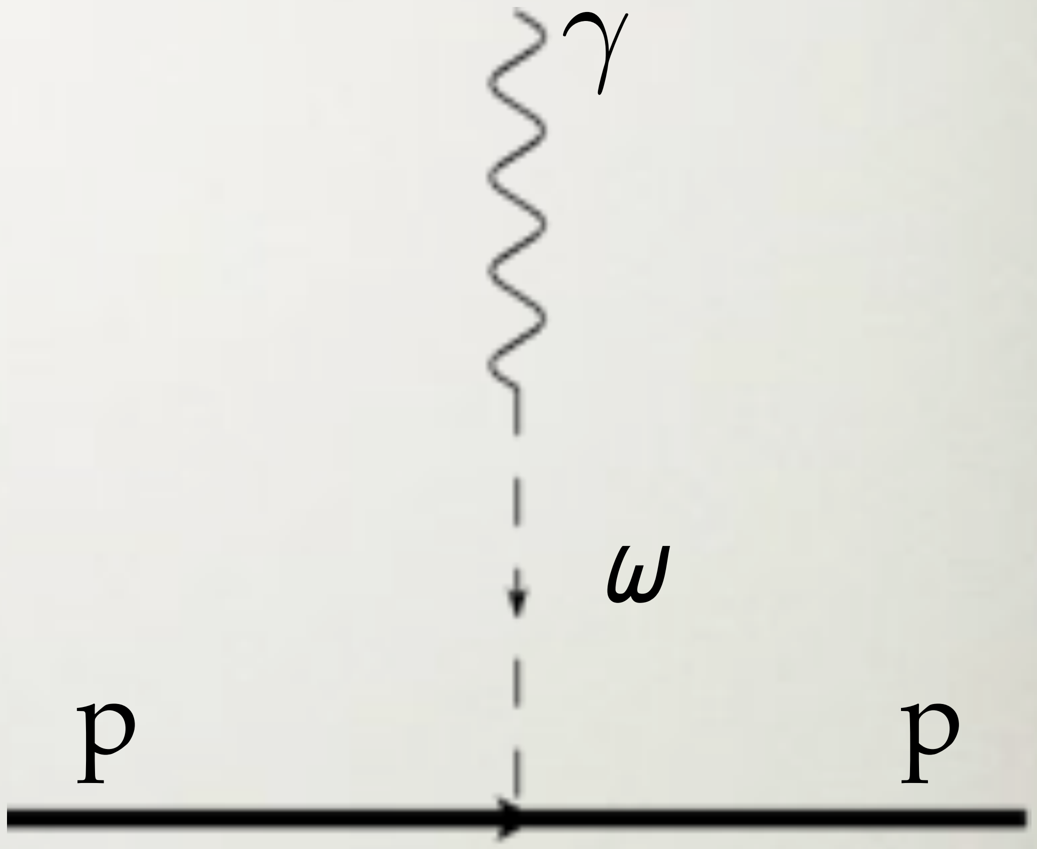
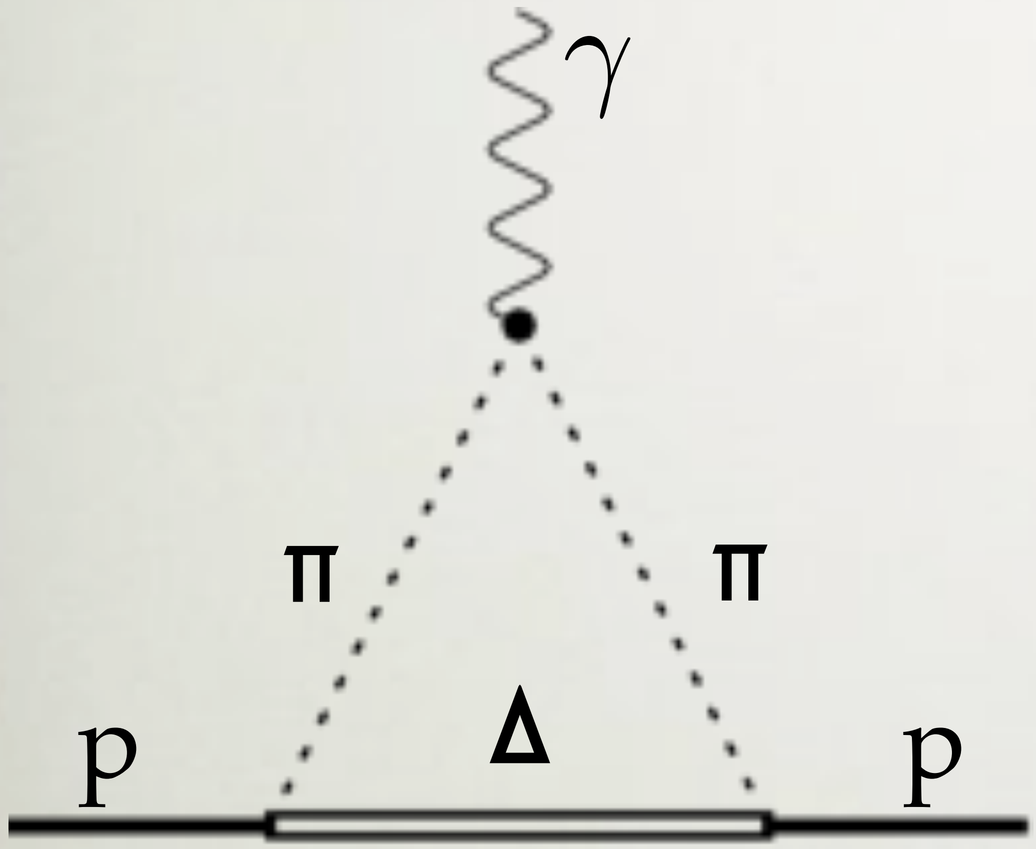


Pions massless =>
Power law in r

vector mesons massive =>
Exponential in r

Apparent behavior of SS model, large R



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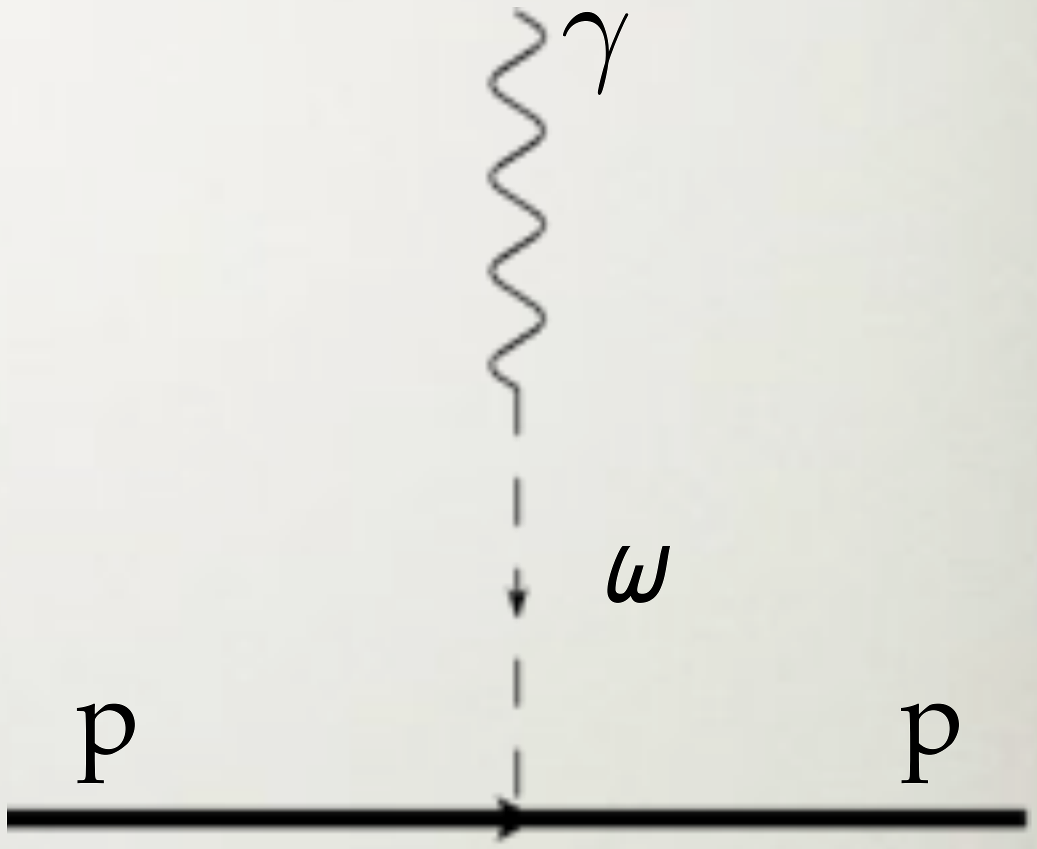

Pions massless =>
Power law in *r*

Down by *1/λ* at large *λ*

vector mesons massive =>
Exponential in *r*

Leading order at large *λ*

Consequences of standard approach

$$\lim_{r\to\infty} r^2 \frac{\tilde{G}_E^{I=0}\tilde{G}_E^{I=1}}{\tilde{G}_M^{I=0}\tilde{G}_M^{I=1}} = \frac{\lambda\sqrt{40/3}}{\pi\rho_1^2}$$

Appears that in SS model large *r* limit and large *λ* limits don't commute

But *1/λ* corrections aren't calculable in the gravity theory
- can't reverse the order of limits.

Sounds bad for the model: very different infrared properties than QCD...

We'd need to work with flavor gauge field
action to all orders in α′ to do better

But is this really right answer?

AC, T. Ishii 2011

Baryons in the SS model, from scratch

How to approach baryons in SS model
without assuming instanton approximation?

In general would need to solve EoMs numerically.

Explicit numerical solution would be in a box with cutoffs on r and z.

Metric breaks flat-space $SO(4)$ symmetry combining x_i, z,
but $YM+CS$ action has an $SO(3)$ spatial rotation symmetry.

Need to work with most general ansatz with $SO(3)$ symmetry.

Also need to find solutions for a slowly-rotating solution to do collective coordinate quantization and pick out e.g. the proton

Given such a solution, we could read off its large r behavior, and thence obtain large behavior of form factors

AC, T. Ishii 2011

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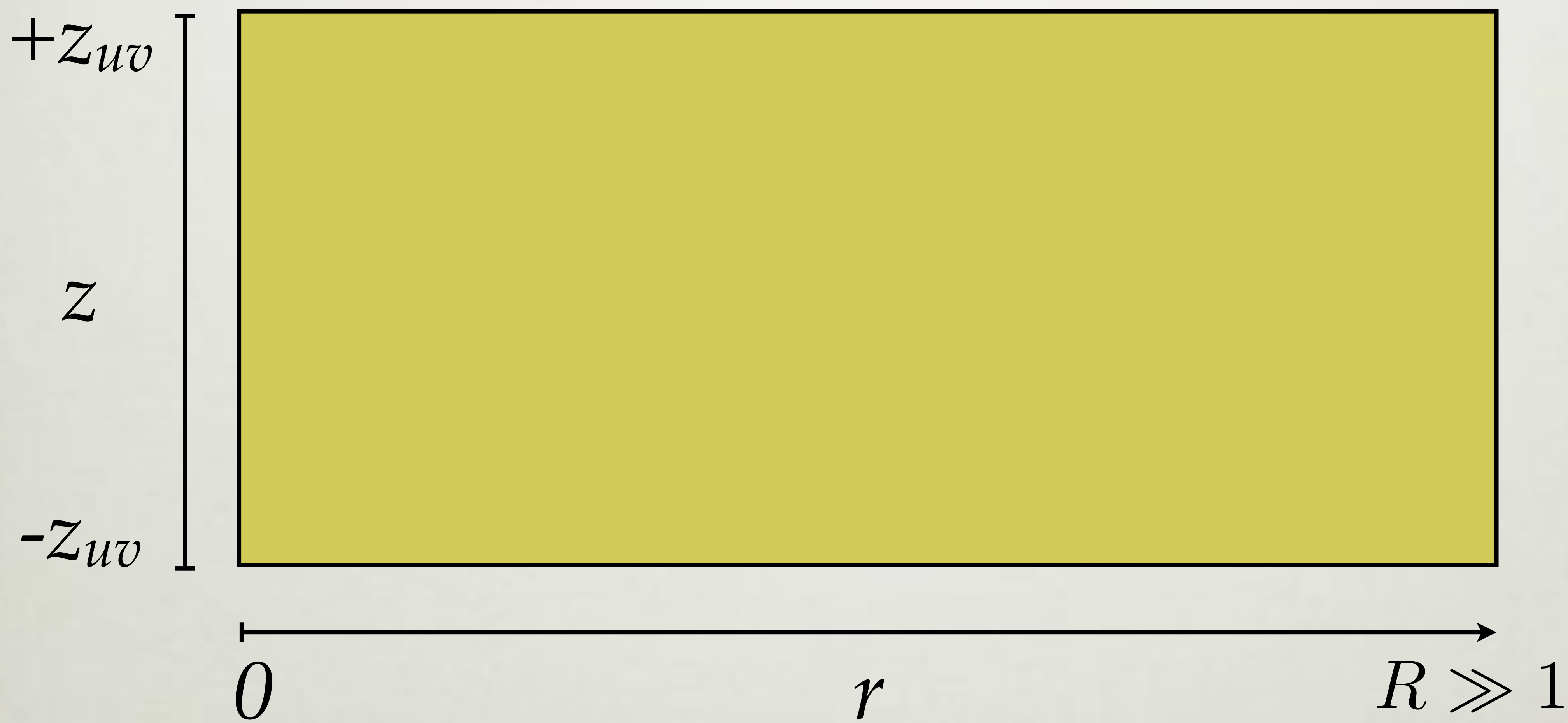
In general would need to solve EoMs numerically.

Explicit numerical solution would be in a box with cutoffs on r and z.



In fact there are other reasons one may entertain thought of z cutoff

Working in a box

$$1/\lambda \ll U M_{\mathrm{KK}} \ll N_c^{4/3}/\lambda$$

Itzhaki et al 1998

Dilaton blows up at large enough U - gravity approximation breaks down

Since large U behavior of bulk fields determines
field theory properties, might want to be careful...

An abundance of caution would suggest putting in a UV
cutoff on z, $|z| < z_{uv}$, and then removing it at the end.

Cutoff would also allow tracking holographic renormalization issues.

Gauge-gravity duality trades UV divergences in field theory
for volume divergences from z integral in gravity theory

Standard approach: work with a cutoff on z, identify
divergences, add boundary terms on z_{uv} to subtract them
off, remove cutoff at the end of a calculation.

Sakai-Sugimoto model: the currents

$$S = -\kappa \int_{-z_{\rm uv}}^{+z_{\rm uv}} d^4x dz \,\mathrm{Tr}\left[\frac{1}{2}h(z)\mathcal{F}_{\mu\nu}^2 + k(z)\mathcal{F}_{\mu z}^2\right] + \frac{N_c}{24\pi^2}\int_{\mathbb{R}^4\times I}\omega_5 + S_b$$

S_b is some gauge-invariant function g of gauge field, determined by rules of holographic renormalization

$$S_b = \int d^4x\, g\left(\mathcal{F}_{\mu\nu}^2(z_{\rm uv}), \mathcal{F}_{\mu z}^2(z_{\rm uv});\ z_{\rm uv}\right)$$

$$J^a_{\mu,I=1} = -\kappa\mathcal{R}_{I=1}(z_{\rm uv})\left[k(z)F^a_{\mu z}\right]_{z=-z_{\rm uv}}^{z=z_{\rm uv}}$$ SU(2) part

Definition of currents in terms of bulk fields may get multiplicative renormalization from S_b

$$J_{\mu,I=0} = -\kappa\mathcal{R}_{I=0}(z_{\rm uv})\left[k(z)\hat{F}_{\mu z}\right]_{z=-z_{\rm uv}}^{z=z_{\rm uv}}$$ U(1) part

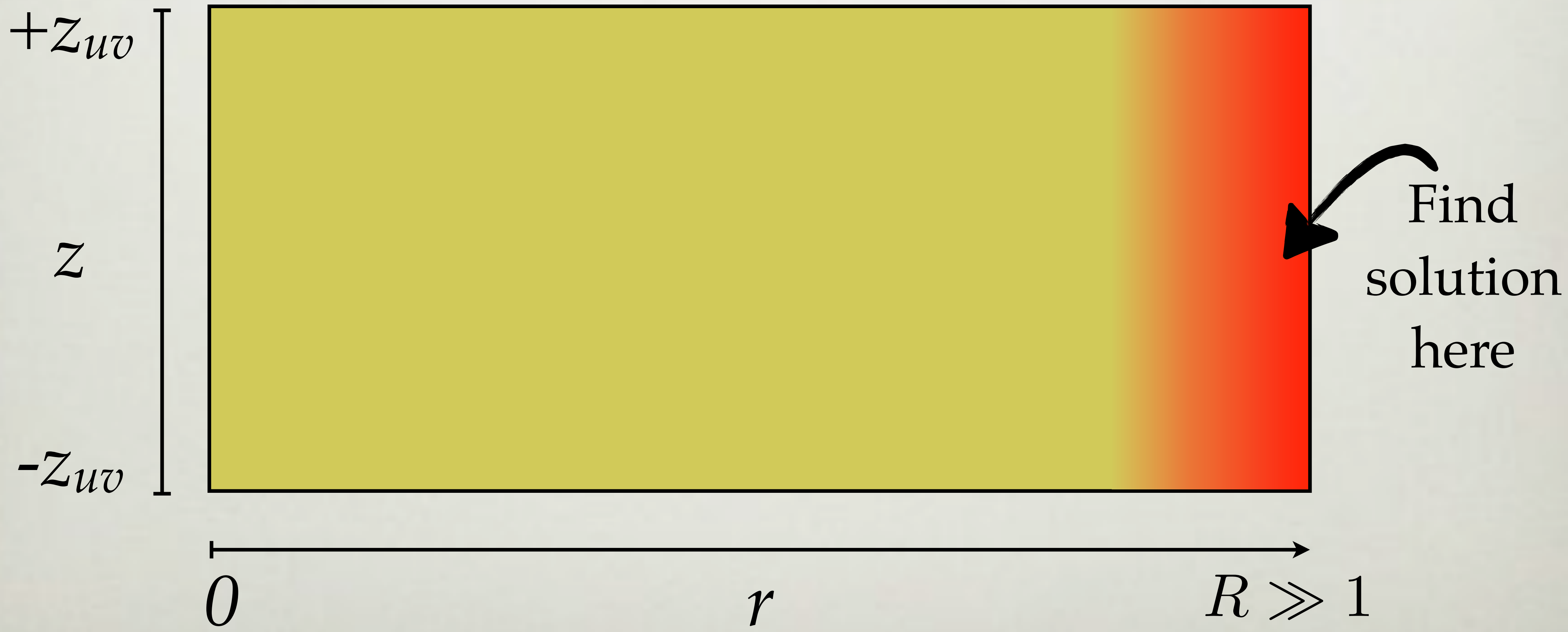
Seek ratios of matrix elements of these currents in baryons

Any renormalization of currents due to S_b will cancel in ratio!

Baryons in the SS model, from scratch

Given a numerical solution, we could read off its large r behavior, and thence obtain large behavior of form factors

Our approach: just solve for large r asymptotics directly.



SO(3) symmetric ansatz

First step: write down most general static ansatz with *SO(3)* symmetry

$$A_j^a = \frac{\phi_2 + 1}{r^2} \epsilon_{jak} x_k + \frac{\phi_1}{r^3} [\delta_{ja} r^2 - x_j x_a] + A_r \frac{x_j x_a}{r^2}$$

$$A_z^a = A_z \frac{x^a}{r}, \quad \hat{A}_0 = s$$

Witten 1977,
Forgacs+Manton 1980
Pomarol-Wulzer 2009

Leaves five functions of two variables r, z to be determined

Action now reduces to a 2D Abelian Higgs model.

$$S = 16\pi\kappa \int_0^\infty dr \int_{-z_{\mathrm{uv}}}^{z_{\mathrm{uv}}} dz \left[h(z)|D_r \phi|^2 + k(z)|D_z \phi|^2 + \frac{1}{4} r^2 k(z) F_{\mu\nu}^2 \right.$$
$$\left. + \frac{1}{2r^2} h(z)(1 - |\phi|^2)^2 - \frac{1}{2} r^2 \left(h(z)(\partial_r s)^2 + k(z)(\partial_z s)^2 \right) \right]$$
$$+ 16\pi\kappa \frac{27\pi}{2\lambda} \int_0^\infty dr \int_{-z_{\mathrm{uv}}}^{z_{\mathrm{uv}}} dz \, s \, \epsilon^{\mu\nu} \left[\partial_\mu (-i\phi^* D_\nu \phi + h.c) + F_{\mu\nu} \right] + S_b$$

Boundary conditions

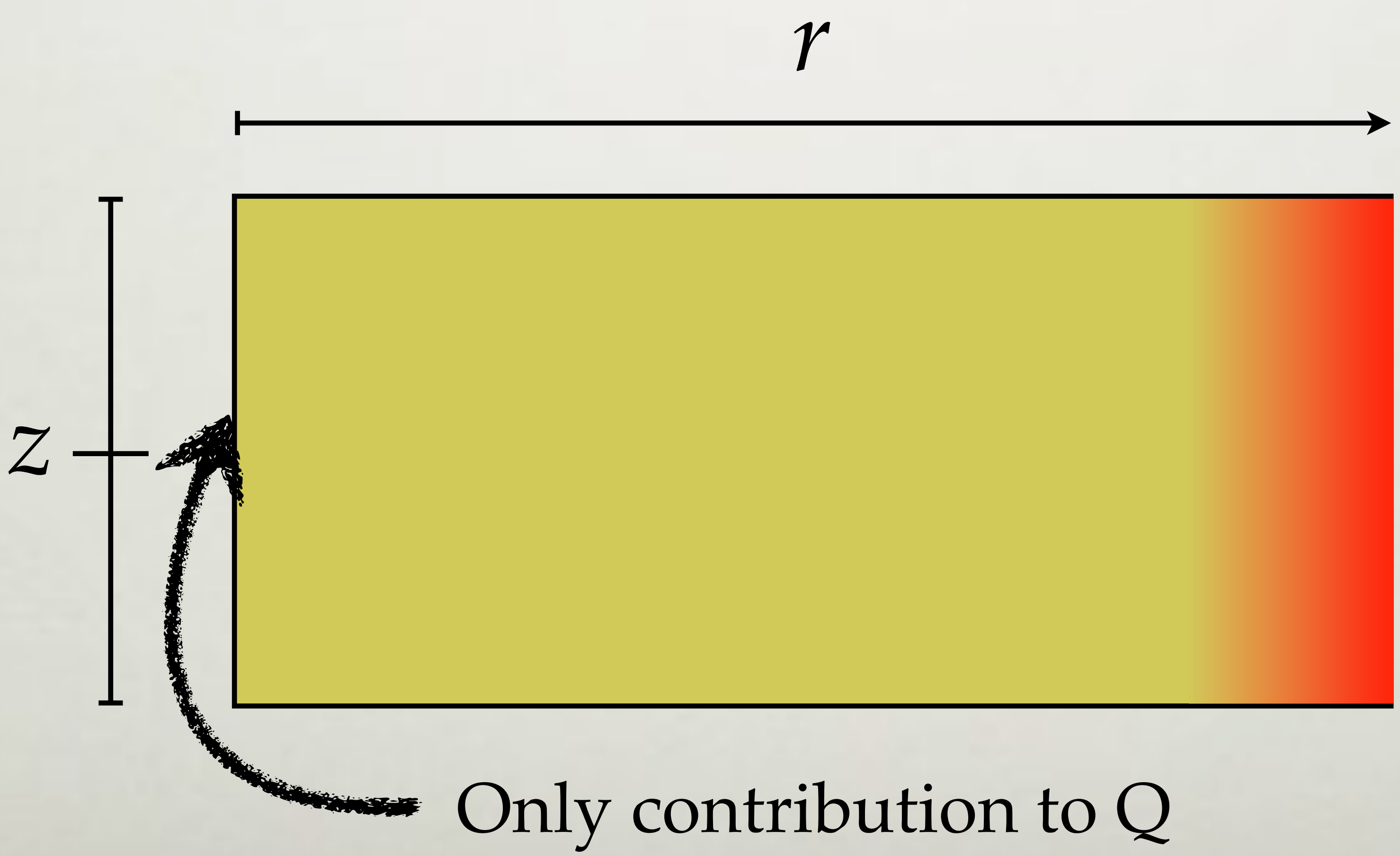
$$Q = \frac{1}{4\pi}\int drdz\ \left(\epsilon^{\mu\nu}\partial_\mu\left[-i\phi^* D_\nu\phi + h.c.\right] + \epsilon^{\mu\nu}F_{\mu\nu}\right)$$

Q gets contributions from the four boundaries of solution domain.

Choose BCs such that Q gets a contribution only from r=0 boundary.

Just because it is convenient for asymptotic analysis!

Q is gauge-invariant, but each individual boundary contribution is not.



Boundary conditions

$$Q = \frac{1}{4\pi} \int drdz \, \left(\epsilon^{\mu\nu} \partial_\mu \left[-i\phi^* D_\nu \phi + h.c. \right] + \epsilon^{\mu\nu} F_{\mu\nu} \right)$$

Q gets contributions from the four boundaries of solution domain.

Choose BCs such that Q gets a contribution only from r=0 boundary.

$r \to \infty$	$r = 0$	$z = \pm z_{\mathrm{uv}}$
$\phi_1 = 0$	$\phi_1 = \sin\left(\frac{\pi z}{z_{\mathrm{uv}}}\right)$	$\phi_1 = 0$
$\phi_2 = -1$	$\phi_2 = -\cos\left(\frac{\pi z}{z_{\mathrm{uv}}}\right)$	$\phi_2 = -1$
$A_z = 0$	$A_z = \frac{\pi}{2 z_{\mathrm{uv}}}$	$\partial_z A_z = 0$
$\partial_r A_r = 0$	$\partial_r A_r = 0$	$A_r = 0$
$s = 0$	$s = 0$	$s = 0$

Large r static solution

Analytically solved EoMs order by order in *1/r*

$$\phi_1(r,z)=\sum_{n=1}^{\infty}\phi_1^{(n)}(z)\frac{1}{r^n},\quad \phi_2(r,z)=-1+\sum_{n=1}^{\infty}\phi_2^{(n)}(z)\frac{1}{r^n},$$

$$A_z(r,z)=\sum_{n=1}^{\infty}A_z^{(n)}(z)\frac{1}{r^n},\quad A_r(r,z)=\sum_{n=1}^{\infty}A_r^{(n)}(z)\frac{1}{r^n},\quad s(r,z)=\sum_{n=1}^{\infty}s_r^{(n)}(z)\frac{1}{r^n}$$

EoM PDEs turn into ODEs determining *z* dependence

Solutions self-consistently show that EoMs can be linearized in a power series in 1/r for large r.

Large r asymptotic solutions must depend on the global solution to the full boundary value problem

Indeed, count of BCs shows that large r solution fixed up to one unknown constant of integration

Large r static solution

Analytically solved EoMs order by order in *1/r*

EoM PDEs turn into ODEs determining *z* dependence

$$\phi_1 = \frac{\beta(z - \frac{z_{\rm uv}\tan^{-1}(z)}{\tan^{-1}(z_{\rm uv})})}{r^2} - \frac{\beta\left(-3\left(-1+z^2\right)z_{\rm uv}\tan^{-1}z + z\left(\left(-3+z^2+2z_{\rm uv}^2\right)\tan^{-1}z_{\rm uv} + 3z_{\rm uv}\log\left[\frac{1+z^2}{1+z_{\rm uv}^2}\right]\right)\right)}{\tan^{-1}[z_{\rm uv}]r^4},$$

$$\phi_2 = -1 + \beta^2\frac{\frac{1}{2}\left(z^2+z_{\rm uv}^2\right) - \frac{z z_{\rm uv}\tan^{-1}[z]}{\tan^{-1}[z_{\rm uv}]}}{r^4},$$

$$A_z = \frac{\beta}{r^2} + \frac{\beta\left(6z z_{\rm uv}\tan^{-1}[z] + \left(3-3z^2-2z_{\rm uv}^2\right)\tan^{-1}[z_{\rm uv}] - 3z_{\rm uv}\left(1+\log\left[\frac{1+z^2}{1+z_{\rm uv}^2}\right]\right)\right)}{\tan^{-1}[z_{\rm uv}]r^4},$$

$$A_r = \frac{-2z\beta + \frac{2z_{\rm uv}\beta\tan^{-1}[z]}{\tan^{-1}[z_{\rm uv}]}}{r^3} + \frac{4\beta\left(-3\left(-1+z^2\right)z_{\rm uv}\tan^{-1}[z] + z\left(\left(-3+z^2+2z_{\rm uv}^2\right)\tan^{-1}[z_{\rm uv}] + 3z_{\rm uv}\log\left[\frac{1+z^2}{1+z_{\rm uv}^2}\right]\right)\right)}{\tan^{-1}[z_{\rm uv}]r^5},$$

$$s = \frac{\beta^3\gamma z_{\rm uv}^3(\tan^{-1}(z)^4 - 6\tan^{-1}(z)^2\tan^{-1}(z_{\rm uv})^2 + 5\tan^{-1}(z)\tan^{-1}(z_{\rm uv})^3)}{2\tan^{-1}(z_{\rm uv})^3 r^9}.$$

Large r static solution

Analytically solved EoMs order by order in *1/r*

EoM PDEs turn into ODEs determining *z* dependence

$$\phi_1 = \frac{\beta(z - \frac{z_{\mathrm{uv}}\tan^{-1}(z)}{\tan^{-1}(z_{\mathrm{uv}})})}{r^2} - \frac{\beta\left(-3\left(-1+z^2\right)z_{\mathrm{uv}}\tan^{-1}z + z\left(\left(-3+z^2+2z_{\mathrm{uv}}^2\right)\tan^{-1}z_{\mathrm{uv}} + 3z_{\mathrm{uv}}\log\left[\frac{1+z^2}{1+z_{\mathrm{uv}}^2}\right]\right)\right)}{\tan^{-1}[z_{\mathrm{uv}}]r^4},$$

$$\phi_2 = -1 + \beta^2\frac{\frac{1}{2}\left(z^2+z_{\mathrm{uv}}^2\right) - \frac{zz_{\mathrm{uv}}\tan^{-1}[z]}{\tan^{-1}[z_{\mathrm{uv}}]}}{r^4},$$

$$A_z = \frac{\beta}{r^2} + \frac{\beta\left(6zz_{\mathrm{uv}}\tan^{-1}[z] + \left(3-3z^2-2z_{\mathrm{uv}}^2\right)\tan^{-1}[z_{\mathrm{uv}}] - 3z_{\mathrm{uv}}\left(1+\log\left[\frac{1+z^2}{1+z_{\mathrm{uv}}^2}\right]\right)\right)}{\tan^{-1}[z_{\mathrm{uv}}]r^4},$$

$$A_r = \frac{-2z\beta + \frac{2z_{\mathrm{uv}}\beta\tan^{-1}[z]}{\tan^{-1}[z_{\mathrm{uv}}]}}{r^3} + \frac{4\beta\left(-3\left(-1+z^2\right)z_{\mathrm{uv}}\tan^{-1}[z] + z\left(\left(-3+z^2+2z_{\mathrm{uv}}^2\right)\tan^{-1}[z_{\mathrm{uv}}] + 3z_{\mathrm{uv}}\log\left[\frac{1+z^2}{1+z_{\mathrm{uv}}^2}\right]\right)\right)}{\tan^{-1}[z_{\mathrm{uv}}]r^5},$$

$$s = \frac{\beta^3\gamma z_{\mathrm{uv}}^3(\tan^{-1}(z)^4 - 6\tan^{-1}(z)^2\tan^{-1}(z_{\mathrm{uv}})^2 + 5\tan^{-1}(z)\tan^{-1}(z_{\mathrm{uv}})^3)}{2\tan^{-1}(z_{\mathrm{uv}})^3r^9}.$$

β is an integration constant that would be fixed by matching to a full solution

Since full solution depends on Q, λ and z_{uv}, $\beta = \beta[Q,\lambda,z_{uv}]$ as well.

Assuming stable non-trivial global solution exists - only expect this for $Q = 1$.

Need for holographic renormalization?

On-shell action takes form

$$S - S_b = \int dtdr \left\{ \frac{6(z_{\rm uv}\beta)^2}{\tan^{-1}(z_{\rm uv}) r^4} + \mathcal{O}(1/r^8) \right\}$$
$$\to \int dtdr \left\{ \frac{12(z_{\rm uv}\beta)^2}{\pi r^4} + \mathcal{O}(1/r^8) \right\}$$

Interpretation depends on dependence of $\boldsymbol{\beta}$ on z_{uv}.

$\beta \sim 1/z_{uv}^n, n \geq 1$

On-shell action finite even without S_b

$\beta \sim \log(z_{\rm uv})$, or $\beta \sim z_{\rm uv}^n, n \geq 0$

Contribution from S_b essential to make on-shell action finite

Need explicit numerical soliton solution to say more...

But fortunately for modest goal here, can proceed without solving the complicated numerical problem.

Form factor *ratio* not sensitive to renormalization of currents

Rotating soliton solution and collective coordinate quantization

Give soliton small constant angular velocity, action becomes

$$\mathcal{L} = -M + \frac{\Lambda}{2} k_a k^a$$ k_a: collective coordinates

Rigid rotor: mass $M \sim \lambda N_c$, moment of inertia $\Lambda \sim \lambda N_c$

Rotating solution described by previous five functions, plus seven new function of (r,z), whose asymptotic form is known but unilluminating...

$SO(3)$ symmetry preserved so long as soliton does not deform under rotation

Rotating soliton solution and collective coordinate quantization

Give soliton small constant angular velocity, action becomes

$$\mathcal{L} = -M + \frac{\Lambda}{2} k_a k^a$$

Quantization of collective coordinates proceeds in standard way, plugging results into *I=0* and *I=1* currents gives form factor expressions

$$\tilde{G}_E^{I=0}(r) = -\frac{4}{N_c}\kappa\left[k(z)\partial_z s\right]_{-z_{\rm uv}}^{z_{\rm uv}} ,$$

$$\tilde{G}_M^{I=0}(r) = -\frac{2}{3N_c\Lambda}\kappa\left[r k(z)\partial_z Q\ \right]_{-z_{\rm uv}}^{z_{\rm uv}} ,$$

$$\tilde{G}_E^{I=1}(r) = \frac{2}{3\Lambda}\kappa\left[k(z)(\partial_z v - 2(\partial_z \chi_2 - A_z \chi_1))\right]_{-z_{\rm uv}}^{z_{\rm uv}} ,$$

$$\tilde{G}_M^{I=1}(r) = -\frac{4}{9}\kappa\left[k(z)(\partial_z \phi_2 - A_z \phi_1)\right]_{-z_{\rm uv}}^{z_{\rm uv}}$$

Large r asymptotics of form factors

Plugging large r solutions into form factor expressions, and sending z_{uv} large, we get

$$\tilde{G}_E^{I=0}(r) \to \frac{432\pi\kappa(z_{\rm uv}\beta)^3}{N_c\lambda r^9}$$

$$\tilde{G}_M^{I=0}(r) \to -\frac{72\pi\kappa(z_{\rm uv}\beta)^3}{\Lambda N_c\lambda r^7}$$

$$\tilde{G}_E^{I=1}(r) \to -\frac{16\kappa(z_{\rm uv}\beta)^2}{3\pi\Lambda r^4}$$

$$\tilde{G}_M^{I=1}(r) \to \frac{16\kappa(z_{\rm uv}\beta)^2}{9\pi r^4}$$

Amusing note: if $\beta \sim 1/z_{uv}$, so that on-shell action becomes finite without S_b, form factors immediately become cutoff-independent...

Result

$$\lim_{r\to\infty} r^2 \frac{\tilde{G}_E^{I=0}\tilde{G}_E^{I=1}}{\tilde{G}_M^{I=0}\tilde{G}_M^{I=1}} = 18$$

All model-dependent factors cancel.

Sakai-Sugimoto model obeys the form factor relation.

The SS model is QCD-like enough to capture
the expected IR properties of baryons after all!

Conclusions and open questions

SS model as defined by $YM+CS$ action, which is leading order in α' expansion, already rich enough to capture relevant physics

Suggests g_{mBB} behavior may be more QCD-like than previously thought.

Key is to construct solitons with asymptotics which explicitly solve large r EoMs

Conclusions and open questions

SS model as defined by $YM+CS$ action, which is leading order in α' expansion, already rich enough to capture relevant physics

Suggests g_{mBB} behavior may be more QCD-like than previously thought.

Key is to construct solitons with asymptotics which explicitly solve large r EoMs

Not yet obvious what's wrong with using flat-space instanton approximation for these questions

Full solution may look rather different - SO(3) vs SO(4) symmetry?

Or is there something subtle in matching instanton approximation solutions between small r,z region and large r,z region?

Our treatment of holographic renormalization was rather cavalier - cries out for a careful investigation...

Does having a renormalized action for mesons automatically handle baryons?

For all of this: need to find full numerical soliton solutions...