SL(2,Z) duality on AdS/BCFT

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Introduction of boundaries in AdS/CFT

- The dynamics of strong-coupling theory can be solved using the AdS/CFT correspondence.

  Maldacena ``97

- We want to add the defects and boundaries in CFT for the application to the condensed matter physics.

- Example: the boundary entropy, edge modes, a local quench

Introduction of boundaries in AdS/CFT

- We can introduce defect in the probe limit.

Karch-Randall ``01

- We understand the defect with backreaction in the context of Randall-Sundrum braneworlds (bottom-up model with thin branes).

Randall-Sundrum, ``99

- String theory duals to defects with backreacting defects and boundaries

D'Hoker-Estes-Gutperle, ``07

Aharony, Berdichevsky, Berkooz, and Shamir, ``11

Introduction to AdS/BCFT

- General construction for boundary CFTs and their holographic duals

Takayanagi ``11

Fujita-Takayanagi-Tonni ``11

- Based on the thin branes

- Including the orientifold planes, which in the context of string theory are described by thin objects with negative tension.

Gravity dual of the BCFT

- We consider the AdS_4 gravity dual on the half plane.

- The 4-dimensional Einstein-Hilbert action with the boundary term

$$I = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}(R-\Lambda) + \frac{1}{8\pi G_N}\int_{Q_1} d^3x\sqrt{-\gamma}(K-T)$$

- The boundary condition $K_{\mu\nu} = (K-T)\gamma_{\mu\nu}$

- The AdS_4 metric

$$ds^2 = \frac{-dt^2+dx^2+dy^2+dz^2}{z^2}$$

➢ This is restricted to the half plane $y>0$ at the AdS boundary.

Gravity dual of the BCFT

- The spacetime dual to the half-plane

$$Q_1 : y = \cot \theta_1 z$$

- the vector normal to Q_I pointing outside of the gravity region $n_\mu = (0, 0, -\sin\theta, \cos\theta)$
- the vector parallel to Q_I $l_\mu = (0, 0, \cos\theta, \sin\theta)$

- The extrinsic curvature $K_{\mu\nu}$ and the tension T

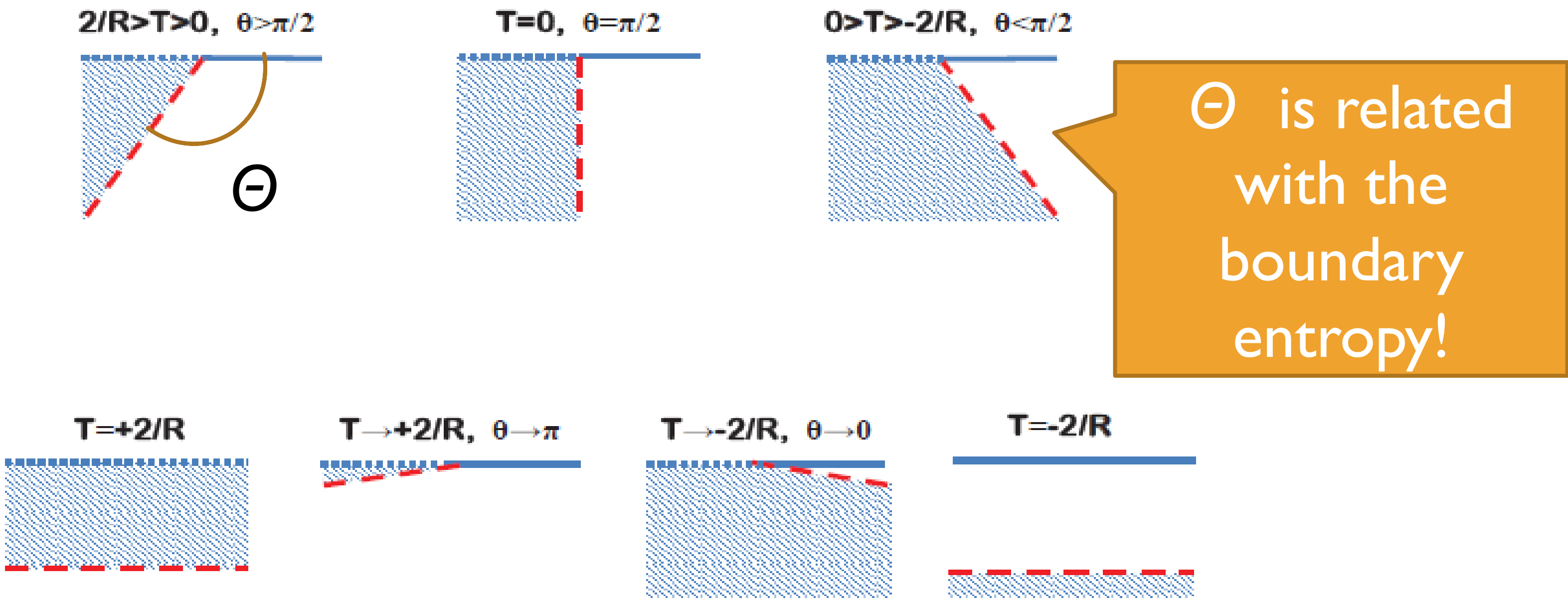
$$K_{\mu\nu} = -\frac{\cos\theta_1}{R}\gamma_{\mu\nu}, \quad T = -\frac{2\cos\theta_1}{R}$$

, where $-2/R \le T \le 2/R$.

The ends of this interval are described by no defect.

Gravity dual of the BCFT

- Embedding of the brane $Q_1 : y = \cot\theta_1 z$ corresponding to various values of the tension.



The effective abelian action

- We introduce the effective abelian action

$$\int c_1 * F \wedge F + c_2 F \wedge F - f(\phi) \int_{Q_1} A \wedge F$$

- The solutions for the EOM: $d(*F) = 0$

$$A_t = Ey + bz, \quad A_x = -By + cz, \quad A_y = dz, \quad A_z = 0$$

- The boundary condition at Q_l : Neumann boundary condition

$$c_1 * F + (c_2 + f(\phi)) F|_{Q_1} = 0$$

- The solutions to the boundary condition

$$b = -\frac{(f(\phi) + c_2) B c_1 - ((f(\phi) + c_2)^2 + c_1^2) \cos\theta_1 \sin\theta_1 E}{{c_1}^2 \cos^2\theta_1 - (f(\phi) + c_2)^2 \sin^2\theta_1},$$

$$c = -\frac{({c_1}^2 + ((f(\phi) + c_2)^2) B \cos\theta_1 \sin\theta_1 - (f(\phi) + c_2) c_1 E}{{c_1}^2 \cos^2\theta_1 - (f(\phi) + c_2)^2 \sin^2\theta_1} \qquad c_1 d = 0$$

The GKPW relation

- The current density derived using the GKPW relation

$$J^t = \frac{\delta S}{\delta A_t}\Big|_b = 2\epsilon_{(3)}(c_1 * F + c_2 F)|_b = 2c_1 b + 2c_2 B,$$

$$J^x = \frac{\delta S}{\delta A_x}\Big|_b = 2\epsilon_{(3)}(c_1 * F + c_2 F)|_b = -2c_1 c + 2c_2 E,$$

$$J^y = \frac{\delta S}{\delta A_y}\Big|_b = 0$$

Gubser-Klebanov-
-Polyakov ``98
Witten ``98

- 3 independent boundary conditions and the AdS boundary conditions determine the current.

Describing the position independent gap and the standard Hall physics

- The conductivity becomes

For $\theta_1 = \pi / 2$

$$J^t = \frac{2c_1^2 B}{f(\phi) + c_2} + 2c_2 B,$$

$$J^x = \frac{2c_1^2 E}{f(\phi) + c_2} + 2c_2 E.$$

For $\theta_1 = 0$

$$J^t = -2Bf(\phi), \qquad J^x = -2Ef(\phi)$$

It describes FQHE for proper coupling constant.

Dirichlet boundary condition on Q_1

- We choose the different boundary condition at Q_1

$$n_\mu \tilde{F}^{\mu\nu}|_{Q_1} = 0$$

- Rewritten as

$$\cos\theta_1 B - c\sin\theta_1 = 0, \quad E\cos\theta_1 + b\sin\theta_1 = 0, \quad F_{tx} = 0$$

- The current at the AdS boundary

$$J^t = -2c_1 \cot\theta_1 E + 2c_2 B, \quad J^x = -2c_1 B\cot\theta + 2c_2 E,$$

$$J^y = -2c_1 d$$

Cf. B appears in the second order of the hydrodynamic expansion.

- The Hall conductivity

$$\sigma_{xy} = -2c_1 \frac{B}{E}\cot\theta + 2c_2 \qquad \sigma_{yy} = \frac{J^y}{E}$$

A duality transformation of D=2 electron gas and the discrete group *SL(2,Z)*

The duality transformation in the d=2 electron gas

- The states of different filling fractions $\nu = J^t/B$ are related by

(i) $$\nu \leftrightarrow \nu + 1,$$

Landau level addition

(ii) $$\nu \leftrightarrow 1 - \nu \quad for \;\; \nu < 1,$$

Particle-hole transition

(iii) $$\frac{1}{\nu} \leftrightarrow \frac{1}{\nu} + 2,$$

Flux-attachment

Girvin ``84, Jain-Kivelson-Trivedi ``93, Jain-Goldman ``92

- ν transforms under the subgroup $\Gamma_0(2) \subset SL(2,Z)$ like the complex coupling τ

$$ST^2S,\; T \in \Gamma_0(2)$$

$SL(2,Z)$ duality in the $d=3$ CFT: review

Action of *SL(2,Z)* on CFT

- *S* transformation is used to describe the d=3 mirror symmetry.

Kapustin-Strassler ``99

- Defining the current of the dual theory $\tilde{J}_i(k) = -(i/2\pi)\varepsilon_{ijr}k_j A_r(k)$ the action after *S*-transformation becomes

$$L'(\Phi, A, B) = \frac{1}{2\pi}\epsilon^{ijk}B_i\partial_j A_k + \tilde{L}(\Phi, A)$$

- *T* action adds the Chern-Simons action

$$\tilde{L}(\Phi, A) \to \tilde{L}(\Phi, A) + \frac{1}{4\pi}\varepsilon^{ijk}A_i\partial_j A_k$$

Modular Action on Current 2-point function

- S operation has been studied in the case of N_f free fermions with U(1) gauge group for large N_f

  Borokhov-Kapustin-Wu ``02

- The effective action becomes weak coupling proportional to $\frac{1}{\sqrt{N_f}}$

- The large N_f theory has the property that the current has nearly Gaussian correlations

$$\langle J_i(k) J_j(-k) \rangle = (\delta_{ij} k^2 - k_i k_j) \frac{t}{2\pi\sqrt{k^2}} + \epsilon_{ijk} k_r \frac{w}{2\pi}$$

- Complex coupling $\tau = w + it$ (t>0)

Modular Action on Current 2-point function

- The effective action of A_i after including gauge fixing $k_iA_i=0$

$$\int d^3k\Big(\frac{t}{4\pi}A_i(k)\sqrt{k^2}A_i(-k)+\frac{w}{4\pi}\epsilon_{ijr}A_i(k)k_rA_j(-k)\Big)$$

- The propagator of A_i is the inverse of the matrix

$$M_{ij}=\frac{t}{2\pi}\sqrt{k^2}\delta_{ij}+\frac{w}{2\pi}\epsilon_{ijr}k_r$$

- The current of the theory transformed by *S*: $\tilde{J}_i(k)=-(i/2\pi)\varepsilon_{ijr}k_jA_r(k)$

- The 2-point function of $\tilde{J}_i(k)$

τ → -1/τ compared with <*J J*>

$$\langle\tilde{J}_i(k)\tilde{J}_j(-k)\rangle=(\delta_{ij}k^2-k_ik_j)\frac{t}{2\pi\sqrt{k^2}(t^2+w^2)}-\epsilon_{ijk}k_r\frac{w}{2\pi(t^2+w^2)}$$

SL(2,Z) action on the conductivity

- The conductivity $\sigma = \sigma_{xy} + i\sigma_{xx}$ has the natural action of S-duality

cf. Kubo formula $\sigma_{ij} = \langle J_i J_j \rangle / i\omega$

- The S action $\tau \to -1/\tau$ and the *T* action $\tau \to \tau + 1$ are rewritten as

$$S : 2\pi\sigma \to -1/2\pi\sigma, \; T : \sigma_{xy} \to \sigma_{xy} + \frac{1}{2\pi}$$

- transforms under *SL*(2,Z) in the standard way

$$2\pi\sigma \to \frac{a(2\pi\sigma)+b}{c(2\pi\sigma)+d}$$

The *SL(2,Z)* version of the law o
states in d=2 electron gas
for $\sigma=\sigma_{xy}=\nu/(2\pi)$

SL(2,Z) duality in the AdS/CFT correspondence

Interpretation of *SL(2,Z)* in the gravity side

- We consider 4-dimensional gravity theory on AdS_4 with the Maxwell field.

$$ds^2 = R^2 \frac{dz^2 + d\vec{x}^2}{z^2}$$

- Its conformal boundary *Y* at $z=0$

- The standard GKPW relation fixes a gauge field $\vec{A}$ at *Y*

- The path integral with boundary conditions is interpreted as the generation functional $\left\langle \exp(i\int_Y d^3x \vec{A}\cdot\vec{J}) \right\rangle$ in the CFT side

Interpretation of S-transformation in the gravity side

- The 3d mirror symmetry ⇔ The S duality in the bulk Maxwell theory *Witten``03*

- The S transformation in $SL(2,Z)$ maps $\vec{B} \to \vec{E},\ \vec{E} \to -\vec{B}$ and the gauge field $\vec{A}$ to $\vec{A}'$. Here, $\vec{B}_i = \varepsilon_{ijk} F_{jk},\ \vec{E}_i = F_{zi}$

- The standard AdS/CFT in terms of $\vec{A}'$ is equivalent in terms of the original $\vec{A}$ to using a boundary condition $\vec{E} = 0$ instead of $\vec{A}$ fixed

Only one linear combination of net electric and magnetic charge corresponds to the conserved quantity in the boundary.

Interpretation of *T*-transformation in the gravity side

- The generator T in $SL(2,Z)$ corresponds to a 2π shift in the theta angle.

- After integration parts, it transforms the generating function by Chern-Simons term.

$$\left\langle \exp(i\int_Y d^3x \vec{A}\cdot\vec{J}) \right\rangle \to \left\langle \exp(i\int_Y d^3x \vec{A}\cdot\vec{J}) \right\rangle \exp(\frac{i}{4\pi}\int d^3x \varepsilon^{ijk} A_i \partial_j A_k)$$

- A contact term $\sim \frac{w}{2\pi}\varepsilon_{ijk}\frac{\partial}{\partial x^j}\delta^3(x-y)$ is added to the correlation functions.

A duality transformation in the AdS/BCFT

A duality transformation on AdS/BCFT

- The $d=4$ Abelian action has the $SL(2,R)$ symmetry

Should be quantized for superstring

- Defining the coupling constant $\tau = 4\pi(c_2 - c_1 i)$

$$I = \frac{1}{8\pi}\mathrm{Im}\int \tau(F + i\tilde{F}) \wedge *(F + i\tilde{F}) = \int \sqrt{-g}d^4x\left(\frac{c_1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{c_2}{2}F_{\mu\nu}\tilde{F}^{\mu\nu}\right)$$

Here, $*$ is the 4-dimensional epsilon symbol and $\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu}{}^{\rho\lambda}F_{\rho\lambda}/2$

- The $SL(2,Z)$ transformation of τ

$$\tau' = \frac{a\tau + b}{c\tau + d} \qquad a, b, c, d \in \mathbb{R} \qquad ad - bc = 1$$

$$S:\ \tau' = -\frac{1}{\tau}, \quad T:\ \tau' = \tau + 1$$

The transformation is accompanied with that of the gauge field.

A duality transformation on AdS/BCFT

- Introduction of the following quantity

$$\tilde{H}^{\mu\nu} \equiv -\frac{4\pi}{\sqrt{-g}}\frac{\delta I}{\delta F_{\mu\nu}} = 4\pi(-c_1 F^{\mu\nu} + c_2 \tilde{F}^{\mu\nu})$$

- Simplified to $\mathcal{H}^{\mu\nu} = \bar{\tau}\mathcal{F}^{\mu\nu}$

, where $\mathcal{F}_{\mu\nu} = F_{\mu\nu} - i\tilde{F}_{\mu\nu}, \quad \mathcal{H}_{\mu\nu} = H_{\mu\nu} - i\tilde{H}_{\mu\nu}$

- $\mathcal{H}^{\mu\nu} = \bar{\tau}\mathcal{F}^{\mu\nu}$ is invariant under the transformation of τ

and following transformation

$$\begin{pmatrix} \mathcal{H}'_{\mu\nu} \\ \mathcal{F}'_{\mu\nu} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathcal{H}_{\mu\nu} \\ \mathcal{F}_{\mu\nu} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} H'_{\mu\nu} \\ F'_{\mu\nu} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} H_{\mu\nu} \\ F_{\mu\nu} \end{pmatrix}$$

A duality transformation on AdS/BCFT

- After the *SL(2,Z)* duality, the coupling constant and the gauge field are transformed to the dual values.

$$c_1,\ c_2,\ E,\ B \to c'_1,\ c'_2,\ E',\ B'$$

- In the case of $\theta_1 = \pi/2$, the *S* transformation gives

$$\sigma'_{xy} = -\frac{E}{4\pi^2(-2c_1c + 2c_2E)} = -\frac{1}{8\pi^2 c_2}$$

- After the *T* transformation, $c_2 \to c_2 + 1/(4\pi)$

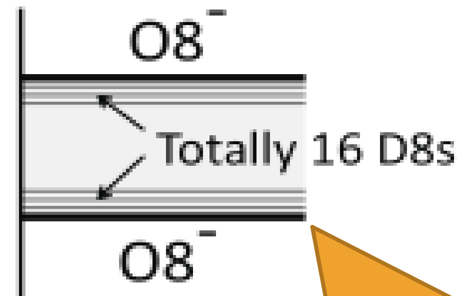
$$\sigma'_{xy} = 2\Big(\frac{c_1^2}{c_2 + 1/(4\pi)} + c_2 + \frac{1}{4\pi}\Big)$$

- The same action is operated for the case of the Dirichlet boundary condition Q_I.

Stringy realization of the Abelian theory

- Type IIA string theory on AdS_4*CP^3 with orientifold 8-plane can realize the AdS/BCFT. *Fujita-Takayanagi-Tonni ``11*
- No orientifold cases are dual to the $d=3$ $N=6$ Chern-Simons theory (ABJM theory) *Aharony-Bergman-Jafferis-Maldacena ``08*
- The 10-dimensional metric of AdS_4*CP^3

$$ds^2 = R^2 \frac{dz^2 - dt^2 + dx^2 + dy^2}{z^2} + 4R^2 ds^2_{CP3}$$



8 chiral fermions for each boundary

- The orientifold projection: $y \rightarrow -y$

  even under the orientifold : Φ, g, C_1, odd under it: B_2, C_3

Stringy realization

- After dimensional reduction to $d=4$, we obtain the Abelian action of the massless gauge fields $A_\mu = C_{\mu mn} \; (\mu \in AdS_4,\ m,n \in CP^3)$

$$\frac{R^2}{12\pi^2}\int F\wedge {}^*F + \frac{M}{4\pi k}\int F\wedge F$$

M/k: number of B_2 flux

for no O-planes, *Hikida-Li-Takayanagi* ``09

- satisfying the Dirichlet boundary condition at the boundaries.

- This system realizes the FQHE and the Hall conductivity

$\sigma_{xy}=M/2\pi k$

- The $SL(2,Z)$ action of σ_{xy}

$$S:\ \sigma_{xy} = \frac{9Mk}{4\pi(2Nk+9M^2)},\quad T:\sigma_{xy} = -\frac{M}{2\pi k} + \frac{1}{2\pi}$$

Discussion

- We analyzed the response of a conserved current to external electromagnetic field in the AdS/BCFT correspondence.
- This allows us to extract the Hall current.
- Analysis of the action of a duality transformation
- String theory embedding of the abelian theory

CFT correlator of $U(1)$ current J_μ in 2+1 dimensions

: a universal number analogous to the level number of the Kac-Moody algebra in 1+1 dimensions

$$\left\langle J_\mu(p) J_\nu(-p) \right\rangle = K\sqrt{p^2}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)$$

K: a universal number analogous to the level number of the Kac-Moody algebra in 1+1 dimensions

Application of Kubo formula shows that

$$\sigma = \frac{4e^2}{h} 2\pi K$$