The Omega Deformation from String and M–Theory

Domenico Orlando

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Collaboration with: **Simeon Hellerman (IPMU) & Susanne Reffert (CERN)**

Outline

Outline

The Ω deformation

The Ω background was introduced by Nekrasov as a way of **regularizing the four-dimensional instanton partition function** and reproduce the results of Seiberg and Witten.

One introduces an appropriate **deformation of the four-dimensional theory**, with parameters ϵ_1 and ϵ_2, breaking rotational invariance of $\mathbb{R}^4$.

The path integrals localize on a discrete set of points.
The *k*-instanton contribution to the prepotential for the original (undeformed) theory is found in the limit $\epsilon_k \to 0$.

Finite ε

In fact this turned out to be a much richer subject.
The partition function in the Ω background has a **meaning also for finite values of ε**.

- In the limit $\varepsilon_1 = -\varepsilon_2 \propto g_s$ the partition function is the same as the one for **topological strings** on a CY related to the spectral curve;
- In the limit $\varepsilon_1 = 0$ the gauge theory is strictly related to **quantum integrable models** with $\hbar = \varepsilon_2$;
- In the general case $\varepsilon_1 \neq \varepsilon_2$, we have the **refinement of topological strings**;
- The **AGT** construction can be understood in terms of compactifications of a six-dimensional theory on the Ω background.

Today

Today I will show you how the Ω–deformation can be understood from the point of view of *old fashioned String Theory*.

Using the fluxtrap background introduced in Susanne's talk I will show how the deformation of the four-dimensional theory can be understood as coming from the bulk.

Advantages:

- Technical: use the **methods of string theory**
- Conceptual: understand the localization in **geometrical terms**
- Conceptual: make some progress towards the understanding of the **six-dimensional** $(2,0)$ **theory**
- Conceptual: understand how the ε parameters are related to a **quantization of the spectral curve**

Conclusions

- We propose a **string realization of the Ω–deformation**, based on NS fields alone. We call it the fluxtrap
- We show how the **effective theory for a $D4$–brane** in this background is the four-dimensional **Ω deformation of $\mathcal{N} = 2$ Seiberg–Witten** introduced by Nekrasov
- We lift the configuration to M–theory and interpret the $D4$–$NS5$ setup in terms of an **$M5$-brane wrapped on a Riemann surface** with flux
- We use a **9/11 flip** to show the geometric interpretation of ε **as a quantization parameter**.

Outline

Melvin construction in field theory

We want to write the String Theoretical analog to a compactification with a Wilson line.
In the Melvin construction one starts with a S^1 **fibration over $\mathbb{R}^4$, with a non-trivial monodromy**

$$\begin{array}{ccc} S^1(\tilde{u}) & \longrightarrow & M \\ & & \downarrow \\ & & \mathbb{R}^4(\rho_k, \theta_k) \end{array} \qquad \begin{cases} \tilde{u} \sim \tilde{u} + 2\pi n_u , \\ \theta_k \sim \theta_k + 2\pi \epsilon_k \tilde{R} n_u , \end{cases} \qquad n_u \in \mathbb{Z}$$

T-duality is the same as a reduction on S^1

The String Theory version

Start from a Ricci-flat metric $ds^2 = g_{ij}\, dx^i\, dx^j + d(\tilde{x}^9)^2$, where $\tilde{x}^9 = \widetilde{R}\tilde{u}$, where g has $N \leq 4$ (non-bounded) rotational isometries generated by ∂_{θ_k}.
Pass to a set of **disentangled variables**

$$\phi_k = \theta_k - \epsilon_k \widetilde{R}\tilde{u}\,,$$

This modifies the boundary conditions from

$$(\tilde{u}, \theta_k) \sim (\tilde{u}, \theta_k) + 2\pi n_u \left(1, \epsilon_k \tilde{R}\right) + 2\pi n_k (0, 1)$$

to

$$(\tilde{u}, \phi_k) \sim (\tilde{u}, \phi_k) + 2\pi n_u \left(1, 0\right) + 2\pi n_k (0, 1)\,.$$

The generic fluxtrap

Now T-dualize in $\tilde{u}$. We get a B-field and a dilaton: the fluxtrap

$$ds^2 = g_{ij}\, dx^i\, dx^j - \frac{\epsilon^2 U_i U_j\, dx^i\, dx^j}{1 + \epsilon^2 U_i U^i} + \frac{(dx^9)^2}{1 + \epsilon^2 U_i U^i},$$

$$B = \epsilon \frac{U_i\, dx^i \wedge dx^9}{1 + \epsilon^2 U_i U^i},$$

$$e^{-\Phi} = \frac{\sqrt{\alpha'}\, e^{-\Phi_0}}{R} \sqrt{1 + \epsilon^2 U_i U^i},$$

In this picture the irrelevant degrees of freedom (rotations around $\tilde{u}$) have been removed (they turn into infinitely heavy winding modes). **All the local degrees of freedom are physical.**

The generic fluxtrap

$$ds^2 = g_{ij}\, dx^i\, dx^j + \frac{(dx^9)^2 - \varepsilon^2 U_i U_j\, dx^i\, dx^j}{1 + \varepsilon^2 U_i U^i},$$

$$B = \varepsilon \frac{U_i\, dx^i \wedge dx^9}{1 + \varepsilon^2 U_i U^i},$$

$$e^{-\Phi} = \frac{\sqrt{\alpha'}\, e^{-\Phi_0}}{R}\sqrt{1 + \varepsilon^2 U_i U^i},$$

- branes will be **trapped in $U = 0$** by the terms in the denominators
- For $\varepsilon = 0$ this is the initial Ricci-flat background
- U is the generator of the **rotational isometries before and after the duality**

$$\varepsilon\, U^i\, \partial_i = \sum_{k=1}^{N} \varepsilon_k\, \partial_{\phi_k}$$

- ε **regularizes the rotation**, which is always bounded if $\varepsilon \neq 0$

$$\|U\|^2_{\text{trap}} = \frac{U_i U^i}{1 + \varepsilon^2 U_i U^i} < \frac{1}{\varepsilon^2}.$$

- the dilaton has a minimum when $U = 0$.

Supersymmetry

The fluxtrap breaks in general all the supersymmetries, unless we impose conditions on the parameters ϵ.

In an appropriate coordinate system, before T-duality the Killing spinors are:

$$\eta_{\mathrm{iib}} = (\mathbb{1} + \Gamma_{11}) \prod_{k=1}^{N} \exp[\frac{\theta_k}{2} \Gamma_{\rho_k \theta_k}] \eta_w$$

Pass to the disentangled coordinates ϕ_k,

$$\eta_{\mathrm{iib}} = \prod_{k=1}^{N} \exp[\frac{\phi_k}{2} \Gamma_{\rho_k \theta_k}] \exp[\frac{\widetilde{R}\tilde{u}}{2} \epsilon_k \Gamma_{\rho_k \theta_k}] \eta_w ,$$

the second exponential is not compatible with the boundary conditions. We need to impose a condition that breaks all supersymmetries:

$$\sum_{k=1}^{N} \epsilon_k \Gamma_{\rho_k \theta_k} \eta_w = 0 .$$

Supersymmetries

Let now

$$\sum_{k=1}^{N} \varepsilon_k = 0 ,$$

the condition becomes:

$$\sum_{k=1}^{N} \varepsilon_k \Gamma_{\rho_k \theta_k} \eta_w = \sum_{k=1}^{N-1} \varepsilon_k \left(\Gamma_{\rho_k \theta_k} - \Gamma_{\rho_N \theta_N} \right) \eta_w = 0 .$$

This is the linear combination of $N-1$ commuting projectors. It is annihilated by the product of all the corresponding orthogonal projectors

$$\Pi^{\text{flux}} = \prod_{k=1}^{N-1} \left(\Gamma_{\rho_k \theta_k} + \Gamma_{\rho_N \theta_N} \right) ,$$

so that the boundary conditions are satisfied by the Killing spinor

$$\eta_{\text{iib}} = \left(\mathbb{1} + \Gamma_{11} \right) \prod_{k=1}^{N} \exp\left[\frac{\phi_k}{2} \Gamma_{\rho_k \theta_k} \right] \Pi^{\text{flux}} \eta_w .$$

Supersymmetry

Since all dependence on $\tilde{u}$ has disappeared from the expression, T–duality maps the Killing spinors η_{iib} into local type iia Killing spinors η_{iia}. Using an appropriate vielbein for the T–dual metric they take the form $\eta_{\text{iia}} = \eta^L_{\text{iia}} + \eta^R_{\text{iia}}$ with

$$\begin{cases} \eta^L_{\text{iia}} = (\mathbb{1} + \Gamma_{11}) \prod_{k=1}^{N} \exp[\frac{\phi_k}{2} \Gamma_{\rho_k \theta_k}] \Pi^{\text{flux}} \eta_0 , \\ \eta^R_{\text{iia}} = (\mathbb{1} - \Gamma_{11}) \, \Gamma_u \prod_{k=1}^{N} \exp[\frac{\phi_k}{2} \Gamma_{\rho_k \theta_k}] \Pi^{\text{flux}} \eta_1 , \end{cases}$$

where Γ_u is the gamma matrix in the u direction normalized to unity. Depending on η_w, **the projector Π^{flux} can either break all supersymmetries or preserve some of them**. In the latter case, at least $1/2^{N-1}$ of the original ones are preserved.

The point

- We look at a **String Theory realization** of the Melvin construction
- T-duality removes the non-physical degrees of freedom
- We find a background where all **local degrees of freedom are physical**
- We can study this background using String Theory
- Supersymmetry in terms of Killing spinors in the bulk

Outline

The gauge theory

Now that we have found the bulk we can try to reproduce the Ω-deformed four-dimensional gauge theory. The idea is to place D–branes a la Hanany–Witten, so that the gauge theory encodes their fluctuations.
In the previous talk we have seen that D2–**branes** suspended between NS5–five branes which are not extended along the directions with identifications receive **twisted mass deformations**.
Here we will be concerned with D4–**branes** which are extended **in the directions of the shifts**.

X	0	1	2	3	4	5	6	7	8	9
fluxbrane	ϵ_1		ϵ_2		ϵ_3		×	×	×	○
NS5	×	×	×	×					×	×
D4	×	×	×	×			×			
ξ	0	1	2	3			4			

The field theory on $\mathbb{R}^4_{\varepsilon_1,\varepsilon_2}$

Start from flat space and use three rotations.

$$U = \varepsilon_1 \partial_{\phi_1} + \varepsilon_2 \partial_{\phi_2} + \varepsilon_3 \partial_{\phi_3}$$

the fluxbrane is:

$$ds^2 = d\mathbf{x}^2_{0\ldots8} + \frac{(dx^9)^2 - \left(\varepsilon_1 \rho_1^2 d\phi_1 + \varepsilon_2 \rho_2^2 d\phi_2 + \varepsilon_3 \rho_3^2 d\phi_3\right)^2}{1 + \varepsilon_1^2 \rho_1^2 + \varepsilon_2^2 \rho_2^2 + \varepsilon_3^2 \rho_3^2},$$

$$B = \frac{\left(\varepsilon_1 \rho_1^2 d\phi_1 + \varepsilon_2 \rho_2^2 d\phi_2 + \varepsilon_3 \rho_3^2 d\phi_3\right) \wedge dx^9}{1 + \varepsilon_1^2 \rho_1^2 + \varepsilon_2^2 \rho_2^2 + \varepsilon_3^2 \rho_3^2},$$

$$e^{-\Phi} = \sqrt{1 + \varepsilon_1^2 \rho_1^2 + \varepsilon_2^2 \rho_2^2 + \varepsilon_3^2 \rho_3^2}.$$

In order to preserve some supersymmetry we impose

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0,$$

and using the general prescription introduced in the previous section it is immediate to see that the background preserves $32/2^2 = 8$ supercharges.

D4–brane embedding

X	0	1	2	3	4	5	6	7	8	9
fluxbrane	ϵ_1		ϵ_2		ϵ_3		×	×	×	○
NS5	×	×	×	×					×	×
D4	×	×	×	×			×			
ξ	0	1	2	3			4			

Consider the **static embedding** defined by

$$f: \xi^0 = x^0, \quad \xi^1 = x^1, \quad \xi^2 = x^2, \quad \xi^3 = x^3, \quad \xi^4 = x^6, \quad v = x^8 + \mathrm{i}\, x^9$$

The Dirac–Born–Infeld action is given by

$$S = -\mu_p \int \mathrm{d}^5 \xi \; e^{-\Phi} \sqrt{-\det(\hat{g} + \hat{B} + 2\pi\alpha' F)} ,$$

The Ω-deformed action

Now we just need to write the determinant expanded at second order in the fields:

$$\mathscr{L}_{\varepsilon_1,\varepsilon_2} = -\frac{1}{4g_4^2}\left(1 + \|F\|^2 + \frac{1}{2}\| \mathrm{d}\varphi + 2\mathrm{i}\,\varepsilon\,\imath_{\hat{U}}F\|^2 + \frac{\varepsilon^2}{8}\|\imath_{\hat{U}}\mathrm{d}(\varphi + \bar{\varphi})\|^2\right),$$

where $\hat{U}$ is the pullback of the vector field U,

$$\varepsilon\,\hat{U} = \varepsilon f^* U = \varepsilon\,\hat{U}^i\partial_{\xi^i} = \varepsilon_1\left(\xi^0\partial_1 - \xi^1\partial_0\right) + \varepsilon_2\left(\xi^2\partial_3 - \xi^3\partial_2\right).$$

This is the Lagrangian of the Ω–deformation of $\mathcal{N} = 2$ SYM.
The advantage is that now **we can understand it as coming from string theory** and we have an algorithmic way to generalize it.

The interpretation

$$\mathscr{L}_{\varepsilon_1,\varepsilon_2} = -\frac{1}{4g_4^2}\left(1 + \|F\|^2 + \frac{1}{2}\|\mathrm{d}\varphi + 2\,\mathrm{i}\,\varepsilon\,\imath_{\hat{U}}F\|^2 + \frac{\varepsilon^2}{8}\|\imath_{\hat{U}}\,\mathrm{d}(\varphi + \bar{\varphi})\|^2\right),$$

- the terms in ε are odd under charge conjugation $A_\mu \to -A_\mu$. This is because they come from the B field. This is the leading deformation of the background
- the terms in ε^2 come from metric and dilaton. They control classical gauge configurations and hence directly to the instanton moduli space
- A **D–instanton is a $D(-1)$ brane**. Its action is

$$\mathscr{L}_{\text{inst}} = e^{-\Phi}$$

a critical point for the action is a **critical point for the dilaton profile:** $U = 0$. This is the **string theoretical version of localization**.

The point

- We realize the deformed four-dimensional gauge theory in terms of **Hanany–Witten branes** (D4 suspended between NS5)
- The fluxtrap background is pulled back on the branes and modifies the theory
- We have a **geometric origin for the new terms in the action**
- **Localization** can be understood in terms of **dilaton gradient**

Outline

Four-dimensional theories from $M5$–branes

The theories we are interested in are deformations of four-dimensional Seiberg–Witten theories.

One can see them in terms of Hanany–Witten configurations of $D4$ suspended between $NS5$.

A very powerful approach consists in **lifting these configurations to M–theory**: in the undeformed case the full brane configuration is represented by a single $M5$ brane wrapped on a Riemann surface Σ.

Σ is the **geometric representation of the Seiberg–Witten curve**.

We want to understand how this picture changes in presence of the Ω deformation.

The lift to eleven dimensions

The lift of a type iia background to M–theory is straightforward:

$$G_{MN}\,\mathrm{d}x^M\,\mathrm{d}x^N = e^{-2\Phi/3} g_{mn}\,\mathrm{d}x^m\,\mathrm{d}x^n + e^{4\Phi/3}\left(\mathrm{d}x^{10} + C_1\right)^2 ,$$
$$A_3 = C_3 + B \wedge \mathrm{d}x^{10} .$$

where Φ is the dilaton, C_1 and C_3 are the Ramond–Ramond forms. The general M–theory fluxtrap is then:

$$\mathrm{d}s^2 = \left(1 + \epsilon^2 U_i U^i\right)^{1/3}\left[g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j + \frac{(\mathrm{d}x^9)^2 + (\mathrm{d}x^{10})^2 - \epsilon^2 U_i U_j\,\mathrm{d}x^i\,\mathrm{d}x^j}{1 + \epsilon^2 U_i U^i}\right],$$
$$A_3 = \epsilon\,\frac{U_i\,\mathrm{d}x^i \wedge \mathrm{d}x^9 \wedge \mathrm{d}x^{10}}{1 + \epsilon^2 U_i U^i} .$$

The directions x^9 **and** x^{10} which have completely different origins (x^9 is the dual of the Melvin circle while x^{10} is the M–circle) **enter the background in a completely symmetric fashion**.

Close to the trap

It is interesting to consider the physics close to the fluxtrap, *i.e.* the limit $\epsilon^2 U_i U^i \ll 1$. The fields become

$$G_{MN}\, dx^M\, dx^N = g_{ij}\, dx^i\, dx^j + (dx^9)^2 + (dx^{10})^2 + \mathcal{O}(\epsilon^2 \|U\|^2)\,,$$
$$A_3 = \epsilon\, U_i\, dx^i \wedge dx^9 \wedge dx^{10} + \mathcal{O}(\epsilon^3 \|U\|^3)\,.$$

The appropriate setting to discuss the gauge theories found in the previous section is obtained by starting from a flat metric $g_{ij} = \delta_{ij}$.

$$G_{MN} = \delta_{MN} + \mathcal{O}(\epsilon^2 \|U\|^2)\,,$$
$$F_4 = 2\,\epsilon\,\omega \wedge dx^9 \wedge dx^{10} + \mathcal{O}(\epsilon^3 \|U\|^3)\,,$$

where ω is the weighted sum over the volume forms of the plane in which the original Melvin identifications have been performed,

$$\epsilon\,\omega \equiv \sum_{k=1}^{N} \epsilon_k \omega_k \qquad \text{(graviphoton field strength)}$$

The M5–brane in the trap

The simplest approach to find the M5–brane embedding **makes use of supersymmetry**.
We look for the simplest bps object that preserves the same supersymmetries as the NS5 and the D4 separately.
Concretely we have projectors of the type:

$$\Pi_{+}^{M5} = \frac{1}{2}\left(\mathbb{1} + \Gamma^{M5}\right)\eta_{m} = 0\,,$$

where η_{m} is the generic Killing spinor preserved by the background and Γ^{M5} is:

$$\Gamma^{M5} = \left(-\mathbb{1} + \frac{1}{3}\hat{\Gamma}^{m_1 m_2 m_3} h_{m_1 m_2 m_3}\right)\Gamma_{(0)}\,, \quad \Gamma_{(0)} = \frac{1}{6!\sqrt{-\hat{g}}}\eta^{m_1 \dots m_6}\hat{\Gamma}_{m_1 \dots m_6}\,,$$

where $\hat{\Gamma}$ are pullbacks and h is the self-dual three-field on the brane.

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$$\Gamma^{M5} = \left(- \mathbb{1} + \frac{1}{3} \hat{\Gamma}^{m_1 m_2 m_3} h_{m_1 m_2 m_3} \right) \Gamma_{(0)} \,, \quad \Gamma_{(0)} = \frac{1}{6! \sqrt{-\hat{g}}} \eta^{m_1 \dots m_6} \hat{\Gamma}_{m_1 \dots m_6} \,,$$

where $\hat{\Gamma}$ are pullbacks and h is the self-dual three-field on the brane.

The NS5 and D4

The NS5 is lifted to an M5–brane extended in $(x^0, \dots, x^3, x^8, x^9)$. The pullback of the four-form flux vanishes $f^*_{NS5} F_4 = 0$:

$$\Pi_+^{NS5} = \frac{1}{2}\left(\mathbb{1} + \Gamma_{012389}\right) ;$$

The D4–brane is lifted to an M5–brane extended in $(x^0, \dots, x^3, x^6, x^{10})$. Also in this case the pullback of the four-form flux vanishes, but we need to take into account the deformed metric:

$$\Pi_+^{D4} = \frac{1}{2}\left(\mathbb{1} + \frac{\Gamma_{026}\left(\Gamma_{13} + \epsilon_1 |u| \Gamma_{39} + \epsilon_2 |w| \Gamma_{19}\right)\Gamma_{10}}{\sqrt{1 + \epsilon_1^2 |u|^2 + \epsilon_2^2 |w|^2}}\right),$$

where $u = x^0 + i x^1$ and $w = x^2 + i x^3$.
The kappa symmetry projectors for the NS5 and D4 commute,

$$\left[\Pi_+^{NS5}, \Pi_+^{D4}\right] = 0,$$

and **each breaks $1/2$ of the supersymmetries. Only two remain.**

Lowest order

It is convenient to tackle this problem **order by order**. Let $s = x^6 + i x^{10}$ and $v = x^8 + i x^9$. The embedding is defined by:

$$\begin{cases} s(z,\bar{z}) = s_0(z,\bar{z}) + \varepsilon\, s_1(z,\bar{z}) + \dots \\ v(z,\bar{z}) = v_0(z,\bar{z}) + \varepsilon\, v_1(z,\bar{z}) + \dots \end{cases}$$

The D4 is $v = 0$. The NS5 is $s = 0$.
At zeroth order this is the classical configuration. The M5**–brane wraps a Riemann surface** in the $\mathbb{C}^2$ space described by (s, v):

$$\begin{cases} s_0(z,\bar{z}) = s_0(z) \\ v_0(z,\bar{z}) = v_0(z) \end{cases}$$

Linear order in ε

At **linear order in ε there is a non-trivial four-form flux**.
The flux has a non-vanishing pullback on the M5, coming from the $(1,1)$ component in the s, v plane:

$$f^*_{M5} F_4 = \frac{i}{4} \left(\bar\partial \bar s_0 \, \partial v_0 - \partial s \, \bar\partial \bar v_0 \right) dz \wedge d\bar z \wedge \left(\varepsilon_1 \, du \wedge d\bar u + \varepsilon_2 \, dw \wedge d\bar w \right) .$$

At this order the selfdual three-form h on the M5 is just the primitive form

$$dh = -\frac{1}{4} f^*_{M5} F_4 .$$

The important extra condition is that h is **selfdual**:

$$h = *h$$

The self-dual three-form

Selfduality breaks h into two pieces, depending on $\varepsilon_\pm = \varepsilon_1 \pm \varepsilon_2$:

$$h = -\mathrm{i}\left(\varepsilon_- \partial K \, \mathrm{d}z \wedge (f^*_{M5}\omega_-) + \varepsilon_+ \bar\partial K \, \mathrm{d}\bar z \wedge (f^*_{M5}\omega_+)\right)$$

where

- $K(z,\bar z)$ is a **Kähler potential on the Riemann surface**:

$$K(z,\bar z) = \frac{1}{8}\Im(v_0(z)\bar s_0(\bar z)) .$$

- $\omega_\pm$ are the symplectic structures associated to $\varepsilon_\pm$:

$$\omega_\pm = \mathrm{d}x^0 \wedge \mathrm{d}x^1 \pm \mathrm{d}x^2 \wedge \mathrm{d}x^3$$

The h field at first order only depends on the embedding at zeroth order.

The solution

Now we can write the pullback of the gamma matrices

$$\Gamma^{M5}_{\mathcal{O}(\varepsilon)} = \left(-\mathbb{1} + \tfrac{1}{3} h_{m_1 m_2 m_3} \hat{\Gamma}^{m_1 m_2 m_3}\right) \Gamma_{(0)} ,$$

and impose the kappa symmetry projection

$$\Pi_{+}^{M5} \Pi_{-}^{NS5} \Pi_{-}^{D4} \eta_{m} = 0 ,$$

which is greatly simplified by the fact that the contribution of the *h* field is projected out by supersymmetry

$$h_{m_1 m_2 m_3} \hat{\Gamma}^{m_1 m_2 m_3} \Gamma_{(0)} \Pi_{-}^{NS5} \Pi_{-}^{D4} \equiv 0 .$$

This means that we are back to

$$\left(\mathbb{1} - \Gamma_{(0)}\right) \Pi_{-}^{NS5} \Pi_{-}^{D4} \eta_{m} = 0 ,$$

which is precisely the same equation as in the $\varepsilon = 0$ case and is satisfied by the same Cauchy–Riemann conditions. **The M5 is still wrapped on a Riemann surface, but there is a non-trivial *h* field.**

The point

- We lifted the fluxtrap to **M–theory**
- The system of D4–NS5 turns into **a single** M5**–brane**
- The M5–brane is wrapped on a **Riemann surface** (at first order in ε)
- A non-trivial h **field** lives on the M5
- A natural **Kähler form** appears on the Riemann surface
- **Self-duality of** h imposes the identification $so(4) \simeq su(2) \oplus su(2)$ under which $(\varepsilon_1, \varepsilon_2)$ are traded for $\varepsilon_\pm$

Outline

Taub–nut

In order to make our construction more transparent it is convenient to start from a **Taub–nut space** and put a fluxtrap in $TN_Q \times S^1 \times \mathbb{R}^5$.

$$\begin{array}{ccc} S^1(\theta) & \longrightarrow & TN \\ & & \downarrow \\ & & \mathbb{R}^3(\mathbf{r}) \end{array}$$

A Taub–nut space is a singular S^1 fibration over $\mathbb{R}^3$

It interpolates between $\mathbb{R}^4$ for $r \to 0$ and $\mathbb{R}^3 \times S^1$ for $r \to \infty$.

Taub–nut

The Taub–nut can also be understood as a **complex two-dimensional manifold** and the metric written in terms of z_1 and z_2. The angle θ that describes the S^1 fibration is precisely the angle that we would use for the **Melvin construction in the limit** $\varepsilon_1 = -\varepsilon_2 = \varepsilon$

$$\theta = \theta_1 - \theta_2$$

we impose the identifications

$$\begin{cases} \tilde{u} \simeq \tilde{u} + 2\pi \,, \\ \theta \simeq \theta + 4\pi \widetilde{R}\, \varepsilon \,, \end{cases}$$

The rotation generator is $U = \partial_\phi$ and its norm is

$$\|U\|^2 = \frac{1}{V(r)} < \lambda^2$$

where λ^2 is the asymptotic radius of the Taub–nut.

The fluxtrap construction **does not break additional supersymmetries**.

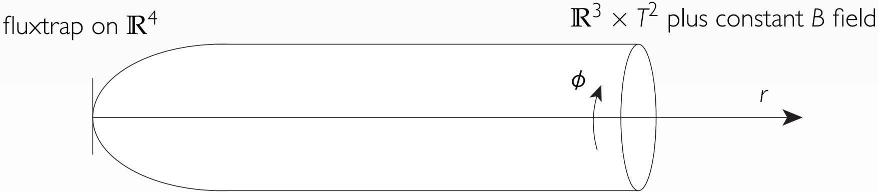
Flux–trap

$$ds^2 = V(r)\,d\mathbf{r}^2 + \frac{1}{V(r) + \varepsilon^2}(d\phi + Q\cos\omega\,d\psi)^2 + \frac{V(r)}{V(r) + \varepsilon^2}(dx^9)^2 + d\mathbf{x}^2_{4\dots8}\,,$$

$$B = \frac{\varepsilon}{V(r) + \varepsilon^2}(d\phi + Q\cos\omega\,d\psi)\wedge dx^9\,,$$

$$e^{-\Phi} = \sqrt{1 + \frac{\varepsilon^2}{V(r)}}\,.$$

This interpolates between the fluxtrap in flat space that we used to reproduce Nekrasov's action and $\mathbb{R}^3 \times T^2$ with a constant B field.



The alternative description

In the limit $r \to \infty$ the Taub–nut becomes $\mathbb{R}^3 \times S^1$ and the fluxtrap is the result of a **T–duality on a torus with shear**, *i.e.* a constant B field.
Putting a D4–brane wrapping the Taub–nut space we obtain the **alternative description of the Ω deformation** proposed by Witten and Nekrasov.

We can calculate the Killing spinors before and after the duality and we find (from the analysis of the bulk) that the supersymmetry generators in the four–dimensional theory are "rotated"

$$\eta_{\varepsilon} = \exp[\frac{\vartheta}{2} \Gamma_{39}] \eta_{\varepsilon=0} .$$

where

$$\tan \frac{\vartheta}{2} = \varepsilon \lambda$$

Non-commutativity

The Ω deformation for $\varepsilon_1 = -\varepsilon_2$ is related to **topological strings**. It has been observed (via the connection to matrix models) that

> "*the **Riemann surface** Σ behaves for many purposes as a **subspace of a quantum mechanical** (s, v) **phase space** where* $g_s = \hbar$. [Aganagic, Dijkgraaf, Klemm, Marino, Vafa]"

Our construction gives a **precise geometrical interpretation** for this observation in terms of Riemann surface on a non-commutative plane.

A **first clue** is given by the **T–duality on a torus with shear**, which is the classical example of Seiberg and Witten. To turn these observations into a precise statement we introduce a 9/11 flip.

M–theory fluxtrap

Lift the background to **M–theory**:

$$ds^2 = \left(1 + \frac{\varepsilon^2}{V(r)}\right)^{1/3} \left[V(r)\, d\mathbf{r}^2 + \frac{(d\phi + Q\cos\omega\, d\psi)^2 + V(r)\left((dx^9)^2 + (dx^{10})^2\right)}{V(r) + \varepsilon^2} + \right.$$

$$C = \frac{\varepsilon}{V(r) + \varepsilon^2}(d\phi + Q\cos\omega\, d\psi) \wedge dx^9 \wedge dx^{10}.$$

This picture becomes particularly clear in the $\varepsilon \to 0$ limit. The metric is $TN_Q \times \mathbb{R}^7$ and the four–form flux is

$$F_4 = \varepsilon\, \omega_{TN} \wedge dx^9 \wedge dx^{10},$$

where ω_{TN} is the unique two-form on the Taub–nut that is invariant under the triholomorphic $U(1)$ isometry:

$$\omega_{TN} = d\left[\frac{d\phi + Q\cos\omega\, d\psi}{V(r)}\right].$$

Reduction

... and now **reduce** it on ϕ

$$
\begin{aligned}
ds^2 &= V(r)^{1/2} dr^2 + V(r)^{-1/2} \left(d\mathbf{x}_{4\dots10}^2 - \frac{\varepsilon^2}{V(r) + \varepsilon^2} \left((dx^9)^2 + (dx^{10})^2 \right) \right) , \\
B &= \frac{\varepsilon}{V(r) + \varepsilon^2} dx^9 \wedge dx^{10} , \\
e^{-\Phi} &= V(r)^{1/4} \sqrt{V(r) + \varepsilon^2} , \\
A_1 &= Q \cos \omega \, d\psi , \\
A_3 &= B \wedge A_1 .
\end{aligned}
$$

These are Q D6**–branes extended in** $(x^4, \dots, x^{10})$ in presence of an Ω–deformation.

The Seiberg–Witten map

An equivalent description is obtained by applying the **Seiberg–Witten map** to the D6–brane theory in order to turn the B–field into a non-commutativity parameter:

$$\left(\hat{g} + \hat{B}\right)^{-1} = \tilde{g}^{-1} + \Theta \,,$$

where $\hat{g}$ and $\hat{B}$ are the pullbacks of metric and B–field on the brane and $\tilde{g}$ is the new effective metric for a non-commutative space satisfying

$$[x^i, x^j] = \mathrm{i}\, \Theta^{ij} \,.$$

Applying this map to our case:

$$\tilde{g}_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j = \mathrm{d}\mathbf{x}^2_{4\ldots10} \,,$$

$$[x^9, x^{10}] = \mathrm{i}\, \varepsilon \,.$$

All dependence on ε disappears from the D6–brane theory and is turned into a constant non-commutativity parameter.

A non-commutative Riemann surface

Let's follow the fate of the **branes** whose dynamics reproduce the Ω–deformed gauge theory.

Start from the configuration of **D4–NS5s, with the D4 wrapping the Taub–nut space**.

In the M–theory lift this configuration turns into a **single M5–brane** extended in the directions $(x^0, \dots, x^3)$ and wrapped **on a Riemann surface** Σ embedded in the (s, v) plane.

Reduction on ϕ turns the M5–brane into an **D4–brane** extended in **r** and **wrapped on** Σ, which is now embedded in the worldvolume of the D6–brane.

For finite ε this picture remains the same, but this time **the Riemann surface** Σ **is embedded in a non-commutative complex plane** where

$$[s, v] = \mathrm{i}\, \varepsilon\ .$$

The point

- We repeat our construction starting from a Taub–nut space in the bulk
- The Taub–trap solution **interpolates** between Nekrasov's **original description** and Nekrasov–Witten's **"alternative" description**
- We lift the IIA background to **M–theory**
- We **reduce** it on the isometry circle.
- The resulting D6 background has a natural **non–commutativity** ε
- The gauge theory describes the dynamics of a D4 wrapped on a Riemann surface living on a non-commutative $\mathbb{C}^2$ plane. This is the geometric interpretation of the **"quantum spectral curve"**.

Outline

Conclusions

- We propose a **string realization of the Ω deformation**, based on NS fields alone. We call it the fluxtrap
- We show how the **effective theory for a $D4$–brane** in this background is the four-dimensional **Ω deformation of $\mathcal{N} = 2$ Seiberg–Witten** introduced by Nekrasov
- We lift the configuration to M–theory and interpret the $D4$–$NS5$ setup in terms of an **$M5$-brane wrapped on a Riemann surface** with flux
- We use a **9/11 flip** to show the geometric interpretation of ε **as a quantization parameter**.

Thank you

for your attention

Us and the topological string

How does the Ω–deformation enter topological strings?
The four-dimensional gauge theory is realized as the dynamics of a **geometrically engineered Calabi–Yau singularity in type iia string theory**.
Then the gauge theory can be lifted to a five-dimensional theory living on the same singularity in M–theory, and finally re-compactified with Melvin boundary conditions to yield an Ω–deformed gauge theory in four dimensions, which is ultraviolet-completed to the topological string on the Calabi–Yau singularity.
The topological string coupling g_{top} is directly proportional to the parameter $\varepsilon = \varepsilon_1 = -\varepsilon_2$ of the Melvin twist.

We have realized the Ω–deformed $\mathcal{N} = 2$ gauge dynamics as the **dynamics of a (p, q) fivebrane web** (the initial type iib picture) compactified on a circle with Melvin boundary conditions

Us and the topological string

Prior to reduction on the Melvin circle, the two five-dimensional theories are not the same and have different properties.

The **R–symmetry groups are different**:

- In the (p, q) fivebrane web there is an exact $SU(2)$ R–symmetry rotating three common transverse coordinates to all the branes
- In the non-compact Calabi–Yau singularities of interest for the study of the topological string, the R–symmetry is generically only a $U(1)$.

Us and the topological string

For small ε **both constructions** can be dualized to a configuration of a **single** M5**–brane wrapped on a Riemann surface in presence of a flux**:

- In our construction there is a non-vanishing pullback of the flux on the M5–brane
- For the topological string the flux has no component on Σ [Dijkgraaf, Hollands, Sulkowski, Vafa]

The type iia reduction of this latter background gives an action that has the **same type of terms in four dimensions, but with different orientations** (there is a non-trivial coupling of both components of the complex scalar field to the gauge field).

It is tempting to think that our embedding **may be related directly by some duality** to the topological string. Whatever the duality, **it cannot be a duality that is realized as geometric in eleven dimensions**, since the flux along the Riemann surface is a geometric invariant.