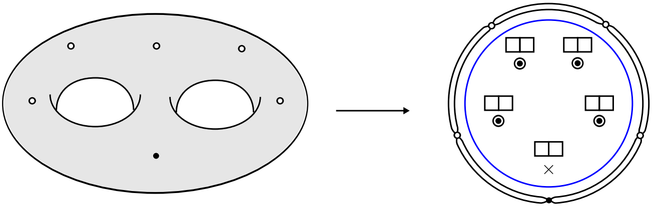
Ramification Points of Seiberg-Witten Curves

Chan Y. Park

California Institute of Technology

March 13, 2012 @ IPMU



Based on: C.Y.P., JHEP07(2011)068 [arXiv:1102.0288].

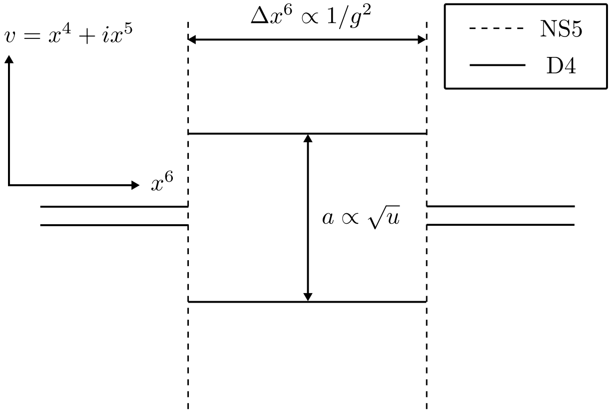
Outline

1. Introduction
   - M-theory description of the Seiberg-Witten theory
   - Multiple M5-branes wrapping a punctured Riemann surface
2. Ramification points of Seiberg-Witten curves
   - Ramification of an M5-brane over a Riemann sphere
   - $SU(2) \times SU(2)$ SCFT and the ramification point
   - $SU(3)$ pure gauge theory and Argyres-Douglas fixed points
3. Conclusion
   - Discussion and Outlook
   - Summary

Outline

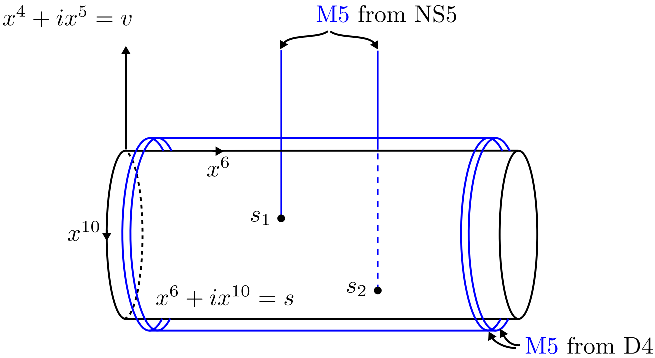
Four-dimensional theory from a IIA brane configuration

- The following brane configuration of type IIA string theory gives a four-dimensional $\mathcal{N} = 2$ $SU(2)$ superconformal field theory (SCFT). [Witten, 1997]



M-theory lift of IIA brane configuration

- When $u = 0$, lifting the brane system to M-theory gives



- After turning on $u \neq 0$, these M5-branes become a single M5-brane wrapping a complex curve $f(t, v) = 0$, where

$$f(t, v) = (t - t_1)(t - t_2)v^2 - ut, \;\; t = \exp(-s).$$

Seiberg-Witten curve and differential from M-theory

Seiberg-Witten curve C_{SW} from an M5-brane

- The M5-brane from the M-theory lift of the IIA brane system wraps a complex one-dimensional algebraic curve $f(t, v) = 0$.
- This curve is identified with C_{SW} of the four-dimensional theory from the IIA brane system.

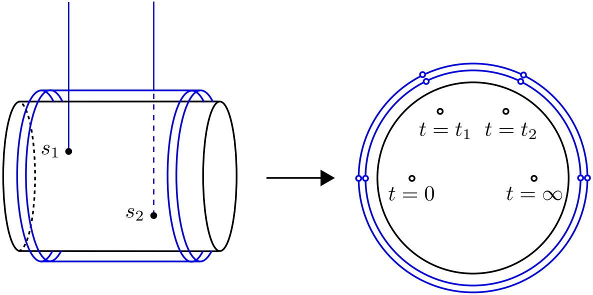
Seiberg-Witten differential λ and M2-branes

- When λ is integrated over a 1-cycle γ on C_{SW}, it gives the mass of the corresponding BPS state.
- The mass can also be calculated from an M2-brane ending on the M5-brane along the boundary γ, which gives us a unique $\lambda = \frac{v}{t} dt$. [Fayyazuddin-Spalinski, 1997] , [Henningson-Yi, 1997] , [Mikhailov, 1997]

Outline

Gaiotto's description of $\mathcal{N} = 2$ gauge theory

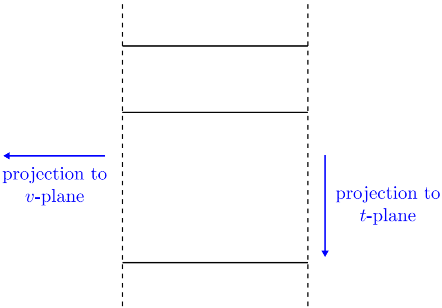
- The four-dimensional $\mathcal{N} = 2$ $SU(2)$ SCFT comes from two M5-branes wrapping a Riemann sphere with four punctures, C_{G}. [Gaiotto, 2009]



- The punctures are at the poles of $\lambda = \frac{v(t)}{t} dt$.
- The locations of the punctures depend only on the gauge coupling parameter.
- The Coulomb branch parameter corresponds to the deformations of the M5-branes along the fiber of $T^* C_{\mathsf{G}}$.

Two different descriptions of a Seiberg-Witten curve

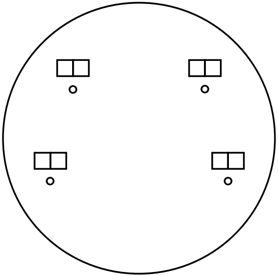
For the C_{SW} from the following IIA brane configuration, there are two ways of describing it as a covering space of a complex plane.



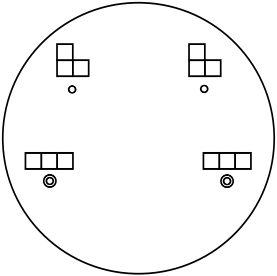
- When the v-plane is the base, C_{SW} is a hyperelliptic curve.
- When the t-plane is the base, C_{SW} is a three-sheeted covering of a Riemann sphere. [Martinec-Warner, 1995] [Hollowood, 1997]

Classification of punctures

- Each puncture can be characterized by a Young tableau.



$SU(2)$ SCFT

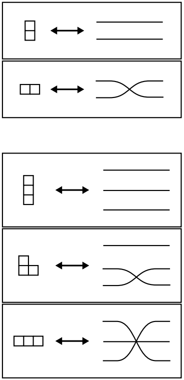


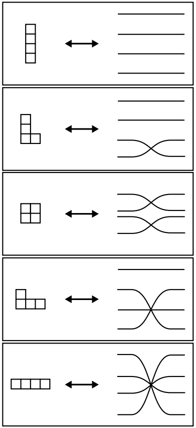
$SU(3)$ SCFT

- For an $SU(N)$ gauge theory, a Young tableau of a puncture has N boxes.

Young tableaux characterize ramification

- Young tableaux can describe the ramification structure of a covering space.





- Question: can we use this to understand the geometric origin of the classification of punctures?

Outline

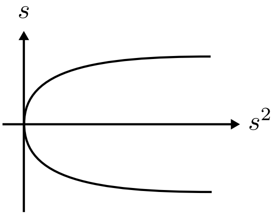
1. Introduction
   - M-theory description of the Seiberg-Witten theory
   - Multiple M5-branes wrapping a punctured Riemann surface

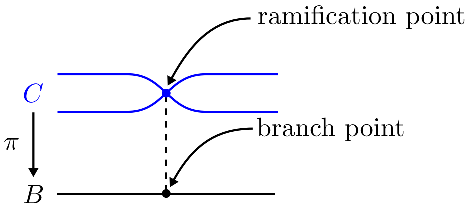
2. Ramification points of Seiberg-Witten curves
   - Ramification of an M5-brane over a Riemann sphere
   - $SU(2) \times SU(2)$ SCFT and the ramification point
   - $SU(3)$ pure gauge theory and Argyres-Douglas fixed points

3. Conclusion
   - Discussion and Outlook
   - Summary

Ramification points and branch points

- Consider a 2-sheeted covering map $\pi : C \to B,\ s \mapsto s^2$.





- When there is a nontrivial ramification in a covering map, there is a **ramification point** in the **covering space**.
- A ramification point is mapped by the covering map to a **branch point** in the **base space**.

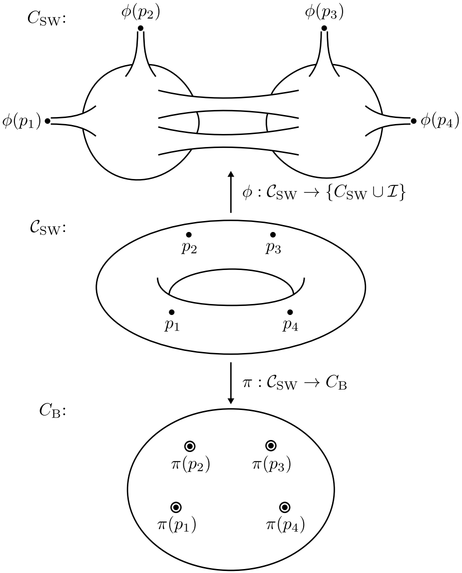
$\mathcal{C}_{\mathsf{SW}}$ wrapping a Riemann sphere

- Consider C_{SW} of $SU(2)$ SCFT,

$$f(t,v) = (t-1)(t-t_1)v^2 - ut.$$

- From the curve we can get $\mathcal{C}_{\mathsf{SW}}$, a compact Riemann surface.
- $\pi \circ \phi^{-1} : C_{\mathsf{SW}} \to C_{\mathsf{B}}$ is the covering map we want.
- We only need a local description of π around each p_i.
- Near a point $p_i \in \mathcal{C}_{\mathsf{SW}}$,

$$\phi_{p_i}(s) = (t(s), v(s))$$
$$\Rightarrow \pi_{p_i}(s) = t(s).$$



Ramification divisor and Riemann-Hurwitz formula

- When analyzing the ramification of an algebraic curve, it is convenient to introduce a ramification divisor R_π,

$$R_\pi = \sum_{p \in \mathcal{C}_{\mathsf{SW}}} (\nu_p(\pi) - 1)[p] = \sum_i (\nu_{p_i}(\pi) - 1)[p_i],$$

where $\nu_p(\pi)$ is the ramification index of a point p,

$$\pi_p(s) - \pi_p(0) \propto s^{\nu_p(\pi)},\ \nu_p(\pi) \in \mathbb{Z}^+.$$

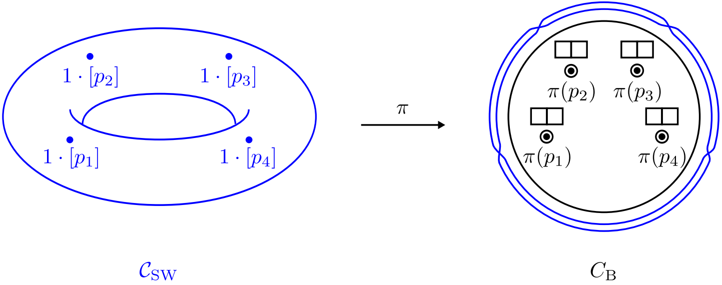
- The Riemann-Hurwitz formula provides a relation between $\deg(\pi)$, R_π, and $g(\mathcal{C}_{\mathsf{SW}})$.

$$\begin{aligned} \chi_{\mathcal{C}_{\mathsf{SW}}} &= \deg(\pi) \cdot \chi_{\mathbb{CP}^1} - \deg(R_\pi) \\ &\Leftrightarrow \deg(R_\pi) = 2(g(\mathcal{C}_{\mathsf{SW}}) + \deg(\pi) - 1). \end{aligned}$$

Example: $SU(2)$ SCFT

$$\pi_{p_1}(s) = s^2,\ \pi_{p_2}(s) = 1 + c_2 s^2,\ \pi_{p_3}(s) = t_1 + c_3 s^2,\ \pi_{p_4}(r^{-1}) = r^2.$$

$$\Rightarrow R_\pi = 1 \cdot [p_1] + 1 \cdot [p_2] + 1 \cdot [p_3] + 1 \cdot [p_4].$$

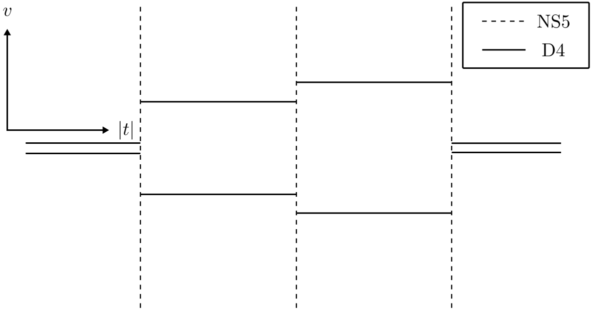


- Each branch point has the same Young tableau as the corresponding puncture!
- Consistent with the Riemann-Hurwitz formula, $\deg(R_\pi) = 1 + 1 + 1 + 1 = 4 = 2(g(\mathcal{C}_{SW}) + \deg(\pi) - 1)$.

Outline

Brane configuration and C_{SW} of $SU(2) \times SU(2)$ SCFT

- Consider the following IIA brane configuration that gives a four-dimensional $\mathcal{N} = 2$ $SU(2) \times SU(2)$ SCFT.

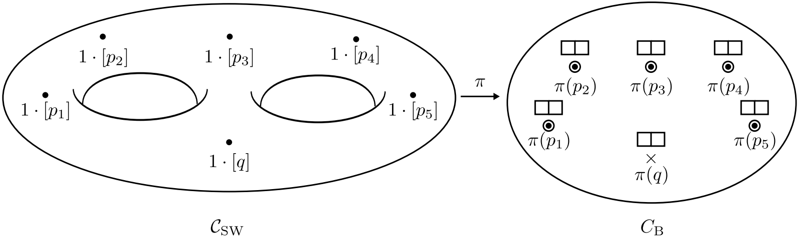


- C_{SW} of the theory is defined by the zero locus of

$$f(t, v) = (t-1)(t-t_1)(t-t_2)v^2 - u_1 t^2 - u_2 t.$$

$\mathcal{C}_{\text{SW}}$ of $SU(2) \times SU(2)$ SCFT and its ramification

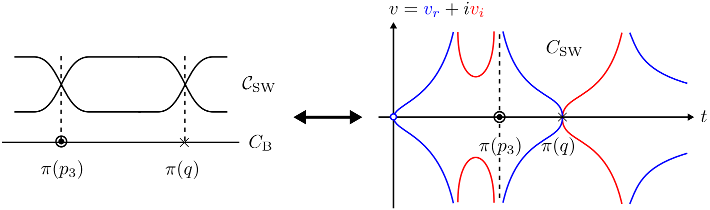
- The ramification of $\mathcal{C}_{\text{SW}}$ over C_{B} is



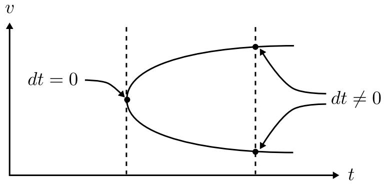
$\pi(q) = -u_2/u_1$: branch point of a different kind

- Comes from solving $dt(q) = 0$ along the curve $f(t, v) = 0$.
- Does not correspond to any puncture.
- Depends on the Coulomb branch parameters $\{u_1,\ u_2\}$
- Consistent with the Riemann-Hurwitz formula,
  $\deg(R_\pi) = 1+1+1+1+1+1 = 6 = 2(g(\mathcal{C}_{SW}) + \deg(\pi) - 1)$.

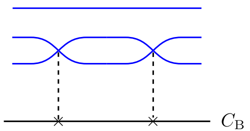
Two different kinds of branch points

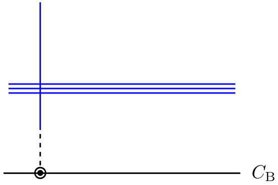


- $\{\pi(p_i)\}$ and $\pi(q)$ are from the ramification points of $\mathcal{C}_{\mathsf{SW}}$.
- When considering C_{SW}, $\{\pi(p_i)\}$ are from the points we added to compactify it, whereas $\pi(q)$ is from the ramification point of C_{SW}, which is located at $dt = 0$ along C_{SW}.



Physical interpretation of the branch points

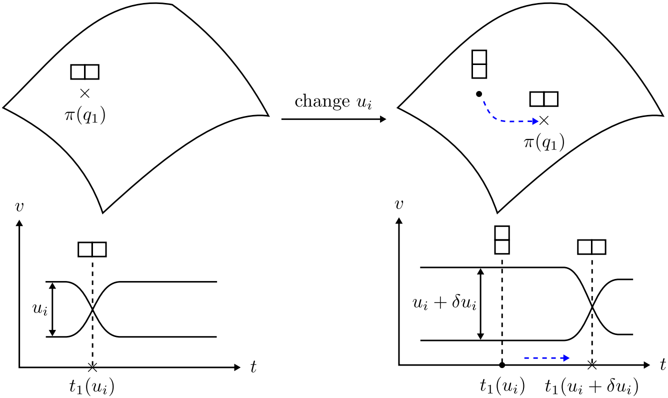




- A branch point from the ramification point of C_{SW} has a physical interpretaion of being a contact point of M5-branes. [Gaiotto-Moore-Neitzke, 2009]
- A puncture is understood as a real codimension two defect on M5-branes from a transversal M5-brane. [Gaiotto, 2009]

Moving the branch point over C_{B}

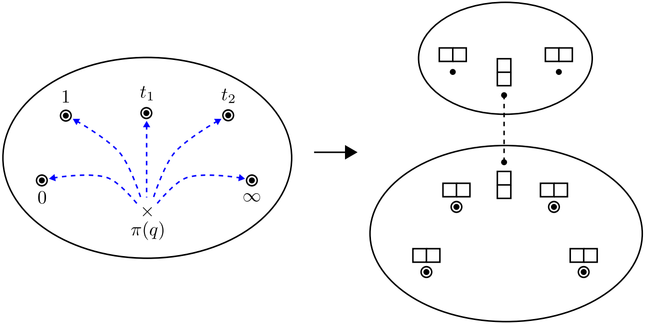
- Changing the Coulomb branch parameters moves the location of $\pi(q)$ on C_{B}.



- We can use this to illustrate various changes of Coulomb branch parameters.

Collision of two branch points

- When $\pi(q)$ collides with $\pi(p_i)$, the original Seiberg-Witten curve factors into two curves.

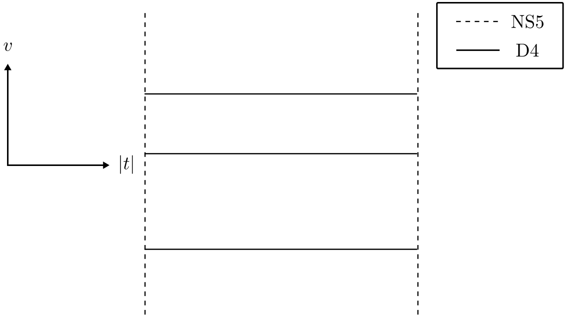


- The new branch point is a useful tool to visualize various interesting limits of Coulomb branch parameters.

Outline

Brane configuration of $SU(3)$ pure gauge theory

- The following IIA brane configuration gives a four-dimensional $\mathcal{N} = 2$ $SU(3)$ pure gauge theory.

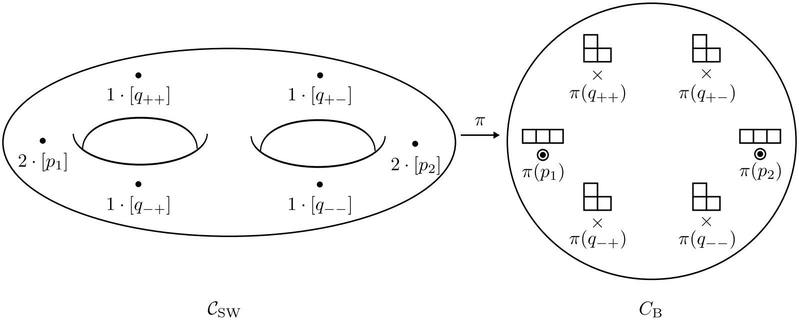


- C_{SW} of the theory is the zero locus of

$$f(t, v) = t^2 + (v^3 - u_2 v - u_3)t + 1.$$

$\mathcal{C}_{\mathsf{SW}}$ of $SU(3)$ pure gauge theory and its ramification

- By analyzing the ramification of $\mathcal{C}_{\mathsf{SW}}$ over C_{B}, we get



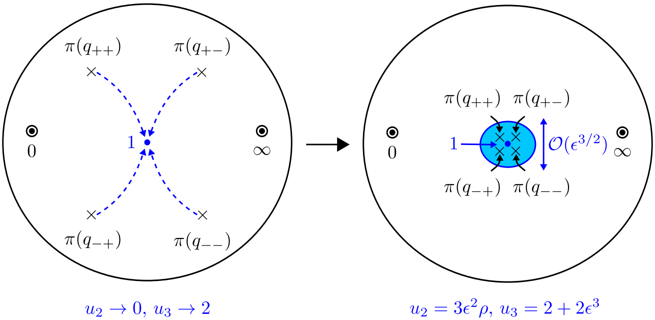
- $\pi(p_1) = 0,\ \pi(p_2) = \infty.$
- $\pi(q_{ab}) = ({v_{2a}}^3 + \frac{u_3}{2}) + b\sqrt{({v_{2a}}^3 + \frac{u_3}{2})^2 - 1},\ v_{2a} = a\sqrt{u_2/3}.$

Branch points near the limit of AD fixed points

- When the Coulomb branch parameters approach one of the Argyres-Douglas fixed points, [Argyres-Douglas, 1995]

$$u_2 = 0,\ \ u_3 = \pm 2,$$

$\{\pi(q_{ab})\}$ gather together around $t = 1$.



Rescaling of parameters

- Zoom in on the part of C_{B} near $t = 1$, that is, redefine the variables as

$$
\begin{aligned}
t &= 1 + i\epsilon^{3/2} w, \\
v &= \epsilon z, \\
u_2 &= 0 + 3\epsilon^2 \rho, \\
u_3 &= 2 + 2\epsilon^3,
\end{aligned}
$$

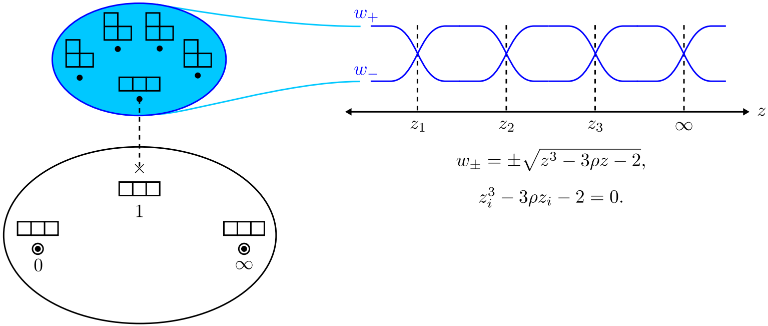
  and take $\epsilon \to 0$.
- Then $f(t, v)$ becomes

$$
\begin{aligned}
f(t, v) &= (t-1)^2 + \epsilon^3 (z^3 - 3\rho z - 2) t \\
&\approx (-w^2 + z^3 - 3\rho z - 2)\epsilon^3 + \mathcal{O}(\epsilon^{9/2}),
\end{aligned}
$$

Appearance of the small torus

- The genus 1 curve given by $w^2 = z^3 - 3\rho z - 2$ is the small torus that appears at the Argyres-Douglas fixed points.

[Argyres-Douglas, 1995]



$$w_\pm = \pm\sqrt{z^3 - 3\rho z - 2},$$

$$z_i^3 - 3\rho z_i - 2 = 0.$$

Outline

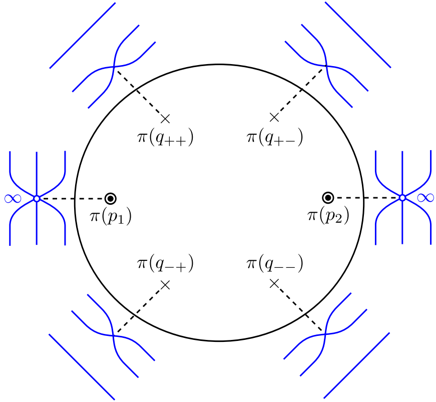
1. Introduction
   - M-theory description of the Seiberg-Witten theory
   - Multiple M5-branes wrapping a punctured Riemann surface

2. Ramification points of Seiberg-Witten curves
   - Ramification of an M5-brane over a Riemann sphere
   - $SU(2) \times SU(2)$ SCFT and the ramification point
   - $SU(3)$ pure gauge theory and Argyres-Douglas fixed points

3. Conclusion
   - Discussion and Outlook
   - Summary

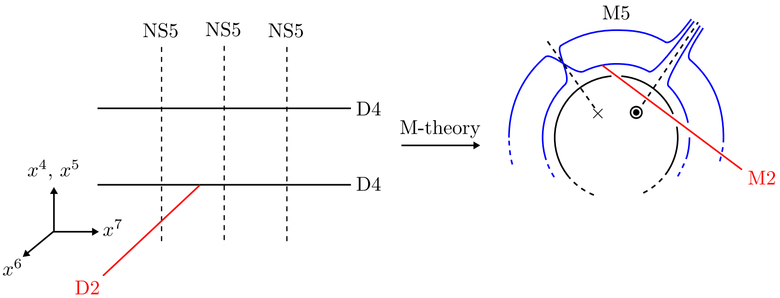
How to construct the global picture of $\mathcal{C}_{\mathsf{SW}}$ from C_{B}

- So far we analyzed only the local picture of $\mathcal{C}_{\mathsf{SW}}$ around its ramification points. How can we have the global description of it?
- How can we build the whole $\mathcal{C}_{\mathsf{SW}}$ from C_{B}?



M2-brane ending at a ramification point

- A D2(M2)-brane ending on the IIA brane system gives a surface operator. [Alday-Gaiotto-Gukov-Tachikawa-Verlinde, 2009]



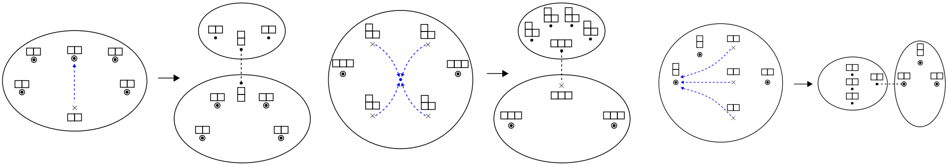
- Use this M2-brane to study the ramification point.
- Take the M2- and M5-brane system and turn it into IIB geometry plus D-brane. [Dijkgraaf-Hollands-Sułkowski-Vafa, 2007]
- Study the effective 2D theory on the M2-brane. [Hanani-Hori, 1997; Dorey, 1998; Gaiotto-Moore-Neitzke, 2011]

AGT conjecture and various limits of branch points

- M5-branes wrapping a punctured Riemann surface give us a correspondence between the four-dimensional $\mathcal{N} = 2$ gauge theory and the two-dimensional CFT on the Riemann surface.

  [Alday-Gaiotto-Tachikawa, 2009]

- Various collisions of branch points give us the factorization of the base Riemann surface. This may be observed, through the AGT conjecture, as $Z \to Z_1 Z_2$, and/or $\mathcal{F} \to \mathcal{F}_1 \mathcal{F}_2$.



Outline

Summary

- When the Seiberg-Witten curve of a four-dimensional $\mathcal{N} = 2$ gauge theory wraps a Riemann surface as a multi-sheeted cover, the curve develops ramification points that are mapped to branch points on the Riemann surface.
- The branch points are different from the punctures of Gaiotto, and their locations on the Riemann surface depend in general on every parameter of the theory, including Coulomb branch parameters.
- These branch points can help us to explore interesting physics in various limits of the parameters, including Argyres-Douglas fixed points.