2-dim McKay Correspondence

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IPMU

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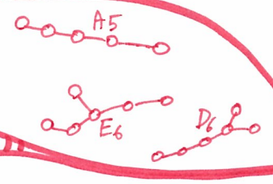
Introduction

the minimal resolution

- Gorenstein
- $\mathbb{C}^2/G$
- singularity
- rational
- $G \subset SL(2,\mathbb{C})$
- finite
- group
- quotient
- irred. ρ_i
- representation
- (conjugacy classes)
- McKay quiver
- $\rho \otimes \rho_i = \sum a_{ij} \rho_j$
- $A = (a_{ij})$
- Cartan matrix
- Representain theory 1979
- $C = 2E - A$

Dynkin diagram



McKay's Observation

simple Lie algebra!

Cartan matrix

$C := 2E - A$

$A := (a_{ij})$

$\bigoplus a_{ij} \rho_j = \rho \otimes \rho_i$

$\{\rho_i\}$ irred. rep. of G

$\rho : G \hookrightarrow SL(2,\mathbb{C})$ nat. rep

the minimal resolution

1983

Algebraic Geometry

$G \subset SL(2,\mathbb{C})$

An cyclic

Dn binary dihedral

E6, E7, E8 binary polyhedral

Theorem [GS-V]

For each $\rho \in \mathrm{Irr}^0(G)(\mathrm{id})$, the sheaf $\mathcal{F}(\rho)$ on $\tilde{X}$ is locally free of rank $\dim(\rho)$.

There is a bijection

$\phi : \mathrm{Irr}^0(G) \to \mathrm{Irr}(G)$ s.t.

$\forall d \in \mathrm{Irr}(G)$, $C_1(\mathcal{F}(\rho)) \cdot d = \begin{cases} 0 & d \neq \phi(\rho) \\ 1 & d = \phi(\rho) \end{cases}$

Moreover

$\phi(\rho_i)\phi(\rho_j) = a_{ij}$,

(for $\forall \rho_i, \rho_j \in \mathrm{Irr}^0(G)$, $\rho_i \neq \rho_j$)

Gonzalez-Sprinberg & Verdier

Singularity

- isolated
  - hypersurface
  - rational double
    - names
      - RDP
      - ADE
      - Du Val
      - Simple
      - ...
- minimal resolution
  - dual
  - Dynkin diagram
    - E_8
    - E_7
    - E_6
    - A_n
    - D_n
  - $\tilde{X} \to X := \mathbb{C}^2/G$
  - $\tilde{X}$: smooth
  - E: exceptional locus $= \bigcup E_i$
    - E_i: rational curves
    - $(E_i \cdot E_j) = \begin{cases} 0 & \\ 1 & \\ -2 & i=j \end{cases}$
    - Intersection Matrix

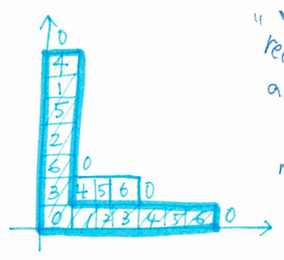
Recent Developments

Iyama-Wemyss

The classification of special Cohen-Macaulay Modules

[Math.Z] arXiv:0809.1958

"Wemyss reconstruction algebra" $\leftrightarrow$ min. resol. of $\mathbb{C}^2/G$.



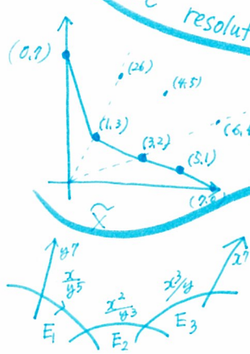
Theorem (Ito) 2001 ~ 2003

ρ_i: special $\Leftrightarrow$ Corresponding monomials is in $B(G)$ but not in $B(G) \setminus L(G)$

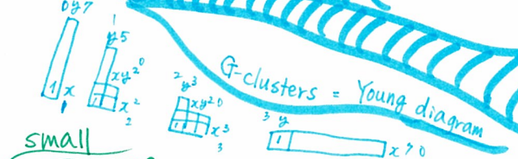
Example

$G = \langle \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^3 \end{pmatrix} \rangle$, $\varepsilon^4 = 1$ special case

Toric resolution = min. resol. $X \to X = \mathbb{C}^2/G$



G-clusters = Young diagram



small 2002 (A. Ishii)

$G \subset GL(2,\mathbb{C})$

2001 Kidoh G: cyclic $\subset GL(2,\mathbb{C})$

Ito-Nakamura $G \subset SL(2,\mathbb{C})$ 2000

1997

G-Hilbert scheme

Def. (G-Hilbert scheme)

$\mathrm{Hilb}^n(\mathbb{C}^2) := \{ I \subset \mathbb{C}[x,y] \mid I \text{ ideal}, \dim \mathbb{C}[x,y]/I = n \}$

$\mathrm{Hilb}^G(\mathbb{C}^2) = \{ I \subset \mathbb{C}[x,y] \mid I : G\text{-inv. ideal}, \mathbb{C}[x,y]/I \cong \mathbb{C}[G] \}$ $|G| = n$

Theorem (I-N, K, I)

$G \subset GL(2,\mathbb{C})$ small. ($\forall \in$ Excep. locus)

- $\mathrm{Hilb}^G(\mathbb{C}^2)$: the minimal resolution of $\mathbb{C}^2/G$
- $I_0/mI_y \cong \begin{cases} \rho_i \otimes \rho_0 & y \in E_i,\ y \notin E_j\ (i \neq j) \\ \rho_i \otimes \rho_j \otimes \rho_0 & y \in E_i \cap E_j \end{cases}$ (m: max. ideal of $0 \in \mathbb{C}^2$ corr. to the origin)

G-Hilb $(\mathbb{C}^2)$ moduli space of G-clusters (Quiver representation)

Special

1988

Theorem (Wunram)

$\exists$ bijection

{"special" non-triv. (irred. rep)'s} $\leftrightarrow$ {irred. components E_i's} $\leftrightarrow$ $\{\rho_i\}$ indecomposable reflexive modules

via $C_1(\mathcal{F}_i) E_j = \delta_{ij}$.

Wunram, Riemenschneider

irred. rep $\{\rho_i\}$

Esnault

$G \subset GL(2,\mathbb{C})$

Small $:\Leftrightarrow$ the action of the group G is free outside 0 (origin)

$\rho : G \to V$ rep.

$G \curvearrowright \mathbb{C}^2 \times V$

a vector bundle on $(\mathbb{C}^2 \setminus \{0\})/G$

$\downarrow$ extend.

a reflexive sheaf $\mathcal{F}$ on $X = \mathbb{C}^2/G$

$\mathcal{F}$: ref. sheaf on X. π: min resol. $\tilde{X} \to X$

$\mathcal{F}_i := \pi^* \mathcal{F}$/torsion

$\mathbb{C}^2/G$

$\exists$ minimal resolution

exceptional curves

$\#\rho_i > \#E_i$

!!

3 DIM McKay Correspondence

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Part II Yukari Ito

Introduction

Recent develop

Nolla de Celis – Sekiya

Flops and mutations for crepant resolutions of polyhedral singularities

arXiv:1108.2352

GCSO(3), flops

modulo sp of rep. of quiver with relations

Sekiya-Takahashi (M.thesis) abelian

GCSL(3,ℂ) "essential rep." {ρi} ⟷ {1}!

a part of quiver

From Superstring Theory

Orbifold Euler characteristic

Dixon-Harvey-Vafa-Witten 1985

$$\chi(M,G)=\frac{1}{|G|}\sum_{gh=hg}\chi(M^{\langle g,h\rangle})$$

M : smooth complex Kähler manifold

G : finite group acting on M

g,h ∈ G.

Vafa's formula !?

$\chi(M,G)=\chi(\widetilde{M/G})$

Conjecture

Hirzebruch-Höfer 1990

crepant resolution

GCSL(3,ℂ) : finite

∃? resolution of sing. $\sigma:\widetilde{X}\to\mathbb{C}^3/G=X$

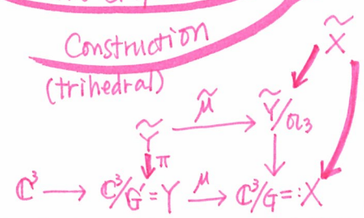
where $K_{\widetilde{X}}\simeq\mathcal{O}_{\widetilde{X}}$ and

χ(X̃) = #{conjugacy classes of G}

Crepant resolution

∃several crepant resolution

Construction (trihedral)



combination of toric resolution

GCSL(3,ℂ) Abelian

Toric geometry

Markushevich 1982

Roan 1989

168 Markushevich 1994

G : simple

G : Trihedral

G=⟨Abelian, (0 1 0 / 0 0 1 / 1 0 0)=T⟩

60 Roan 1994

Ito 1995

Other cases

Monomial Ito 1995

Roan 1996

McKay Correspondence

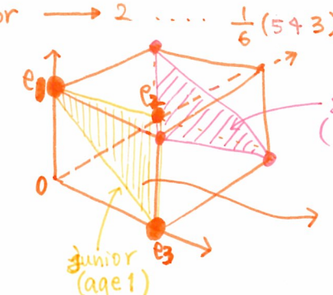
Ito – Reid 1996

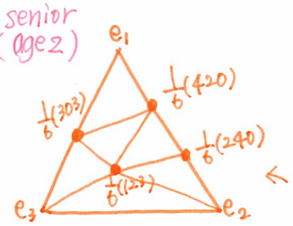
toric resolution $(\frac{1}{6}(1,2,3))$

$$G=\left\langle\begin{pmatrix}\varepsilon&0&0\\0&\varepsilon^2&0\\0&0&\varepsilon^3\end{pmatrix},\varepsilon^6\right\rangle=\frac{1}{6}(1,2,3)$$

$N:=\mathbb{Z}^3+\frac{1}{6}(1,2,3)\mathbb{Z}$

age	0	$\frac{1}{6}(0,0,0)$
junior	1	$\frac{1}{6}(1,2,3)$ $\frac{1}{6}(2,4,0)$, $\frac{1}{6}(3,0,3)$ $\frac{1}{6}(4,2,0)$
senior	2	$\frac{1}{6}(5,4,3)$





← a crepant resolution of ℂ3/G

Correspondence

{conjugacy class of age i} ↕ {base of $H^{2i}(\widetilde{X},\mathbb{Q})$}

for any crepant resol. X̃

age

$g^r=E$

$g\in G$, $g\sim\begin{pmatrix}\varepsilon^{a_1}&&0\\&\ddots&\\0&&\varepsilon^{a_n}\end{pmatrix}$ $0\le a_i\le r-1$

$age(g):=\frac{1}{r}\sum a_i$

G-Hilbert scheme

Quiver representation [Craw-Maclagan-Thomas] 2007

G-Hilb(ℂn) ≅ Mθ(Q,R)

moduli space of McKay quiver with relations

θ : 0-generated

Craw-Reid 2002

abelian case

How to calculate A-Hilb ℂ3

Difficult to see non-abelian examples

G-Hilb(ℂ3) = HilbG(ℂ3)

G-Hilb(ℂn) := the moduli space of G-clusters

G-cluster Z := G-invariant subscheme Z ⊂ ℂn

s.t. $H^0(\mathcal{O}_Z)\cong R_G$ regular rep of G as ℂ[G]-modules.

Theorem (BKR)

(G ⊂ Aut X : finite subgroup)

(X : nonsing. quasiprojective cpx 3-fold)

Y := G-Hilb(X) is irreducible.

and Y → X/G is a crepant resolution

Moreover Φ : D(G-Hilb(X)) → DG(X) is an equivalence of derived categories.

s.t. the stabilizer subgp of ∀pt x∈X acts on the tangent sp TxX as a subgp. of SL(TxX)

~2001~

HilbG(ℂ3) crepant

Nakamura

GCSL(3,ℂ) Abelian

Bridgeland-King-Reid

2001

∀GCSL(3,ℂ)

Fourier Mukai

Derived Equivalence

D(G-Hilb(ℂ3)) ≅ DG(ℂ3)

2010

Ishii-Ito-Nolla de Celis arXiv:1108.2310

Hilb of Hilb

crepant resol by [BKR]

Hilb of Hilb

Example

G-constellations

$M_{23}=\{1,\boxed{x},\eta_2,\eta_2\boxed{x},\eta_2^2,\eta_2^2\boxed{x}\}=\{1,x,x^2,x^3,x^4,x^5\}$

$M_{11}=\{1,\boxed{z},\xi_1,\xi_1\boxed{z},\xi_1^2,\xi_1^2\boxed{z}\}=\{1,z,y,yz,y^2,y^2z\}$

$M_{12}=\{1,\boxed{z},\xi_1,\xi_1\boxed{z},\varepsilon_1,\varepsilon_1\boxed{z}\}=\{1,z,y,yz,\frac{x}{z},x\}$

$M_{13}=\{1,\boxed{z},\xi_1^2,\xi_1^2\boxed{z},\varepsilon_1,\varepsilon_1\boxed{z}\}=\{1,z,\frac{x^2}{z^2},\frac{x^2}{z},\frac{x}{z},x\}$

$M_{21}=\{1,\boxed{x},\xi_2,\xi_2\square,\xi_2^2,\xi_2^2\square\}=\{1,x,\frac{y}{z},xy,y^2,xy^2\}$

$M_{22}=\{1,\varepsilon,\varepsilon_2^2,\boxed{x},\varepsilon\boxed{x},\varepsilon_2^2\boxed{x}\}=\{1,x,\frac{z}{y},z,\frac{z^2}{y},\frac{z^2}{x}\}$

G = 1/6(123)

G-Hilb(ℂ3)

G = 1/3(1,0,1) × 1/3(1,2,0)

ℤ3⊗ℤ-Hilb (ℤ3⊗ℤ-Hilb(ℂ3))

1/3(1,0,2)-Hilb(U1)

1/3(1,1,1)-Hilb(U2)

1/2(1,0,0)-Hilb(ℂ3)

1/3(102)

1/3(1,1,1)

Extend.

For G ⊲ N1 ⊲ N2,

G/N1-Hilb(N1/N2-Hilb(N2-Hilb(ℂ3))) can be considered.

Theorem (IIN)

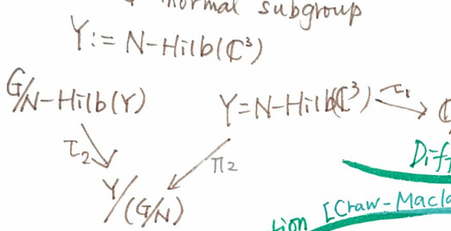
G/N-Hilb(N-Hilb(ℂ3))

• ≅ Moduli space of G-constellations F

• ≇ G-Hilb(ℂ3) in general

For X := G-equivariant coherent sheaf on X s.t. $H^0(\mathcal{F})\cong R_G$ as ℂ[G]-modules

Def. GCSL(3,ℂ)

N ⊲ G normal subgroup

Y := N-Hilb(ℂ3)