Twistor transform, instantons and rational curves

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Plan

1. Hyperkähler and quaternionic-Kähler manifolds and their twistor spaces
2. Chern connection
3. Hyperholomorphic bundles and twistor transform
4. Twistor transform for mathematical instantons

Hyperkähler manifolds

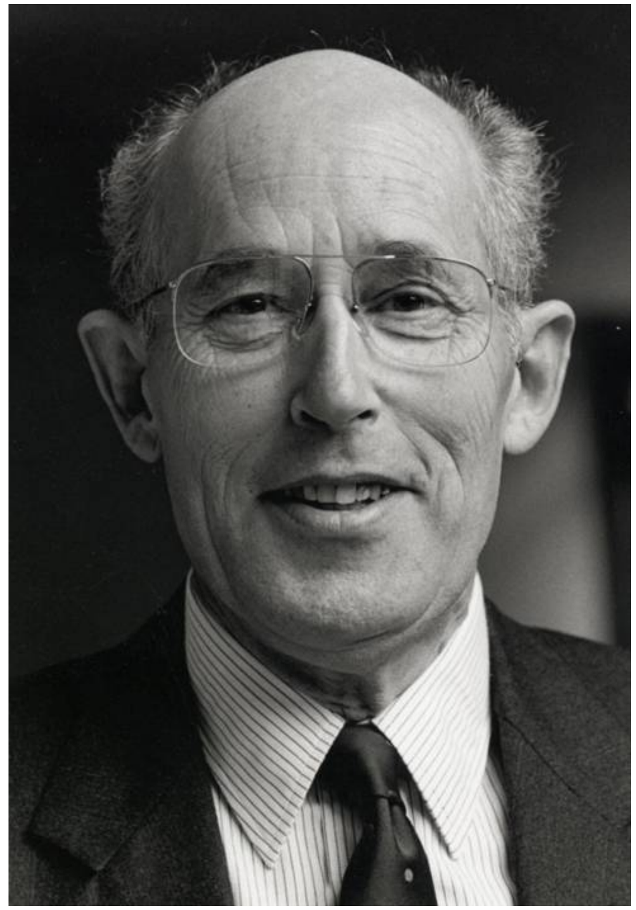
DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$**,** $\omega_J := g(J\cdot, \cdot)$**,** $\omega_K := g(K\cdot, \cdot)$**.**

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the Levi-Civita connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).



Marcel Berger

Classification of holonomies

THEOREM: (de Rham) A complete, simply connected Riemannian manifold with non-irreducible holonomy **splits as a Riemannian product.**

THEOREM: (Berger's theorem, 1955) Let G be an irreducible holonomy group of a Riemannian manifold which is not locally symmetric. **Then G belongs to the Berger's list:**

Berger's list	
Holonomy	*Geometry*
$SO(n)$ acting on $\mathbb{R}^n$	Riemannian manifolds
$U(n)$ acting on $\mathbb{R}^{2n}$	Kähler manifolds
$SU(n)$ acting on $\mathbb{R}^{2n}$, $n > 2$	Calabi-Yau manifolds
$Sp(n)$ acting on $\mathbb{R}^{4n}$	hyperkähler manifolds
$Sp(n) \times Sp(1)/\{\pm 1\}$ acting on $\mathbb{R}^{4n}$, $n > 1$	quaternionic-Kähler manifolds
G_2 acting on $\mathbb{R}^7$	G_2-manifolds
$Spin(7)$ acting on $\mathbb{R}^8$	$Spin(7)$-manifolds

Quaternionic-Kähler manifolds

DEFINITION: A **quaternionic-Kähler manifold** is a Riemannian (M,g) manifold with holonomy in $Sp(n) \times Sp(1)/\{\pm 1\}$. Equivalently, it is a Riemannian manifold **equipped with a 3-dimensional sub-bundle $E \subset \mathfrak{so}(TM)$** satisfying the following

1. E is closed with respect to the commutator, and isomorphic to $\mathfrak{so}(3)$ acting as imaginary quaternions at each point of M

2. $\nabla E \subset E \otimes \Lambda^1 M$.

REMARK: A quaternionic-Kähler manifold is **Einstein**, that is, **satisfies** $\operatorname{Ric}(M) = \lambda g$, for some constant $\lambda \in \mathbb{R}$ (here, $\operatorname{Ric}(M) \in \operatorname{Sym}^2 T^*M$ is a Ricci curvature).

REMARK: Whenever the constant λ is equal 0, M is hyperkähler, otherwise it's **not hyperkähler**. Even if **hyperkähler manifolds are always quaternionic-Kähler,** when people say "quaternionic-Kähler" they actually mean "quaternionic-Kähler with $\lambda \neq 0$."

Further on, all quaternionic-Kähler manifolds will be non-Kähler.

Twistor spaces

DEFINITION: Induced complex structures on a hyperkähler manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$ **They are usually non-algebraic**. Indeed, if M is compact, for generic a, b, c, (M, L) has no divisors (Fujiki).

DEFINITION: A **twistor space** $\operatorname{Tw}(M)$ of a hyperkähler manifold is **a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$.** More formally:

Let $\operatorname{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on M induced by $J \in S^2 \subset \mathbb{H}$. Let I_J denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\operatorname{Tw}} = I_m \oplus I_J : T_x \operatorname{Tw}(M) \to T_x \operatorname{Tw}(M)$ satisfies $I_{\operatorname{Tw}}^2 = -\operatorname{Id}$. **It defines an almost complex structure on $\operatorname{Tw}(M)$.** This almost complex structure is known to be integrable (Obata, Salamon)

EXAMPLE: If $M = \mathbb{H}^n$, $\operatorname{Tw}(M) = \operatorname{Tot}(\mathcal{O}(1)^{\oplus n}) \cong \mathbb{C}P^{2n+1} \backslash \mathbb{C}P^{2n-1}$

REMARK: For M compact, $\operatorname{Tw}(M)$ never admits a Kähler structure.

Twistor spaces for quaternionic-Kähler manifolds

DEFINITION: A **twistor space** $\operatorname{Tw}(M)$ of a quaternionic-Kähler manifold (M, g, E) is a total space of a unit sphere bundle on E, equipped with a complex structure as above.

EXAMPLE: If $M = \mathbb{H}P^n$, then $\operatorname{Tw}(M) = \mathbb{C}P^{2n+1}$. In particular, $\operatorname{Tw}(S^4) = \mathbb{C}P^3$.

REMARK: Consider a compact quaternionic-Kähler manifold (M, g) with $\operatorname{Ric}(M) = \lambda g$, $\lambda > 0$. Then $\operatorname{Tw}(M)$ **is a holomorphically contact Fano manifold.**. Conversely, **any Kähler-Einstein holomorphically contact Fano manifold is a twistor space of a compact quaternionic-Kähler manifold (M, g) with $\operatorname{Ric}(M) = \lambda g$, $\lambda > 0$.**

One can say that **hyperkähler geometry is holomorphic symplectic geometry, and quaternionic-Kähler is holomorphic contact geometry**

A holomorphic structure operator

DEFINITION: Let $d = d^{0,1} + d^{1,0}$ be the Hodge decomposition of the de Rham differential on a complex manifold, $d^{0,1}: \Lambda^{p,q}(M) \longrightarrow \Lambda^{p,q+1}(M)$ and $d^{1,0}: \Lambda^{p,q}(M) \longrightarrow \Lambda^{p+1,q}(M)$. The operators $d^{0,1}$, $d^{1,0}$ are denoted $\overline{\partial}$ and ∂ and called **the Dolbeault differentials**.

REMARK: From $d^2 = 0$, one obtains **$\overline{\partial}^2 = 0$ and $\partial^2 = 0$.**

REMARK: The operator $\overline{\partial}$ is $\mathcal{O}_M$-linear.

DEFINITION: Let B be a holomorphic vector bundle, and $\overline{\partial}: B_{C^\infty} \longrightarrow B_{C^\infty} \otimes \Lambda^{0,1}(M)$ an operator mapping $b \otimes f$ to $b \otimes \overline{\partial} f$, where $b \in B$ is a holomorphic section, and f a smooth function. This operator is called **a holomorphic structure operator** on B. **It is correctly defined, because $\overline{\partial}$ is $\mathcal{O}_M$-linear.**

REMARK: The kernel of $\overline{\partial}$ **coincides with the set of holomorphic sections** of B.

The $\overline{\partial}$-operator on vector bundles

DEFINITION: A $\overline{\partial}$-operator on a smooth bundle is a map $V \xrightarrow{\overline{\partial}} \Lambda^{0,1}(M) \otimes V$, satisfying $\overline{\partial}(fb) = \overline{\partial}(f) \otimes b + f\overline{\partial}(b)$ for all $f \in C^\infty M, b \in V$.

REMARK: A $\overline{\partial}$-operator on B can be extended to

$$\overline{\partial}: \Lambda^{0,i}(M) \otimes V \longrightarrow \Lambda^{0,i+1}(M) \otimes V,$$

using $\overline{\partial}(\eta \otimes b) = \overline{\partial}(\eta) \otimes b + (-1)^{\tilde{\eta}} \eta \wedge \overline{\partial}(b)$, where $b \in V$ and $\eta \in \Lambda^{0,i}(M)$.

REMARK: If $\overline{\partial}$ is a holomorphic structure operator, then $\overline{\partial}^2 = 0$.

THEOREM: (Atiyah-Bott) Let $\overline{\partial}: V \longrightarrow \Lambda^{0,1}(M) \otimes V$ be a $\overline{\partial}$-operator, satisfying $\overline{\partial}^2 = 0$. **Then $B := \ker \overline{\partial} \subset V$ is a holomorphic vector bundle of the same rank.**

DEFINITION: $\overline{\partial}$-operator $\overline{\partial}: V \longrightarrow \Lambda^{0,1}(M) \otimes V$ on a smooth manifold is called **a holomorphic structure operator**, if $\overline{\partial}^2 = 0$.

Connections and holomorphic structure operators

DEFINITION: let (B, ∇) be a smooth bundle with connection and a holomorphic structure $\overline{\partial}\ B \longrightarrow \Lambda^{0,1}(M) \otimes B$. Consider a Hodge decomposition $\nabla = \nabla^{0,1} + \nabla^{1,0}$,

$$\nabla^{0,1} :\ B \longrightarrow \Lambda^{0,1}(M) \otimes B, \quad \nabla^{1,0} :\ B \longrightarrow \Lambda^{1,0}(M) \otimes B.$$

We say that ∇ is **compatible with the holomorphic structure** if $\nabla^{0,1} = \overline{\partial}$.

DEFINITION: A Chern connection on a holomorphic Hermitian vector bundle is a connection compatible with the holomorphic structure and preserving the metric.

THEOREM: On any holomorphic Hermitian vector bundle, **the Chern connection exists, and is unique.**

REMARK: The curvature of a Chern connection on B is an End(B)-valued (1,1)-form: $\Theta_B \in \Lambda^{1,1}(\mathrm{End}(B))$.

REMARK: A converse is true, by Atiyah-Bott theorem. Given a Hermitian connection ∇ on a vector bundle B with curvature in $\Lambda^{1,1}(\mathrm{End}(B))$, we obtain a holomorphic structure operator $\overline{\partial} = \nabla^{0,1}$. Then, **$\nabla$ is a Chern connection of $(B, \overline{\partial})$.**

Hyperholomorphic connections

REMARK: Let M be a hyperkähler manifold. **The group $SU(2)$ of unitary quaternions acts on $\Lambda^*(M)$ multiplicatively.**

DEFINITION: A **hyperholomorphic connection** on a vector bundle B over M is a Hermitian connection with $SU(2)$-invariant curvature $\Theta \in \Lambda^2(M) \otimes \operatorname{End}(B)$.

REMARK: Since the invariant 2-forms satisfy $\Lambda^2(M)_{SU(2)} = \bigcap_{I \in \mathbb{C}P^1} \Lambda_I^{1,1}(M)$, **a hyperholomorphic connection defines a holomorphic structure on B** for each I induced by quaternions.

REMARK: Let M be a compact hyperkähler manifold. Then $SU(2)$ preserves harmonic forms, hence **acts on cohomology.**

CLAIM: All Chern classes of hyperholomorphic bundles are $SU(2)$-invariant.

Proof: Use $\Lambda^2(M)_{SU(2)} = \bigcap_{I \in \mathbb{C}P^1} \Lambda_I^{1,1}(M)$. ■

REMARK: **Converse is also true** (for stable bundles). See the next slide.

Kobayashi-Hitchin correspondence

DEFINITION: Let F be a coherent sheaf over an n-dimensional compact Kähler manifold M. Let

$$\text{slope}(F) := \frac{1}{\text{rank}(F)} \int_M \frac{c_1(F) \wedge \omega^{n-1}}{vol(M)}.$$

A torsion-free sheaf F is called **(Mumford-Takemoto) stable** if for all subsheaves $F' \subset F$ one has $\text{slope}(F') < \text{slope}(F)$. If F is a direct sum of stable sheaves of the same slope, F is called **polystable**.

DEFINITION: A Hermitian metric on a holomorphic vector bundle B is called **Yang-Mills** (Hermitian-Einstein) if the curvature of its Chern connection satisfies $\Theta_B \wedge \omega^{n-1} = \text{slope}(F) \cdot \text{Id}_B \cdot \omega^n$. A Yang-Mills connection is a Chern connection induced by the Yang-Mills metric.

REMARK: Yang-Mills connections minimize the integral

$$\int_M |\Theta_B|^2 \text{Vol}_M$$

Kobayashi-Hitchin correspondence: (Donaldson, Uhlenbeck-Yau). Let B be a holomorphic vector bundle. **Then B admits Yang-Mills connection if and only if B is polystable.** Moreover such a connection is **unique**.

Kobayashi-Hitchin correspondence and hyperholomorphic bundles

CLAIM: Let M be a hyperkähler manifold. Then for any $SU(2)$-invariant 2-form $\eta \in \Lambda^2(M)$, one has $\eta \wedge \omega^{n-1} = 0$.

COROLLARY: **Any hyperholomorphic bundle is Yang-Mills** (hence polystable).

REMARK: This implies that **a hyperholomorphic connection on a given holomorphic vector bundle is unique** (if exists). Such a bundle is called **hyperholomorphic**.

THEOREM: Let B be a polystable holomorphic bundle on (M, I), where (M, I, J, K) is hyperkähler. Then the (unique) **Yang-Mills connection on B is hyperholomorphic if and only if the cohomology classes $c_1(B)$ and $c_2(B)$ are $SU(2)$-invariant.**

COROLLARY: The moduli space of stable holomorphic vector bundles with $SU(2)$-invariant $c_1(B)$ and $c_2(B)$ is a hyperkähler manifold.

COROLLARY: Let (M, I, J, K) be a hyperkähler manifold, and $L = aI + bJ + cK$ a generic induced complex structure. Then any stable bundle on (M, L) is hyperholomorphic.

Twistor transform and hyperholomorphic bundles 1: direct twistor transform

CLAIM: Let σ : $\operatorname{Tw}(M) \longrightarrow M$ be the standard projection, where M is hyperkähler or quaternionic-Kähler, and $\eta \in \Lambda^2 M$ a 2-form. Then **$\sigma^*\eta$ is a (1,1)-form iff η is $SU(2)$-invariant.**

COROLLARY: Let (B, ∇) be a bundle with connection, and $\sigma^* B, \sigma^* \nabla$ its pullback to $\operatorname{Tw}(M)$. **Then $(\sigma^* B, \sigma^* \nabla)$ has (1,1)-curvature iff ∇ has $SU(2)$-invariant curvature.**

REMARK: This construction produces a holomorphic vector bundle on $\operatorname{Tw}(M)$ starting from a connection with $SU(2)$-invariant curvature. It is called **direct twistor transform**. The **inverse twistor transform** produces a bundle with connection on M from a holomorphic bundle on $\operatorname{Tw}(M)$.

DEFINITION: **A non-Hermitian hyperholomorphic connection** on a complex vector bundle B is a connection (not necessarily Hermitian) which has $SU(2)$-invariant curvature.

Twistor transform and hyperholomorphic bundles 2: inverse twistor transform

DEFINITION: Let M be a hyperkähler or quaternionic-Kähler manifold, and $\sigma: \operatorname{Tw}(M) \longrightarrow M$ its twistor space. For each point $x \in M$, $\sigma^{-1}(x)$ is a holomorphic rational curve in $\operatorname{Tw}(M)$. It is called **a horizontal twistor line**.

THEOREM: (The inverse twistor transform; Kaledin-V.) Let B be a holomorphic vector bundle on $\operatorname{Tw}(M)$, which is trivial on any horizontal twistor line. Denote by B_0 the C^∞-bundle on M with fiber $H^0(B\big|_{\sigma^{-1}(x)})$ at $x \in M$. **Then B_0 admits a unique non-Hermitian hyperholomorphic connection ∇** such that B is isomorphic (as a holomorphic vector bundle) to its twistor transform $(\sigma^* B_0, (\sigma^* \nabla)^{0,1})$.

REMARK: The condition of being trivial on any horizontal twistor line is **open.** Therefore, **a holomorphic bundle on a $\operatorname{Tw}(M)$ is "more or less the same" as a bundle with non-Hermitian hyperholomorphic connection on M.**

QUESTION: What can be said about the geometry of the moduli of holomorphic bundles on $\operatorname{Tw}(M)$?

Rational curves on twistor spaces

From now on, we always assume that M **is hyperkähler** (and not quaternionic-Kähler).

DEFINITION: Denote by $\operatorname{Sec}(M)$ **the space of holomorphic sections** of the twistor fibration $\operatorname{Tw}(M) \xrightarrow{\pi} \mathbb{C}P^1$. For each point $m \in M$, one has **a horizontal section** $C_m := \{m\} \times \mathbb{C}P^1$ of π. The space of horizontal sections is denoted $\operatorname{Sec}_{hor}(M) \subset \operatorname{Sec}(M)$

REMARK: The space of horizontal sections of π is identified with M. The normal bundle $NC_m = \mathcal{O}(1)^{\dim M}$. Therefore, **some neighbourhood of** $\operatorname{Sec}_{hor}(M) \subset \operatorname{Sec}(M)$ **is a smooth manifold of dimension** $2\dim M$**.**

Let B be a (Hermitian) hyperholomorphic bundle on M, and W the deformation space of B, which is hyperkähler. Denote by $\tilde{B}$ the holomorphic bundle on $\operatorname{Tw}(M)$, obtained as a twistor transform of B. **Any deformation** $\tilde{B}_1$ **of** $\tilde{B}$ **gives a holomorphic map** $\mathbb{C}P^1 \longrightarrow \operatorname{Tw}(W)$ mapping $L \in \mathbb{C}P^1$ to a bundle $\tilde{B}_1\big|_{(M,L)} \subset \operatorname{Tw}(M)$, considered as a point in (W, L).

THEOREM: (Kaledin-V.) This construction identifies **deformations of** $\tilde{B}$ **(with appropriate stability conditions) and rational curves** $S \in \operatorname{Sec}(W)$**.** The twistor transforms of Hermitian hyperholomorphic bundles on M correspond to $\operatorname{Sec}_h(W) \subset \operatorname{Sec}(W)$.

Holomorphic bundles on $\mathbb{C}P^3$ and twistor sections

DEFINITION: An instanton on $\mathbb{C}P^2$ is a stable bundle B with $c_1(B) = 0$. **A framed instanton** is an instanton equipped with a trivialization $B|_C$ for a line $C \subset \mathbb{C}P^2$.

THEOREM: (Nahm, Atiyah, Hitchin) The space $\mathcal{M}_{r,c}$ of framed instantons on $\mathbb{C}P^2$ is **smooth, connected, hyperkähler.**

The main result of today's talk can be stated as follows

METATHEOREM: There is a similar correspondence between the holomorphic bundles on $\operatorname{Tw}(\mathbb{H}) = \mathbb{C}P^3 \backslash \mathbb{C}P^1$, with appropriate stability and framing conditions, and twistor sections in $\operatorname{Sec}(\mathcal{M}_{r,c})$.

Mathematical instantons

DEFINITION: A **mathematical instanton bundle** on $\mathbb{C}P^n$ is a locally free coherent sheaf E on $\mathbb{C}P^n$ with $c_1(E) = 0$ satisfying the following cohomological conditions:
1. for $n \geqslant 2$, $H^0(E(-1)) = H^n(E(-n)) = 0$;
2. for $n \geqslant 3$, $H^1(E(-2)) = H^{n-1}(E(1-n)) = 0$;
3. for $n \geqslant 4$, $H^p(E(k)) = 0$, $2 \leqslant p \leqslant n-2$ and $\forall k$;

The integer $c = -\chi(E(-1)) = h^1(E(-1)) = c_2(E)$ is called **the charge** of E.
A framed instanton is a mathematical instanton equipped with a trivialization of $B|_\ell$ for some fixed line $\ell = \mathbb{C}P^1 \subset \mathbb{C}P^n$.

REMARK: Mathematical instantons of rank 2 **are always stable** (follows from the monad description below).

REMARK: The space $\mathbb{M}_{r,c}$ of framed instantons with charge c and rank r **is a principal $SL(2)$-bundle** over the space of all mathematical instantons trivial on ℓ.

THEOREM: (Jardim–V.) The space $\mathbb{M}_c$ of framed rank r mathematical instantons on $\mathbb{C}P^3$ **is naturally identified with the space of twistor sections Sec($\mathcal{M}_{r,c}$).**

Monads and mathematical instantons

DEFINITION: A monad is a sequence of vector bundles $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$ which is exact in the first and the last term. **The cohomology** of a monad is $\ker j / \operatorname{im} i$.

THEOREM: Let B be a holomorphic bundle of rank 2 on $\mathbb{C}P^n$, $c_1(B) = 0$, $c_2(B) = c$. **Then the following conditions are equivalent.**

(i) B is a mathematical instanton.

(ii) B is a cohomology of a monad

$$0 \longrightarrow V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}P^k}(-1) \longrightarrow W \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}P^k} \longrightarrow U \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}P^k}(1) \longrightarrow 0$$

with $\dim V = \dim U = c$ and $\dim W = 2c + 2$.

ADHM construction

DEFINITION: Let V and W be complex vector spaces, with dimensions c and r, respectively. The **ADHM data** is maps

$$A, B \in \mathrm{End}(V), I \in \mathrm{Hom}(W, V), J \in \mathrm{Hom}(V, W).$$

We say that ADHM data is

stable,

if there is no subspace $S \subsetneq V$ such that $A(S), B(S) \subset S$ and $I(W) \subset S$;

costable,

if there is no nontrivial subspace $S \subset V$ such that $A(S), B(S) \subset S$ and $S \subset \ker J$;

regular,

if it is both stable and costable.

The ADHM equation is $[A, B] + IJ = 0$.

THEOREM: (Atiyah, Drinfeld, Hitchin, Manin) Framed rank r, charge c instantons on $\mathbb{C}P^2$ are in bijective correspondence with the set of equivalence classes of regular ADHM solutions. In other words, **the moduli of instantons on $\mathbb{C}P^2$ is identified with moduli of the corresponding quiver representation.**

The multi-dimensional ADHM construction

DEFINITION: Let V and W be complex vector spaces, with dimensions c and r, respectively. The **d-dimensional ADHM data** is maps

$$A_k, B_k \in \mathrm{End}(V), I_k \in \mathrm{Hom}(W,V), J_k \in \mathrm{Hom}(V,W), (k=0,\dots,d)$$

Choose homogeneous coordinates $[z_0 : \cdots : z_d]$ on $\mathbb{C}P^d$ and define

$$\tilde{A} := A_0 \otimes z_0 + \cdots + A_d \otimes z_d \quad \text{and} \quad \tilde{B} := B_0 \otimes z_0 + \cdots + B_d \otimes z_d.$$

We say that d-dimensional ADHM data is **globally regular**, if $(\tilde{A}_p, \tilde{B}_p, \tilde{I}_p, \tilde{J}_p)$ is regular for every $p \in \mathbb{C}P^d$. The **d-dimensional ADHM equation** is $[\tilde{A}_p, \tilde{B}_p] + \tilde{I}_p\tilde{J}_p = 0$, for all $p \in \mathbb{C}P^d$

THEOREM: (Marcos Jardim, Igor Frenkel) Let $C_d(r,c)$ denote the set of globally regular solutions of the d-dimensional ADHM equation. Then **there exists a 1-1 correspondence between equivalence classes of globally regular solutions of the d-dimensional ADHM equations and isomorphism classes of rank r instanton bundles** on $\mathbb{CP}^{d+2}$ framed at a fixed line ℓ, where $\dim W = \mathrm{rk}(E)$ and $\dim V = c_2(E)$.

The multi-dimensional ADHM construction for $d = 1$

For $d = 1$, we obtain that the d-dimensional ADHM solutions are families of solutions of ADHM parametrized by $\mathbb{C}P^3$. Also, the space of 1-dimensional ADHM data is the space of sections of

$$\mathcal{O}(1) \otimes_{\mathbb{C}} \left[\mathrm{Hom}(W, V) \oplus \mathrm{Hom}(V, W) \oplus \mathrm{End}(V) \oplus \mathrm{End}(V) \right]$$

over $\mathbb{C}P^1$, that is, the twistor space of a qquaternionic vector space $U = \mathrm{Hom}(W, V) \oplus \mathrm{Hom}(V, W) \oplus \mathrm{End}(V) \oplus \mathrm{End}(V)$. Now, the hyperkähler structure on 0-dimensional ADHM solutions for each $p \in \mathbb{C}P^1$ is compatible with the hyperkähler structure on U, because the space of 0-dimensional ADHM solutions is obtained from U by hyperkähler reduction. **This is used to prove the theorem about instantons on $\mathbb{C}P^3$ and twistor sections.**