Landau-Ginzburg / Calabi-Yau Correspondence of all genera for elliptic orbifold $\mathbb{P}^1$.

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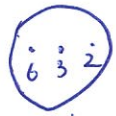
0. Elliptic orbifold $\mathbb{P}^1$ with 3 orbifold points.

← moduli of elliptic curves

E_0 E_1

Quotient of elliptic curves: $E_0/\mathbb{Z}/3\mathbb{Z}$ $E_1/\mathbb{Z}/4\mathbb{Z}$ $E_0/\mathbb{Z}/6\mathbb{Z}$

$\cong$ $\cong$ $\cong$



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elliptic orbifold $\mathcal{X} = \mathbb{P}^1_{3,3,3}$, $\mathbb{P}^1_{4,4,2}$, $\mathbb{P}^1_{6,3,2}$

object: Gromov-Witten theory of $\mathcal{X}$.

1. Landau-Ginzburg / Calabi-Yau correspondence.

Data: (W, G)

In this talk:

- $\underline{W: \mathbb{C}^N \longrightarrow \mathbb{C}}$ nondegenerate quasi-homogeneous polynomial of CY type.

  (1) $\exists ! \, q_i \in \mathbb{Q}_+$, s.t. $W(\lambda^{q_1}x_1, \cdots, \lambda^{q_N}x_N) = \lambda (W(x_1, \cdots, x_N))$, $\forall \lambda \in \mathbb{C}^*$.

  (2) $0 \in \mathbb{C}^N$ is an isolated critical point,

  (3) $\sum_{i=1}^{N} q_i = 1$. (CY-condition)

- $\underline{G: \text{symmetry group}}$ $\langle J \rangle < G < Aut(W) = G_W$.

$$Aut(W) = \left\{ diag(\lambda_1, \cdots, \lambda_N) \,\middle|\, W(\lambda_1 x_1, \cdots, \lambda_N x_N) = W(x_1, \cdots, x_N) \right\}$$

$$J = diag\left(e^{2\pi i q_1}, \cdots, e^{2\pi i q_N}\right).$$

CY-side : $G \xrightarrow{\text{acts}} X_W = \{ W = 0 \} \xhookrightarrow{\text{hypersurface}} W\mathbb{P}^{N-1}(q_1 d, \cdots, q_N \cdot d)$, $\langle J \rangle \xrightarrow{\text{trivial}} X_W$

$q_i \cdot d \in \mathbb{N}$.

CY-orbifold $\mathcal{X} = X_W / \hat{G}$, $\hat{G} = G / \langle J \rangle$.

E.g. $W_1 = x_1^3 + x_2^3 + x_3^3$
$W_2 = x_1^2 x_2 + x_2^3 + x_1 x_3^2$ $\quad G = \mathrm{Aut}(W_i) \Longrightarrow \mathcal{X} =$ $\mathbb{P}^1$-orbifolds in Section 0.
$W_3 = x_1^3 + x_2^3 + x_1 x_3^2$

LG/CY correspondence $=$ FJRW theory of (W, G) V.S. GW theory of $X_W / \hat{G}$

2. GW V.S. FJRW.

GW of $\mathcal{X} = X_W / \hat{G}$.

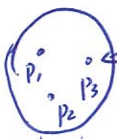
- $H^*_{CR}(\mathcal{X})$ Chen-Ruan cohomology

$= H^{\hat{*}}(\Lambda \mathcal{X})$ inertial orbifold $\Lambda \mathcal{X} = \coprod_{\substack{(g) \\ g \in \hat{G}}} (X_W)^g / C(g)$

$(g) \leftarrow$ conjugacy class, $(X_W)^g \leftarrow$ g-fixed locus, $C(g) \leftarrow$ centralizer

e.g. $H^*_{CR}(\mathbb{P}^1_{3,3,3}) = \bigoplus_{i=1}^{3} \bigoplus_{j=1}^{2} \mathbb{C} \cdot \Delta_{i,j} \oplus \mathbb{C} \cdot \mathbb{1} \oplus \mathbb{C} \cdot D$

$\mathbb{1} \in H^0_{CR}(\mathbb{P}^1_{3,3,3})$, $D \in H^2_{CR}(\mathbb{P}^1_{3,3,3})$



$\Delta_{3,j}$. $j \in \mathbb{Z}/3\mathbb{Z}$.

Moduli space of stable maps

- $\overline{\mathcal{M}}_{g,n,\beta}(\mathcal{X}) = \{ f : \Sigma \xrightarrow{\text{stable}} \mathcal{X} \mid f_*([\Sigma]) = \beta \in H^2(\mathcal{X}, \mathbb{Z}) \}$

$\Sigma \uparrow$ orbi-curve.

$$\overline{\mathcal{M}}_{g,n} \xleftarrow{\text{forgetful}} \overline{\mathcal{M}}_{g,n,\beta}(\mathcal{X}) \xrightarrow{\text{evaluation}} \Lambda \mathcal{X}$$

$$(\Sigma) \xleftarrow{p} [f, \Sigma \to \mathcal{X}] \xmapsto{ev_i} f(p_i)$$

GW Ancestor correlators: $\alpha_i \in \mathcal{H}^*_{CR}(\mathcal{X})$.

$$\langle \tau_{l_1}(\alpha_1), \cdots, \tau_{l_n}(\alpha_n) \rangle_{g,n,\beta} = \int_{[\overline{\mathcal{M}}_{g,n,\beta}(\mathcal{X})]^{vir}} \prod_{i=1}^{n} (p^*\psi_i)^{l_i} \prod_{i=1}^{n} ev_i^*(\alpha_i)$$

$$= \int_{\overline{\mathcal{M}}_{g,n}} \prod \psi_i^{l_i} \cdot \Lambda^{\mathcal{X}}_{g,n,\beta}$$

Cohomological Field Theory (CohFT)

$$\Lambda^{\mathcal{X}}_{g,n,\beta} : \mathcal{H}^*_{CR}(\mathcal{X})^{\otimes n} \longrightarrow H^*(\overline{\mathcal{M}}_{g,n})$$

GW-classes.

GW Correlation function: $\mathbf{t} \in \mathcal{H}^*_{CR}(\mathcal{X})$

$$\langle\langle \cdots \rangle\rangle_{g,n,\beta}(t) = \sum_{k \geq 0} \frac{1}{k!} \langle \cdots, \underbrace{t, \cdots, t}_{k} \rangle_{g,n+k,\beta} \quad \text{formal series}$$

Ancestor generating function:

$$\mathcal{A}^{GW}(\mathcal{X}) = e^{\sum_{g\geq 0} \hbar^{2g-2} \mathcal{F}_g^{GW}(\mathcal{X})}.$$

where
$$\mathcal{F}_g^{GW}(\mathcal{X}) = \sum_{\alpha_{i_j}, l_i, \beta, n} \langle \tau_{l_1}(\alpha_{i_1}), \cdots \rangle_{g,n,\beta} \frac{Q^\beta}{n!} q_{i_1}^{l_1} \cdots q_{i_n}^{l_n}$$

Similarly $\mathcal{A}^{GW}(\mathcal{X})(t)$ via $\langle\langle \cdots \rangle\rangle(t)$.

- FJRW of (W, G)
  - FJRW vect space. $\mathcal{H}^*_{FJRW}(W,G) = \bigoplus_{g\in G} \left(H^*(\mathbb{C}_g, W_g^\infty, \mathbb{C}) \right)^G = \bigoplus_g \mathcal{H}_g$.

    G-inv part of rel cohomology.

    $\mathbb{C}_g$: fixed locus of g.

    W_g^∞: $\mathrm{Re}^{-1} W_g\,(-\infty, -M)$ for $M \gg 0$.
  - $N_g = \dim(\mathbb{C}_g)$. $N_g = 0$. narrow sectors. $N_g \neq 0$ broad sectors.

e.g. $W = X_1^3 + X_2^3 + X_3^3$

$$\mathcal{H}^*_{FJRW}(W, G_W) = \bigoplus_{1 \le i,j,k \le 2} \mathbb{C} \cdot e_{(i,j,k)} \qquad e_{(i,j,k)} \text{ generator.}$$

• <u>W-structure</u>: $W = \sum_{j=1}^{s} \prod_{i=1}^{N} x_i^{b_{ij}}$

$$\mathcal{W}_{g,n} = \left\{ (\Sigma, \mathcal{L}_1, \cdots, \mathcal{L}_N) \,\middle|\, \bigotimes_{i=1}^{N} \mathcal{L}_i^{\otimes b_{ij}} \overset{\varphi_j}{\simeq} K_{\Sigma,\log} \text{ for } \forall j \right\} \Big/ \sim$$

• $K_{\Sigma,\log} := K(\Sigma - \{p_1, \cdots, p_n\})$

• $\mathcal{L}_i \to \Sigma$: orbi-line bundle $\qquad r_{p_i}: \underset{\substack{\uparrow \\ \text{isotropy group for } p_i}}{G_{p_i}} \longrightarrow \mathrm{Aut}(\mathcal{L})$

<u>Decomposition</u>: $\mathcal{W}_{g,n} = \coprod \mathcal{W}_{g,n}(\gamma_1, \cdots, \gamma_n)$, $\quad \gamma_i = r_{p_i}(1) \in G$.

$$\mathcal{W}_{g,n} \xrightarrow{st} \overline{\mathcal{M}}_{g,n}$$

<u>Virtual cycle</u>. $\left[\mathcal{W}_{g,n}(\gamma_1, \cdots, \gamma_n)\right]^{vir} \in H_*\left(\mathcal{W}_{g,n}(\gamma_1, \cdots, \gamma_n)\right) \otimes \prod_{i=1}^{n} \underset{\text{dual space}}{H_{\gamma_i}^*}$

<u>Coh FT</u>: $\Lambda^{FJRW}_{g,n}(\alpha_1, \cdots, \alpha_n) = \dfrac{|G|^g}{\deg(st)} PD\, st_*\left(\left[\mathcal{W}_{g,n}(\gamma_1, \cdots, \gamma_n)\right]^{vir} \cap \prod_{i=1}^{n} \alpha_i \right)$.

$\alpha_i \in H_{\gamma_i}$.

Use $\Lambda^{FJRW}_{g,n}$ to define $\langle\ \rangle^{FJRW}_{g,n}$, $\mathcal{A}^{FJRW}(W, G)$, etc.

3. Ruan's Conj of LG/CY correspondence.

(I) $H^*_{CR}(X_W/\hat{G}) \cong H^*_{FJRW}(W, G)$ as graded vect space

(II) $\exists$ symplectic transformation U (up to analytic continuation)

s.t. $U: \mathcal{L}_{FJRW}(W, G) \xrightarrow{\cong} \mathcal{L}_{GW}(X_W/\hat{G})$

- Both $\mathcal{L}_{FJRW}$, $\mathcal{L}_{GW}$ are Lagrangian in $H[[z, z^{-1}]]$.
- This is a statement for genus $= 0$ theory

(III) $\exists$ Quantization $\hat{U}$: $\mathcal{A}^{FJRW}(W, G) \longrightarrow \mathcal{A}^{GW}(X_W/\hat{G})$

Some results:

Chiodo-Ruan: (I) is true for all (W, G)

Chiodo-Ruan: (II) is true for W: Fermat quintic, $G = \langle J \rangle$

Chiodo-Iritani-Ruan: (II) is true for all W = Fermat, $G = \langle J \rangle$.

Our Main Theorem: [Krawitz - S] + [Milanov - Ruan]

Ruan's Conj (III) is true for $\mathcal{X} = \mathbb{P}^1_{3,3,3}, \mathbb{P}^1_{4,4,2}, \mathbb{P}^1_{6,3,2}$ and their FJRW counterparts.

- (W, G) for $\mathcal{X}$ are listed in section 1.

Reconstruction
- $\langle \quad \rangle^{\mathcal{X}}_{g,n,\beta}$ can be reconstructed from Chen-Ruan product & $\langle \Delta_{1,1}\Delta_{2,1}\Delta_{3,1}\rangle_{0,3,1}$

Convergence
- $\langle\langle \quad \rangle\rangle^{FJRW}_{g,n}(t)$ converges to an analytic function near $t=0$

  $\langle\langle \quad \rangle\rangle^{GW}_{g,n}(t\phi_i^{<2} + sD)$ converges near $t=0$, $Re(s) \ll 0$.

4. Idea of proof (Global Mirror symmetry)

- Berglund-Hübsch-Krawitz mirror
  - Invertible singularity $W = \sum_{j=1}^{N} \prod_{i=1}^{N} x_i^{b_{ij}}$ $\quad E_W = (b_{ij})_{N\times N}$

$(W, G) \xrightarrow{\text{mirror}} (W^T, G^T)$ $\quad E_{W^T} = (E_W)^T$ transpose.

G^T constructed by Krawitz. $\quad G_W^T = \{1\}$.

- Miniversal deformation of singularities

$W^T = W_i^T \quad i=1,2,3$

$W^T_\sigma = W^T + \sigma xyz$. $\quad W^T_\sigma(\vec{s}, \vec{x}) = W^T_0 + \sum s_i\phi_i \quad \{\phi_i\}$ basis of $\mathcal{Q}_{W^T}$.

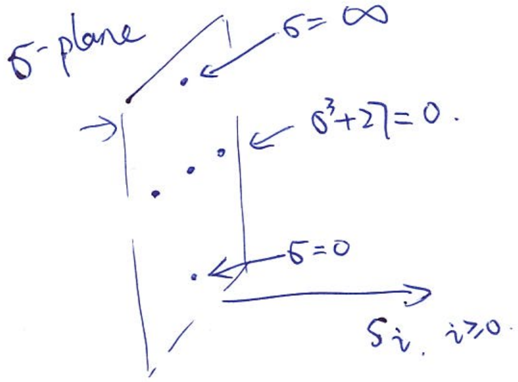
$s_{-1} = \sigma, \quad \phi_{-1} = xyz$

• K. Saito $\Rightarrow$ Frobenius mfd structure on $\vec{s} \in S$.

$\Rightarrow$ define $\langle\langle \quad \rangle\rangle^{SG}_{0,n}(\vec{s})$ primary. $H = \mathcal{Q}_{W^T}$ Jacobian ring.

For generic $\vec{s}$. Givental $\Longrightarrow$ $\langle\langle \cdots \rangle\rangle^{SG}_{g,n}(\vec{s})$ Ancestor correlators.

Global mirror symmetry (Picture)



$\mathcal{A}^{SG}_{W^T_\infty} \xlongequal{LG\text{-to-}CY} \mathcal{A}^{GW}(X_W/\widetilde{G}_W)$

$\mathcal{A}^{SG}_{W^T_0} \xlongequal{LG\text{-to-}LG} \mathcal{A}^{FJRW}(W, G_W)$

• $\mathcal{A}^{SG}_{W^T_\sigma}(\vec{s})$ well defined for generic $\vec{s}$.

• Reconstruction + Computation $\Longrightarrow$ mirror symmetry holds for semi-simple $\vec{s}$

• Convergence $\Rightarrow$ mirror symmetry holds for non-semi simple point.

$\Rightarrow$ LG-to-CY & LG-to-LG hold.

• B-model: [MR] $\mathcal{Q}\ \mathcal{A}^{SG}_{W^T_0} \xrightarrow[+\text{ Analytic Cont.}]{\text{Quantization}} \mathcal{A}^{SG}_{W^T_\infty}$

5. Global mirror Symmetry (Invertible simple elliptic singularities)

[Milanov - S]

• Classification of simple elliptic singularities (K. Saito)

$\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$

• Invertible normal forms. 13 types.

• 1-parameter family. 29 types.

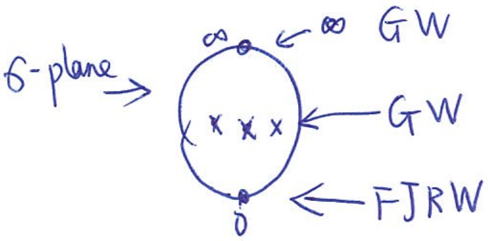
e.g. $W^T = x^3 z + y^3 + z^2 \quad \in \widetilde{E}_8$

$W^T + \sigma xyz$ or $W^T + \sigma x^4 y$.

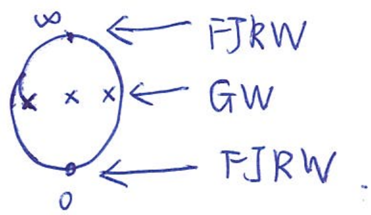
• Singular points: $\sigma = 0, \infty$, cubic (or quadratic) roots.

$\Rightarrow \lambda = f(\sigma)$. $\lambda = 0, 1, \infty$.

e.g. (1) W^T in section 1.



(2) $W^T_\sigma = x^6 + y^3 + z^2 + \sigma x^4 y$



6. Technical details.

• Reconstruction. $\langle \ \rangle_{g,n,d}$ $\quad g, n, d \downarrow$

• g-reduction.

Lemma: (Ionel, Pandharipande)

$P(\psi, \kappa)$ polynomial of ψ-classes & κ-classes, if $\deg P \begin{cases} \geq g & \text{for } g \geq 1 \\ \geq 1 & \text{for } g = 0 \end{cases}$

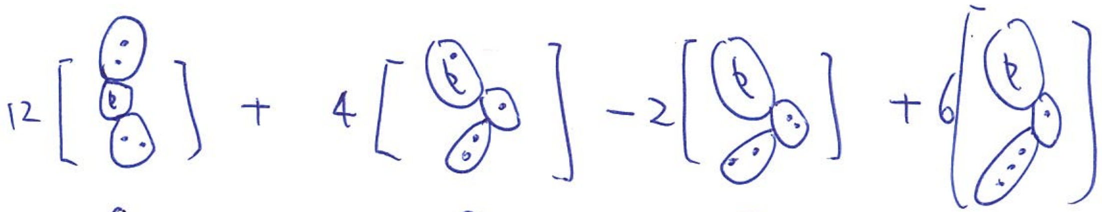
then $P(\psi, \kappa) = \sum PD[\Gamma_i] \cdot P_i(\psi, \kappa)$. Γ_i - dual graph with edge ≥ 1

$\int_{\overline{\mathcal{M}}_{g,n}} P(\psi, \kappa) \Lambda^{*}_{g,n,\beta} \xrightarrow{\text{split into}}$ lower g & $\deg P_i \leq \deg P$

+ dimension formula of $\langle \ \rangle^{*}_{g,n,d} \neq 0 \Rightarrow \begin{cases} g \leq 1 \\ \sum l_i = 0 \end{cases}$

- $g=1$. $\langle D\rangle^{\mathcal{X}}_{1,1,d}$ D: divisor

Getzler's Relation $H^*(\overline{M}_{1,4})$



= genus 0 graph

$\delta_{2,2}$ $\delta_{2,3}$ $\delta_{2,4}$ $\delta_{3,4}$

For $\mathcal{X} = \mathbb{P}^1_{3,3,3}$. Choose $\Delta_{x,2}\ \Delta_{x,2}\ \Delta_{y,1}\ \Delta_{z,1}$ as insertions

$$\int \delta_{3,4}\, \Lambda^{GW}_{1,4,d+1}(\Delta_{x,2}, \Delta_{x,2}, \Delta_{y,1}, \Delta_{z,1}) = \frac{4}{3}\langle D\rangle_{1,1,d} + \cdots$$

$\langle D\rangle_{1,1,d}$ leading term. $\int_{\delta_{2,2}}\Lambda = \int_{\delta_{2,3}}\Lambda = \int_{\delta_{2,4}}\Lambda = 0$.

$\Rightarrow \langle D\rangle_{1,1,d}$ reconstructible.

- $g=0$. Use WDVV.

- <u>Estimation</u>. (Use Reconstruction) For $\alpha_i = D$, $l_i \neq 0$, $\chi = 2g-2+n$, $L=\sum l_i$

$$\left|\langle \tau_{l_1}(\alpha_1), \cdots, \tau_{l_n}(\alpha_n)\rangle_{g,n,d}\right| \leq \begin{cases} d^{\chi-2}\, C(\chi)^{\chi+(g+L+1)d-2}, & d>0 \\ C(\chi)^{\chi-1}, & d=0 \end{cases}$$

$C(\chi)$: constant depends only on χ.

- Convergence.

$$\left|\langle\langle \tau_{l_1}(\alpha_1), \cdots, \tau_{l_n}(\alpha_n)\rangle\rangle_{g,n}(s\cdot D)\right| \overset{\substack{\text{divisor}\\ \text{axiom}}}{=} \left|\sum_{d,k}\langle\ \rangle_{g,n,d}\frac{s^k d^k}{k!}\right|$$

$$\leq \sum_d |e^s|^d\, d^{\chi-2}\, C(\chi)^{\chi+(g+L+1)\cdot d-2}.$$

Convergent $|e^s|\cdot C(\chi)^{g+L+1} \longrightarrow 0$. i.e. $\operatorname{Re} s \to -\infty$. $\square$