2d Gauge/Bethe correspondence from String Theory

Susanne Reffert
CERN

based on work with with D. Orlando and S. Hellerman
(arXiv:1106.2097, 1108.0644, 1111.4811)

The Gauge/Bethe Correspondence

Relates susy gauge theories in 2d/4d to (quantum) integrable systems.

The susy vacua of the gauge theory correspond to a sector of the Bethe spectrum of the spin chain.

Generators of chiral ring correspond to commuting Hamiltonians.

Nekrasov, Shatashvili

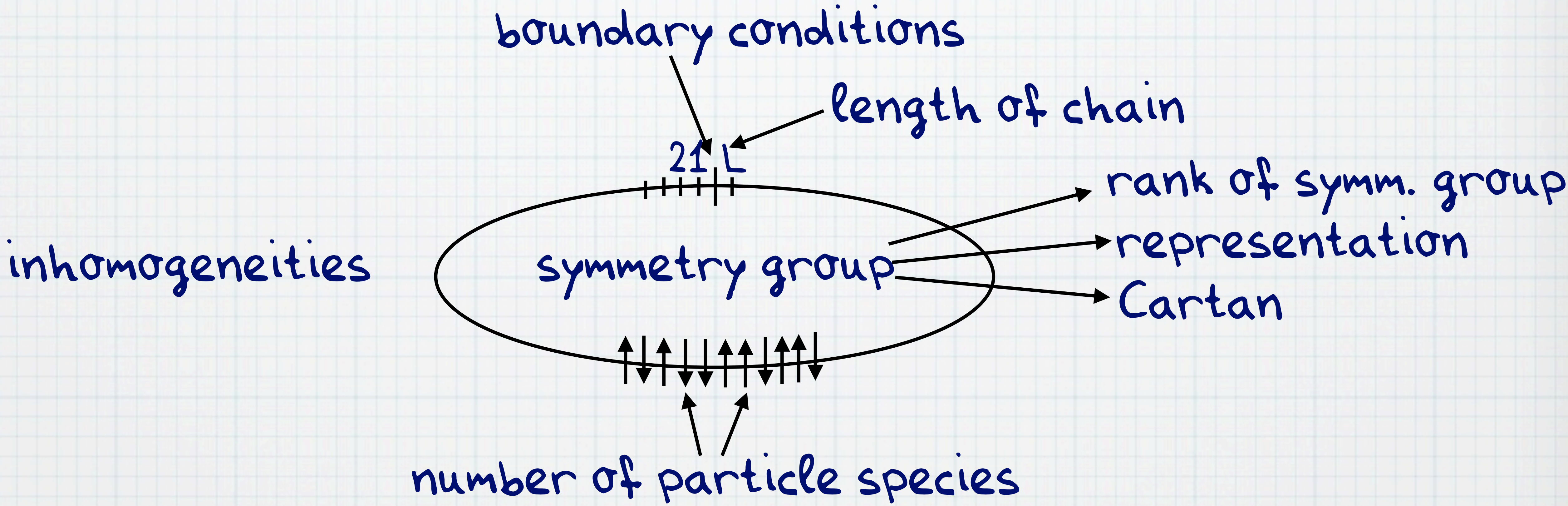
Integrable model: spectrum determined by Bethe equations.

Gauge theory: ground states determined by eff. twisted superpotential.

Correspondence works for all Bethe solvable integrable models.

The Gauge/Bethe Correspondence

What are the parameters of a spin chain?



spectrum is given by solutions of

rapidities

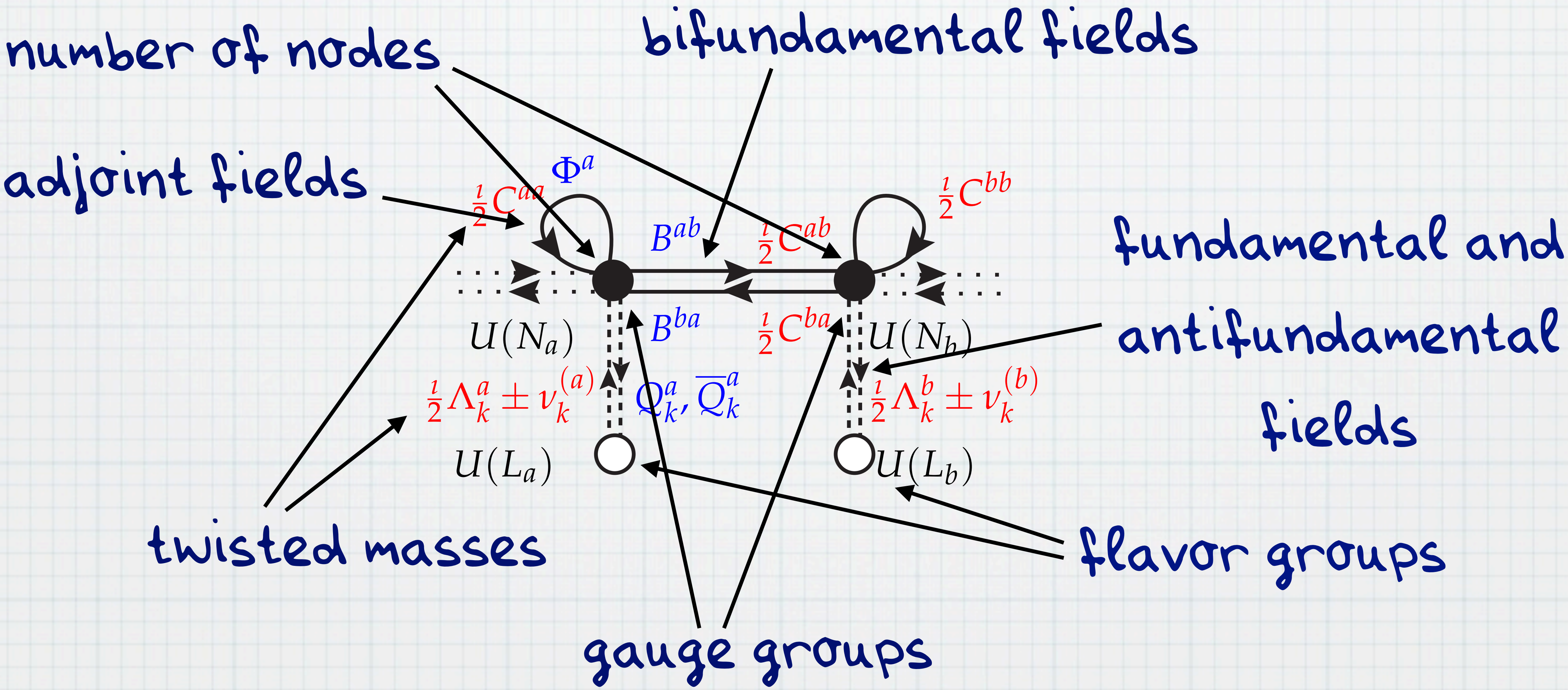
$$e^{2\pi i \, \mathrm{d}Y(\lambda)} = 1$$

Yang counting fn (potential for Bethe equations)

The Gauge/Bethe Correspondence

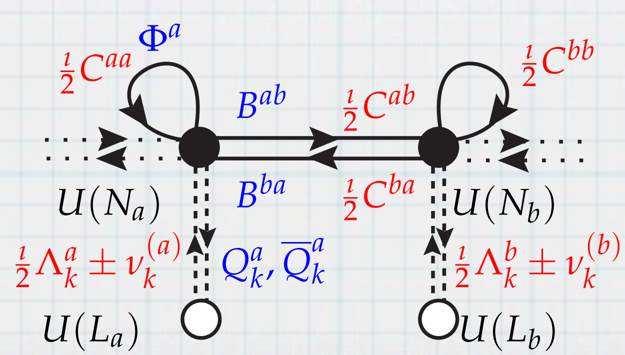
General case: quiver gauge theories

What are the parameters?

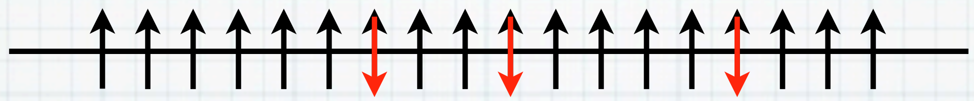


The Dictionary

gauge theory			integrable model
number of nodes in the quiver	r	r	rank of the symmetry group
gauge group at a–th node	$U(N_a)$	N_a	number of particles of species a
effective twisted superpotential	$\widetilde{W}_{\mathrm{eff}}(\sigma)$	$Y(\lambda)$	Yang–Yang function
equation for the vacua	$e^{2\pi \mathrm{d}\widetilde{W}_{\mathrm{eff}}} = 1$	$e^{2\pi \imath \mathrm{d}Y} = 1$	Bethe ansatz equation
flavor group at node a	$U(L_a)$	L_a	effective length for the species a
lowest component of the twisted chiral superfield	$\sigma_i^{(a)}$	$\lambda_i^{(a)}$	rapidity
twisted mass of the fundamental field	$\widetilde{m}^{\mathrm{f}}{}_k^{(a)}$	$\frac{1}{2}\Lambda_k^a + \nu_k^{(a)}$	highest weight of the representation and inhomogeneity
twisted mass of the anti–fundamental field	$\widetilde{m}^{\bar{\mathrm{f}}}{}_k^{(a)}$	$\frac{1}{2}\Lambda_k^a - \nu_k^{(a)}$	highest weight of the representation and inhomogeneity
twisted mass of the adjoint field	$\widetilde{m}^{\mathrm{adj}\,(a)}$	$\frac{1}{2}C^{aa}$	diagonal element of the Cartan matrix
twisted mass of the bifundamental field	$\widetilde{m}^{\mathrm{b}\,(ab)}$	$\frac{1}{2}C^{ab}$	non–diagonal element of the Cartan matrix
FI–term for $U(1)$–factor of gauge group $U(N_a)$	τ_a	$\hat{\vartheta}^a$	boundary twist parameter for particle species a



The Big Picture



For the integrable system, all N magnon sectors must be considered together.

This is usually not done for U(N) gauge theories with different N.

Yet, there is evidence that it would make sense to do so.

Study the simplest example of the spin 1/2 XXX spin chain (Heisenberg model).

The low energy limit of the corresponding gauge theory is given by the NLSM with target space the cotangent bundle of Gr(N,L).

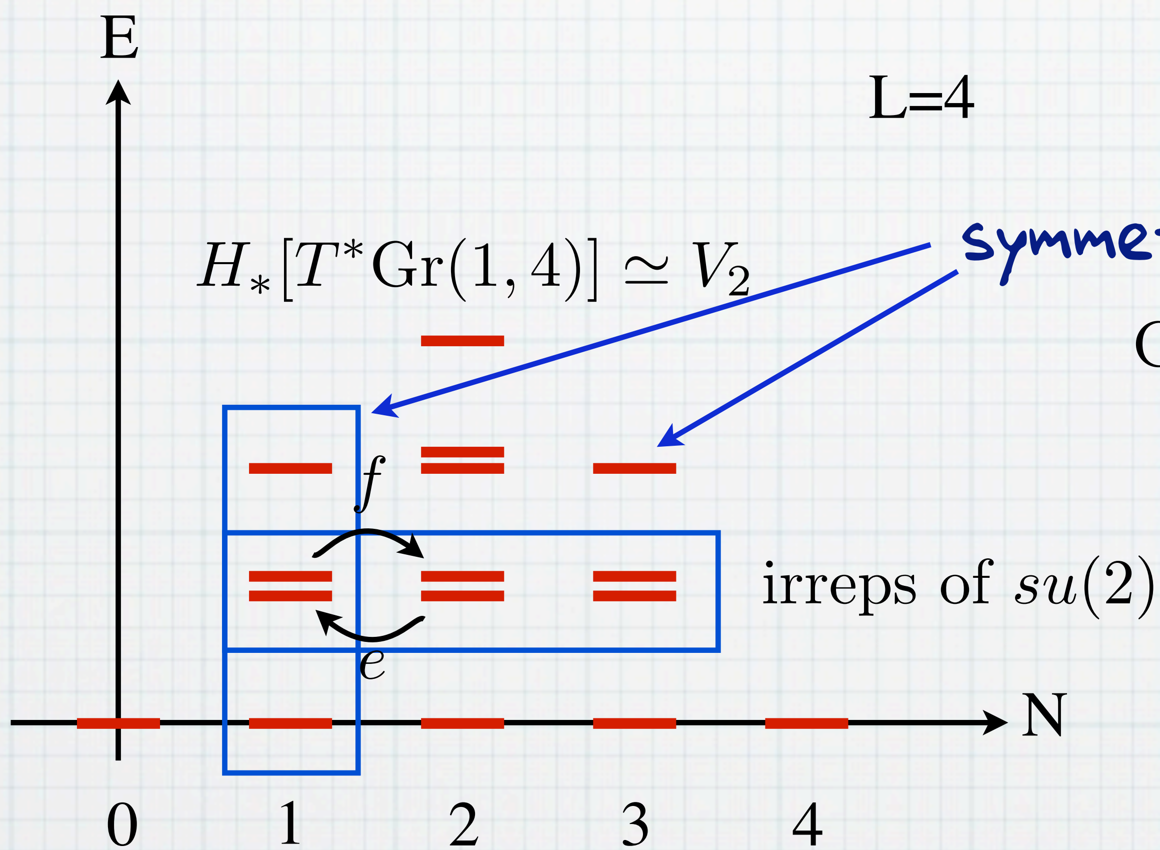
$$\mathrm{Gr}(N,L) = \{W \subset \mathbb{C}^L | \mathrm{dim} W = N\}\,,$$

$$T^*\mathrm{Gr}(N,L) = \{(X,W), W \in \mathrm{Gr}(N,L), X \in \mathrm{End}(\mathbb{C}^L) | X(\mathbb{C}^L) \subset W, X(W) = 0\}$$

Its ground states are given by the cohomology of T*Gr(N,L)

The Big Picture

Study the spectrum of L=4 spin 1/2 XXX spin chain:



symmetry $N, \quad L-N$

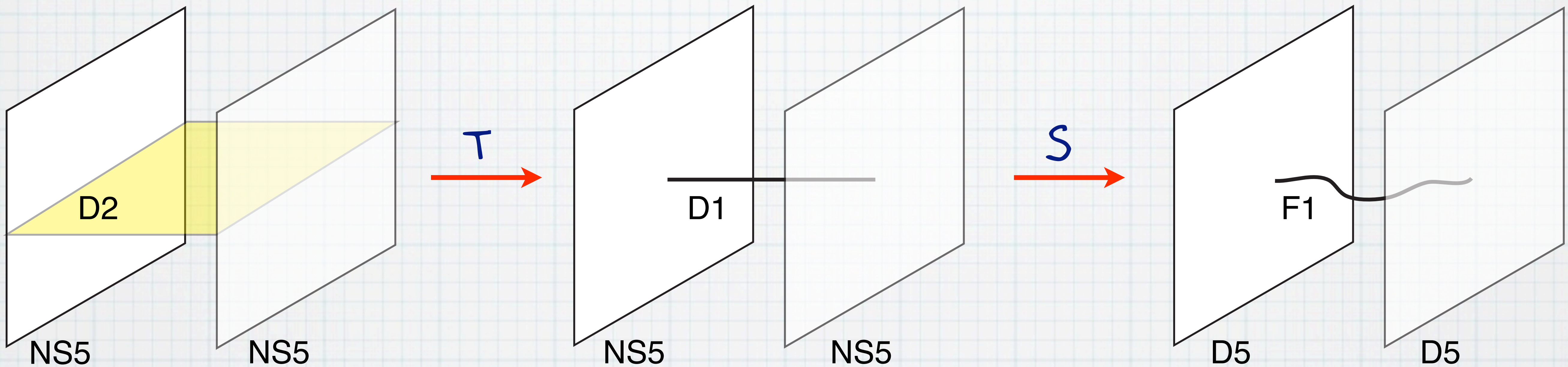
$\mathrm{Gr}(1,4), \quad \mathrm{Gr}(3,4)$

Grassmannian duality!

The Gauge/Bethe correspondence relates gauge theories with different gauge groups!

A Brane Realization

Realize 2d gauge theories of the correspondence in string theory.



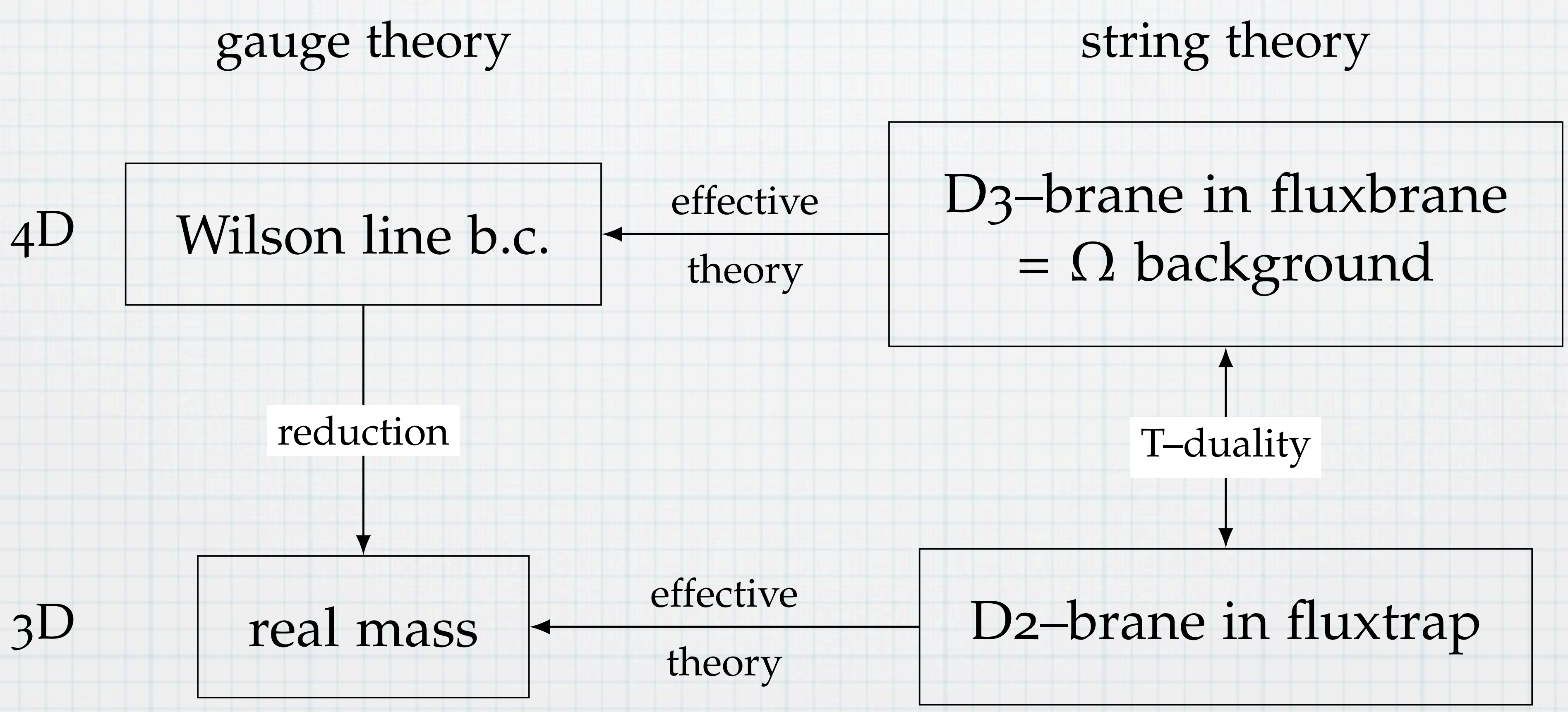
In the last configuration, the fundamental strings are charged under the enhanced symmetry $su(2)$ for coinciding D5 branes.

The symmetry of the integrable model becomes manifest in the D brane set up in the limit of coincident NS5 branes.

Twisted masses from bulk

How to turn on twisted masses in the gauge theory on the worldvolume of the D2 branes?

Answer: they are inherited from the background!



Summary

Gauge/Bethe correspondence relates susy gauge theories in 2d/4d to (quantum) integrable systems.

Find/study string theory realization!

fluxbrane BG

↓ T duality

fluxtrap BG

\+ D2, NS5 ↙ ↘ + D4, NS5

2d N=2 gauge theory
w. tw. masses
(2d Gauge/Bethe corr.)

4d N=2 gauge theory
in Omega BG
(4d Gauge/Bethe corr.)

↓

M Theory lift

Outline

- Introduction/Overview
  - Gauge/Bethe correspondence
- The Bulk
  - fluxbrane
  - fluxtrap
  - generalizations
- Branes
  - 2d gauge theory
  - D brane setup
  - SU(r) symmetry
  - SO(2r) symmetry
- Summary

The Bulk

The Fluxbrane Background

Use a flat bulk background with identifications:

$$(x_0, \ldots, x_3, \rho_1, \theta_1, \rho_2, \theta_2, \widetilde{u}, x_9)$$

$$\theta_1 \simeq \theta_1 + 2\pi\, k_2 \,, \qquad \theta_2 \simeq \theta_2 + 2\pi\, k_3$$

$$x_8 = \widetilde{R}\widetilde{u} \,, \qquad \widetilde{u} \simeq \widetilde{u} + 2\pi k_1$$

impose identifications

$$\begin{cases} \widetilde{u} \simeq \widetilde{u} + 2\pi\, k_1 \,, \\ \theta_1 \simeq \theta_1 + 2\pi m \widetilde{R}\, k_1 \,, \\ \theta_2 \simeq \theta_2 - 2\pi m \widetilde{R}\, k_1 \,, \end{cases}$$

fluxbrane parameter

This corresponds to the well known Melvin or fluxbrane background.

To disentangle periodicities, introduce new coordinates:

$$\begin{cases} \phi_1 = \theta_1 - m\widetilde{R}\widetilde{u} \,, \\ \phi_2 = \theta_2 + m\widetilde{R}\widetilde{u} \,, \end{cases}$$

The Fluxbrane Background

Write down metric:

$$\widetilde{g}_{\mu\nu}\mathrm{d}\widetilde{X}^{\mu}\widetilde{X}^{\nu} = \mathrm{d}\vec{x}_{0\dots3}^{2} + \sum_{i=4}^{7}\left(\mathrm{d}x_i + mV^i\mathrm{d}x_8\right)^2 + \mathrm{d}x_8^2 + \mathrm{d}x_9^2$$

$$V^i\partial_i = -x^5\partial_{x_4} + x^4\partial_{x_5} + x^7\partial_{x_6} - x^6\partial_{x_7} = \partial_{\phi_1} - \partial_{\phi_2}$$

This corresponds to the Omega deformation of flat space in the (4567) directions with $\varepsilon_1 = -\varepsilon_2 = m$

Nekrasov, Okounkov

Locally, the metric is still flat, but some of the rotation symmetries are broken.

Eliminate degrees of freedom which are incompatible with the identifications via T duality.

The Fluxtrap Background

To arrive at the fluxtrap background, we perform a T duality in the 8 direction.

The bulk fields after T duality are

not anymore flat

$$\mathrm{d}s^2 = \mathrm{d}\vec{x}^2_{0\dots3} + \mathrm{d}\rho_1^2 + \mathrm{d}\rho_2^2 + \rho_1^2\mathrm{d}\phi_1^2 + \rho_2^2\mathrm{d}\phi_2^2 + \frac{-m^2\left(\rho_1^2\mathrm{d}\phi_1 - \rho_2^2\mathrm{d}\phi_2\right)^2 + \mathrm{d}x_8^2}{1 + m^2\left(\rho_1^2 + \rho_2^2\right)} + \mathrm{d}x_9^2\,,$$

$$B = m\,\frac{\rho_1^2\mathrm{d}\phi_1 - \rho_2^2\mathrm{d}\phi_2}{1 + m^2\left(\rho_1^2 + \rho_2^2\right)} \wedge \mathrm{d}x_8\,,$$

B field has appeared

$$e^{-\Phi} = \frac{\sqrt{1 + m^2\left(\rho_1^2 + \rho_2^2\right)}}{g_3^2\sqrt{\alpha'}}$$

This background now only contains the physical degrees of freedom.

The Fluxtrap Background

Study supersymmetries preserved by fluxbrane/fluxtrap BG.

The Killing spinor for a flat BG in type IIB is

$$K^{IIB} = \exp[\tfrac{1}{2}\phi_1\Gamma_{45} + \tfrac{1}{2}\phi_2\Gamma_{67}]\exp[\frac{m\widetilde{R}\widetilde{u}}{2}(\Gamma_{45} - \Gamma_{67})]\epsilon_0$$

← cplx Weyl spinor

In order for this to be preserved in the BG with identifications, we must additionally impose the projector $\Pi_{\pm}^{flux} = \frac{1}{2}(\mathbb{1} \pm \Gamma_{4567})$

$$K^{IIB} = \Pi_{-}^{flux}\exp[\tfrac{1}{2}\phi_1\Gamma_{45} + \tfrac{1}{2}\phi_2\Gamma_{67}]\epsilon_0$$

Half of the supersymmetries are left (16 real supercharges).

T dualize to type IIA: $K^{IIA} = \epsilon_L + \epsilon_R$

$$\begin{cases} \epsilon_L = e^{-\Phi/8}(\mathbb{1} + \Gamma_{11})\,\Pi_{-}^{flux}\exp[\tfrac{1}{2}\phi_1\Gamma_{45} + \tfrac{1}{2}\phi_2\Gamma_{67}]\,\epsilon_0\,, \\ \epsilon_R = e^{-\Phi/8}(\mathbb{1} - \Gamma_{11})\,\Gamma_u\Pi_{-}^{flux}\exp[\tfrac{1}{2}\phi_1\Gamma_{45} + \tfrac{1}{2}\phi_2\Gamma_{67}]\,\epsilon_1\,, \end{cases}$$

← const. Majorana spinors

$$\Gamma_u = \frac{m\rho_1}{\Delta}\Gamma_5 - \frac{m\rho_2}{\Delta}\Gamma_7 + \frac{1}{\Delta}\Gamma_8$$ Gamma matrix in u direction

General Fluxtrap Backgrounds

It is also possible to construct more general fluxtrap backgrounds. So far, we used

$$m_1 = -m_2 = m \in \mathbb{R}$$

It is however possible to construct backgrounds with

$$m \in \mathbb{C} \quad \text{or} \quad m_1 \neq -m_2$$

1. Complex fluxtrap BG:

   We need now two periodic variables giving rise to two shift parameters and perform two T dualities:

$$\widetilde{x}_8 = \widetilde{R}_8\, \widetilde{u}\,, \qquad \widetilde{x}_9 = \widetilde{R}_9\, v$$

$$\begin{cases} \widetilde{u} \simeq \widetilde{u} + 2\pi\, k_1\,, \\ \theta_1 \simeq \theta_1 + 2\pi\, m_1 \widetilde{R}_8\, k_1\,, \\ \theta_2 \simeq \theta_2 - 2\pi\, m_1 \widetilde{R}_8\, k_1\,, \end{cases} \qquad \begin{cases} \widetilde{v} \simeq \widetilde{v} + 2\pi\, k_2\,, \\ \theta_1 \simeq \theta_1 + 2\pi\, m_2 \widetilde{R}_9\, k_2\,, \\ \theta_2 \simeq \theta_2 - 2\pi\, m_2 \widetilde{R}_9\, k_2\,, \end{cases}$$

   Preserves same amount of susy as real fluxtrap.

   Corresponds to Omega BG with $m = \frac{1}{2}(m_1 - i m_2)$

General Fluxtrap Backgrounds

2. Fluxtrap BG with $m_1 + m_2 \neq 0$:

For m_1, m_2 to be independent while still preserving some susy, we need to introduce identifications in another plane with a third identification parameter which fulfills

$$m_1 + m_2 + m_3 = 0$$

$$\begin{cases} \widetilde{u} \simeq \widetilde{u} + 2\pi\, k_1 \,, \\ \theta_1 \simeq \theta_1 + 2\pi\, m_1 \widetilde{R}_8\, k_1 \,, \\ \theta_2 \simeq \theta_2 + 2\pi\, m_2 \widetilde{R}_8\, k_1 \,, \\ \theta_3 \simeq \theta_3 - 2\pi\, (m_1 + m_2)\, \widetilde{R}_8 k_1 \,. \end{cases}$$

Refined fluxtrap BG.

The refined fluxtrap preserves only half the susy of the fluxtrap (8 real supercharges)

It is possible in the same way to also construct a complex refined fluxtrap, however the possible brane configurations which can be realized in it are very limited.

Branes

2d Gauge Theories

Let's first discuss the properties of the N=(2,2) gauge theories in 2d we want to realize on the branes:

Vector multiplet:

vector field — cplx scalar — real aux. field

$$
\begin{aligned}
V \;&=\; \theta^-\bar\theta^-(A_0 - A_1) + \theta^+\bar\theta^+(A_0 + A_1) - \theta^-\bar\theta^+\sigma - \theta^+\bar\theta^-\overline{\sigma}\\
&\quad + i\theta^-\theta^+(\bar\theta^-\overline{\lambda}_- + \bar\theta^+\overline{\lambda}_+) + i\bar\theta^+\bar\theta^-(\theta^-\lambda_- + \theta^+\lambda_+) + \theta^-\theta^+\bar\theta^-D
\end{aligned}
$$

Dirac fermion

Chiral multiplet:

cplx scalar — cplx aux. field

$$\Phi = \phi(y^\pm) + \theta^\alpha\psi_\alpha(y^\pm) + \theta^+\theta^-F(y^\pm)$$

Dirac fermion

$$y^\pm = x^\pm - i\theta^\pm\bar\theta^\pm$$

$$x^\pm = x_0 \pm x^1$$

Twisted chiral multiplet:

$$\Sigma = \sigma(\tilde y^\pm) + i\theta^+\overline{\lambda}_+(\tilde y^\pm) - i\bar\theta^-\lambda_-(\tilde y^\pm) + \theta^+\bar\theta^-[D(\tilde y^\pm) - iA_{01}(\tilde y^\pm)] + \ldots$$

$$A_{01} = \partial_0 A_1 - \partial_1 A_0 + [A_0, A_1] \qquad \tilde y^\pm = x^\pm \mp i\theta^\pm\bar\theta^\pm$$

2d Gauge Theories

Action: D terms, F terms, twisted F terms

twisted superpotential

Twisted F term: $$\int \mathrm{d}^2x\, \mathrm{d}\bar{\theta}^- \mathrm{d}\theta^+\, \widetilde{W}\Big|_{\bar{\theta}_+=\theta^-=0} + \mathrm{h.c.}$$

Kinetic term of action:

$$L_{\mathrm{kin}} = \int \mathrm{d}^4\theta \left(\sum_k X_k^\dagger\, e^V X_k - \frac{1}{2e^2}\mathrm{Tr}(\Sigma^\dagger\Sigma) \right),$$

Twisted masses:

$$L_{\mathrm{tw}} = \int \mathrm{d}^4\theta\, (X^\dagger e^{\theta^-\bar{\theta}^+\widetilde{m}_X + \mathrm{h.c.}} X)$$

Want to consider the Coulomb branch.

Calculate eff. action for slowly varying σ fields

Integrate out all massive matter fields.

2d Gauge Theories

Most general action (at most 4 fermions, 2 derivatives):

$$S_{\text{eff}}(\Sigma) = -\int \mathrm{d}^4\theta\, K_{\text{eff}}(\Sigma, \overline{\Sigma}) + \frac{1}{2}\int \mathrm{d}^2\theta\, \widetilde{W}_{\text{eff}}(\Sigma) + \text{h.c.}\,.$$

Integrate out massive fields (S is quadratic in Q)

$$e^{iS_{\text{eff}}(\Sigma)} = \int \mathcal{D}Q\, e^{iS(\Sigma,Q)}$$

This calculation is exact (protected by supersymmetry).

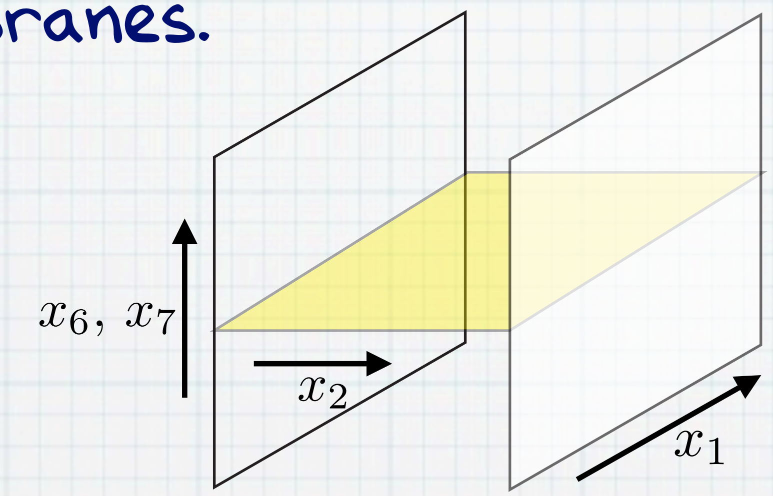
$$\widetilde{W}_{\text{eff}}(\Sigma) = \frac{1}{2\pi}\,(\Sigma - \widetilde{m}_Q)\,(\log(\Sigma - \widetilde{m}_Q) - 1) - \imath\tau\,\Sigma$$

Vacuum equation:

$$\exp\left[2\pi \frac{\partial \widetilde{W}_{\text{eff}}(\sigma)}{\partial \sigma_i}\right] = 1$$

2d Gauge Theories

We can construct N=2 gauge theories in 2d by studying the low energy theory on the worldvolume of D2 branes suspended between NS5 branes.



							ϕ		σ	
direction	0	1	2	3	4	5	6	7	8	9
NS5	×	×					×	×	×	×
fluxtrap	×	×	×	×	m		$-m$		○	×
D2	×	×	×							
D4	×	×		×	×	×				

Separation of NS5s in 3 direction: FI term

Separation of NS5s in 2 direction: $1/g^2$

2d Gauge Theories

Why is the fluxtrap called a fluxtrap?

In the static embedding, $x^0 = \zeta^0,\ x^1 = \zeta^1,\ x^2 = \zeta^3$ the e.o.m. are solved for the D2 branes sitting in $x_3 = x_4 = x_5 = x_6 = x_7 = 0$

The D2s are trapped at the origin.

Adding only D2 branes to the fluxtrap preserves 8 supercharges (static embedding).

Adding also NS5 branes preserves 4 supercharges, N=(2,2)

$$\begin{cases} \epsilon_L = e^{-\Phi/8} \left(\mathbb{1} + \Gamma_{11}\right) \Pi_-^{NS5} \Pi_-^{flux} \Gamma_{1208} \exp[\tfrac{1}{2} \left(\phi_1 + \phi_2\right) \Gamma_{67}] \epsilon \,, \\ \epsilon_R = e^{-\Phi/8} \left(\mathbb{1} - \Gamma_{11}\right) \Gamma_u \Pi_+^{NS5} \Pi_-^{flux} \exp[\tfrac{1}{2} \left(\phi_1 + \phi_2\right) \Gamma_{67}] \epsilon \,. \end{cases}$$

$$\Pi_\pm^{NS5} = \tfrac{1}{2} \left(\mathbb{1} \pm \Gamma_{2345}\right)$$

2d Gauge Theories

The fluxtrap deformation gives rise to the twisted masses!

Start with (kappa fixed) DBI action (democratic formulation):

$$S = -\mu_2 \int \mathrm{d}^3\zeta \, e^{-\Phi} \sqrt{-\det(g_{\alpha\beta} + B_{\alpha\beta})} \left[1 - \frac{1}{2} \bar{\psi} \left((g+B)^{\alpha\beta} \Gamma_\beta D_\alpha + \Delta^{(1)} \right) \psi \right]$$

$$D_\alpha = \partial_\alpha X^\mu \left(\nabla_\mu + \frac{1}{8} H_{\mu m n} \Gamma^{mn} \right) ,$$

$$\Delta^{(1)} = \frac{1}{2} \Gamma^m \partial_m \Phi - \frac{1}{24} H_{mnp} \Gamma^{mnp}$$

After expanding to quadratic order in the fields, we get

dilaton — B field

$$S = -\frac{1}{8\pi^2 g_3^2 (\alpha')^2} \int \mathrm{d}^3\zeta \left[-\dot{X}^\sigma \dot{X}_\sigma + m^2 \left(\rho_1^2 + \rho_2^2 \right) + \bar{\psi} \, \Gamma_0 \dot{\psi} + \frac{m}{2} \bar{\psi} \left(\Gamma_{45} - \Gamma_{67} \right) \Gamma_8 \psi \right] + \dots$$

twisted mass terms!

Realizing the global symmetries

An important ingredient of the Gauge/Bethe correspondence is the symmetry group of the integrable system, which also relates gauge theories with different gauge groups.

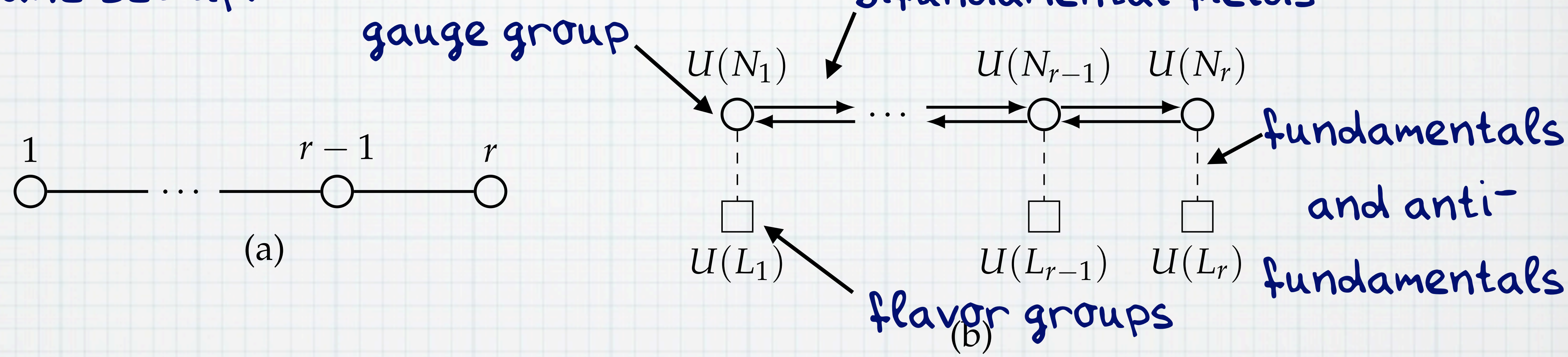
The example with two NS5 branes treated so far corresponds to the simplest case with symmetry group su(2).

Spin chains can have any Lie group as symmetry, even supergroups. Can we realize all those via a brane construction?

So far, we are able to reproduce the A and D series.

SU(r) Quiver Gauge Theories

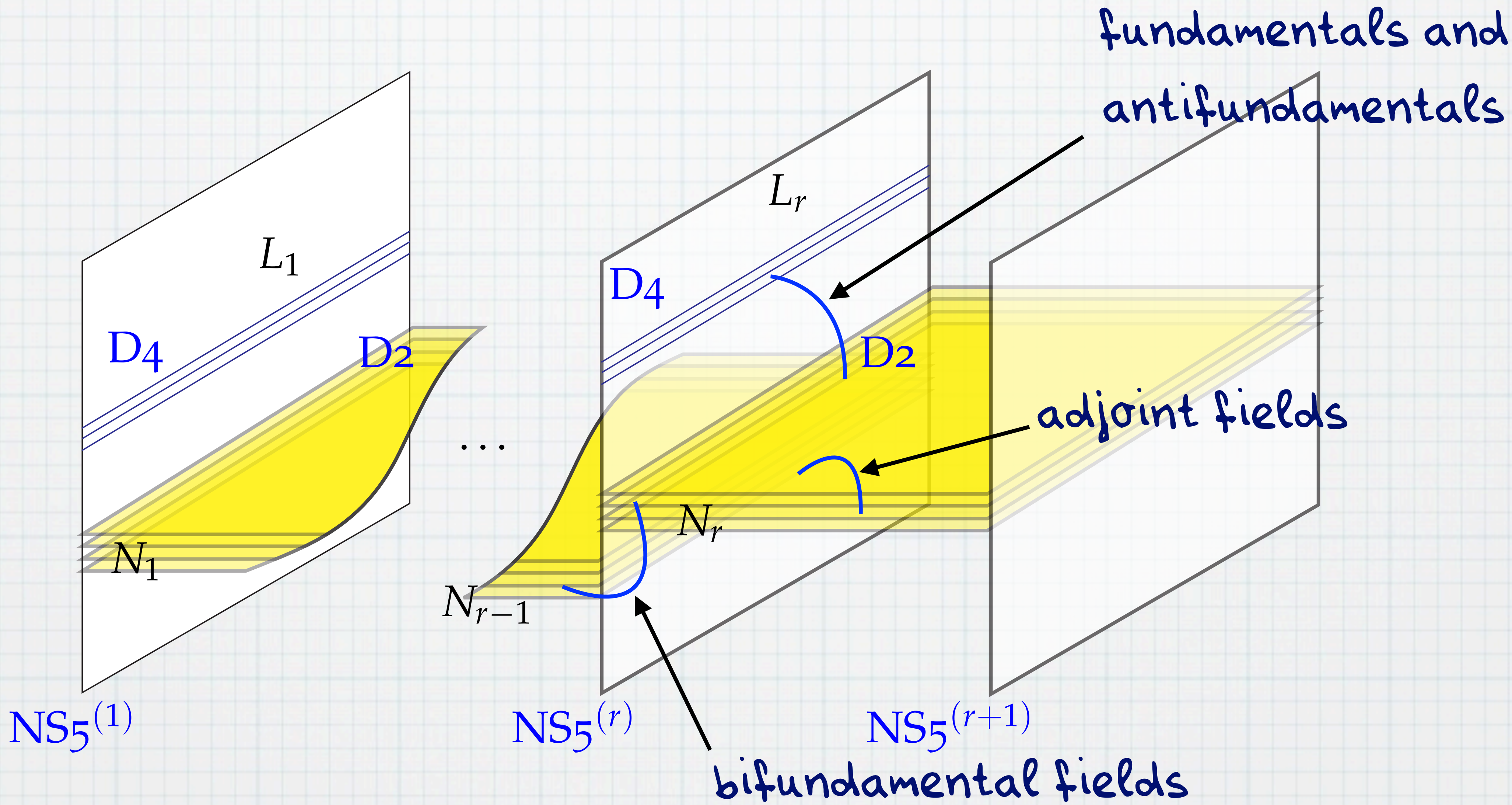
An SU(r) quiver gauge theory corresponds to a spin chain with SU(r) symmetry. Such a theory can be constructed by varying the brane set up.



		0	1	2	3	4	5	6	7	8	9
	fluxbrane	×	×	×	×						×
	NS5	×	×					×	×	×	×
	D2	×	×	×							
$\mathcal{N}=(2,2)$	D4	×	×		×	×	×				
$\mathcal{N}=(1,1)$	D4′	×	×		×					×	×

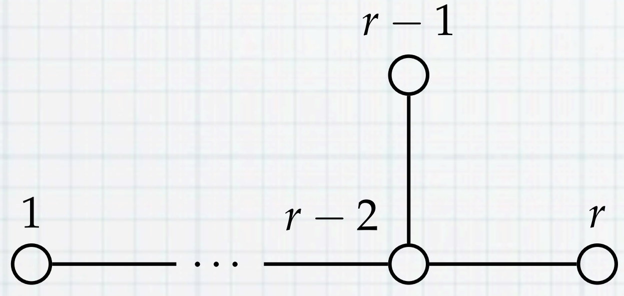
SU(r) Gauge Theories

r+1 NS5 with stacks of D2s suspended in between.

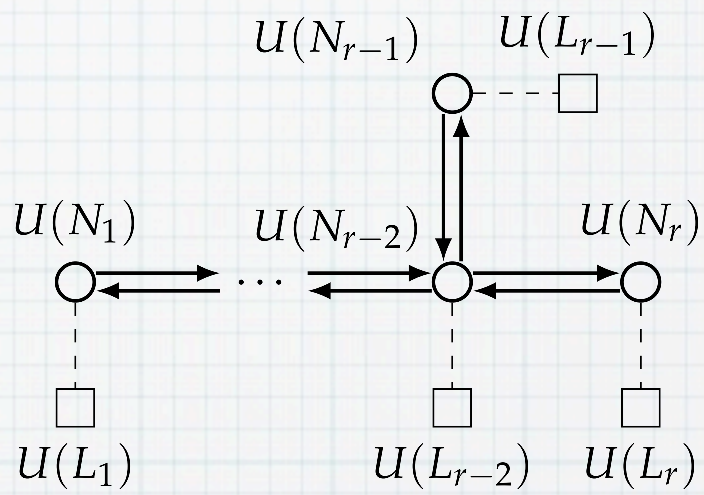


SO(2r) Quiver Gauge Theories

SO(2r) quiver gauge theories can be constructed from SU(r) theories with a further variation.



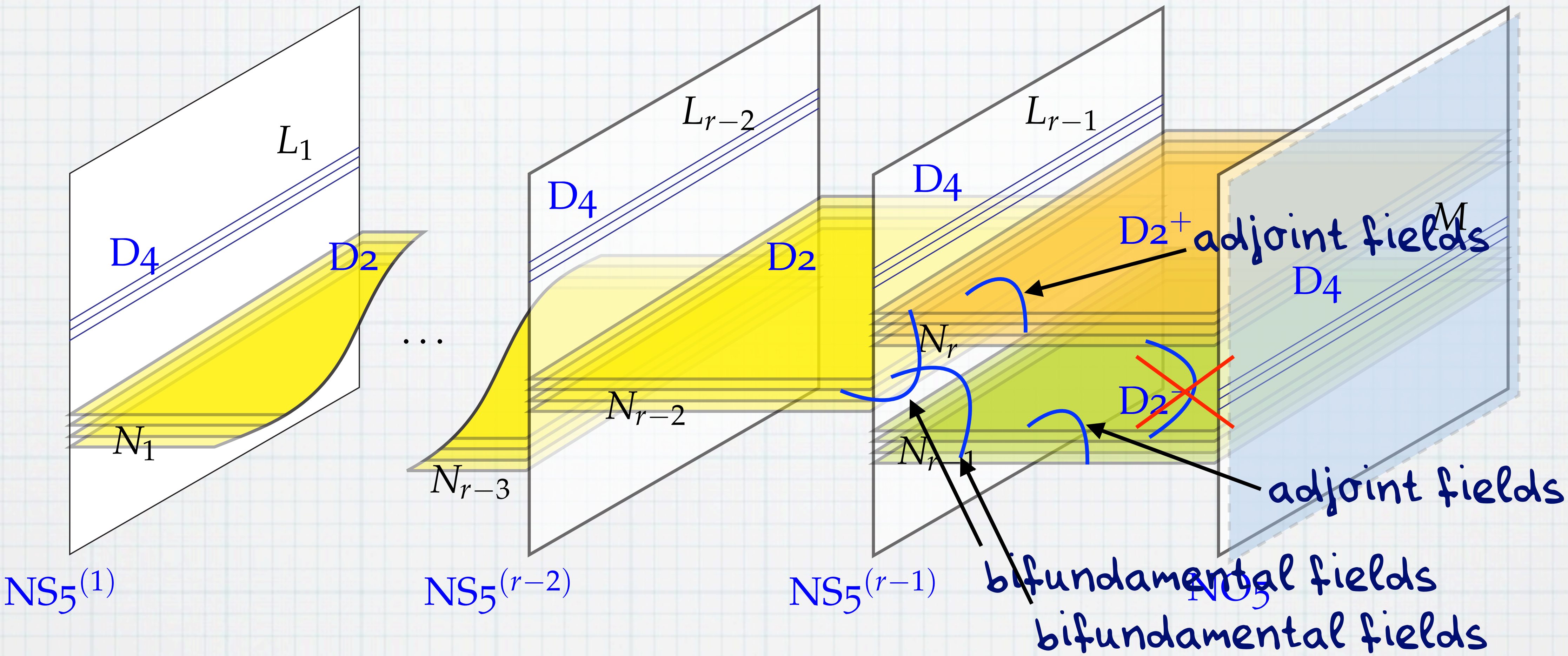
(a)



(b)

		0	1	2	3	4	5	6	7	8	9
	fluxbrane	×	×	×	×						×
	NS5	×	×					×	×	×	×
	NO5	×	×					×	×	×	×
	D2	×	×	×							
$\mathcal{N}=(2,2)$	D4	×	×		×	×	×				
$\mathcal{N}=(1,1)$	D4′	×	×		×					×	×

SO(2r) Gauge Theories



NO5: S dual of a D5 coincident with an O5, here $\mathrm{NO5} : \mathcal{I}_4 \times (-1)^{F_L}$

Preserves same supersymmetries as NS5

6d theory on NO5 has SO(2) symmetry: +,− charged D2s

$$U(N_{r-1} + N_r) \to U(N_{r-1}) \times U(N_r)$$

Sen; Kapustin; Hanany, Zaffaroni

Fluxbrane identifications are compatible with orbifold action.

Summary

Summary

We choose a flat background with identifications (fluxbrane/Melvin BG). After T Duality, it turns into a fluxtrap background.

By placing D2 branes suspended between NS5 branes into the background, we arrive at an N=(2,2) gauge theory in 2d, with twisted masses, as studied in the Gauge/Bethe correspondence.

The symmetry group of the corresponding integrable system is encoded in the brane configuration.

By instead placing D4 branes into the fluxtrap, we get an N=2 gauge theory in 4d in the Omega BG (4d Gauge/Bethe).

This configuration can be lifted to M Theory.

Summary

Gauge/Bethe correspondence relates susy gauge theories in 2d/4d to (quantum) integrable systems.

Find/study string theory realization!

fluxbrane BG

↓ T duality

fluxtrap BG

+ D2, NS5 ↙ ↘ + D4, NS5

2d N=2 gauge theory w. tw. masses (2d Gauge/Bethe corr.)

4d N=2 gauge theory in Omega BG (4d Gauge/Bethe corr.)

↓

M Theory lift

Thank you for your attention!