Asymptotic flatness in higher dimensional spacetimes

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Based on

K.Tanabe, N. Tanahashi, T.Shiromizu, J.Math.Phys, 51, 062502(2010), ibid, 52, 032501(2011),
K.Tanabe, S.Kinoshita, T.Shiromizu, Phys. Rev. D84, 044055(2011), 1203.0452[PRD85(2012)]

K.Tanabe(Barcelona Univ.), N.Tanahashi(UC Davis) S.Kinoshita(Osaka City Univ.)

Content

1. Introduction/4D BH

Why higher dimensions?

- String theory
- Gauge/gravity correspondence
- Black holes in LHC(?)
- A famous cartoon? - “4D” pocket, Doraemon -

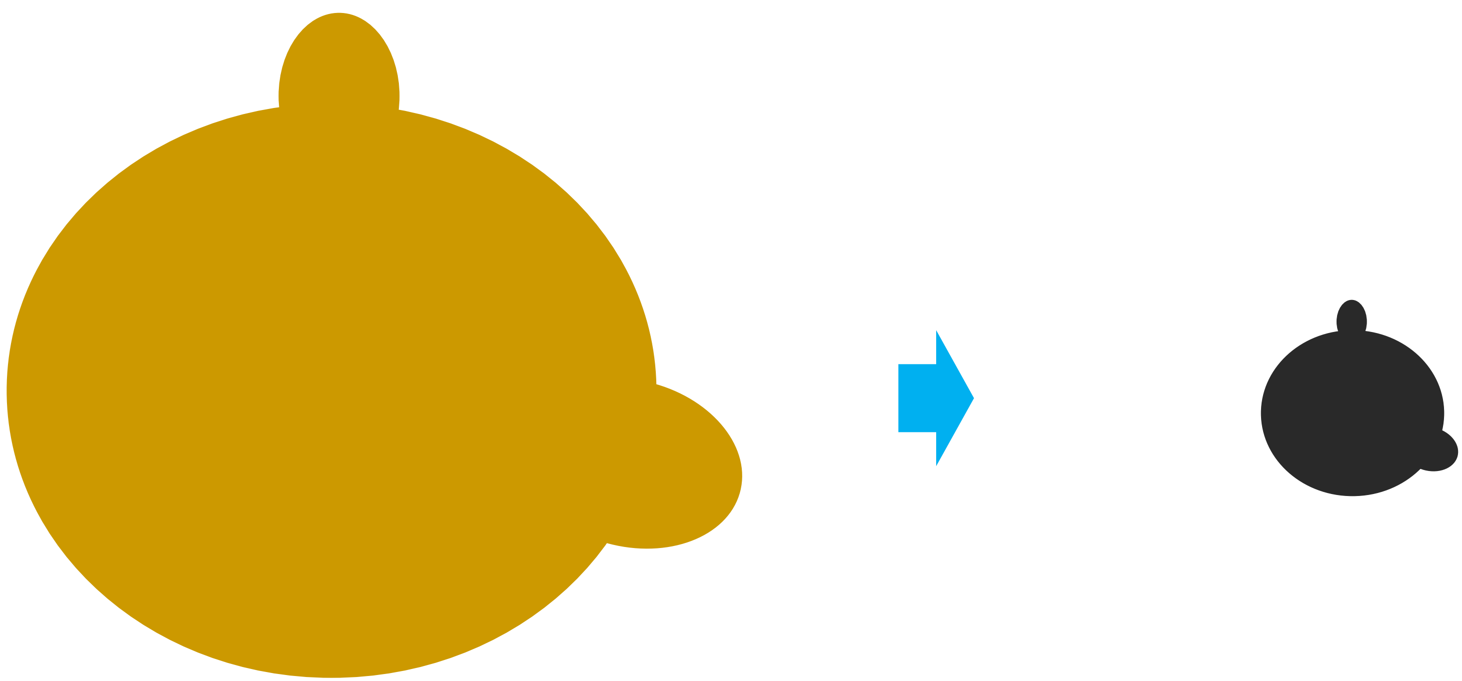
BHs in 4D

- After gravitational collapse, black holes will be formed.
- Then spacetime will be stationary.
- The black hole is described by the Kerr solution.

The Golden age in 4D

- The topology of stationary black holes is 2-sphere. [Hawking 1972]
- It is shown that stationary black holes are axisymmetric(rigidity). [Hawking 1972]
- Stationary and axisymmetric black hole is unique. The Kerr solution! [Carter 1973,…]
- The Schwarzschild solution is unique static BH. [Israel 1967]

Then



The second Golden age?

A lot of remaining issues for higher dimensional BHs.

It is a chance to have a “gold”.

Focus on higher dimensional GR

- Vacuum Einstein equation
- Asymptotically flat
- Cosmic censorship conjecture

  [no naked singularities]

3. Higher dimensional black holes

~ brief review ~

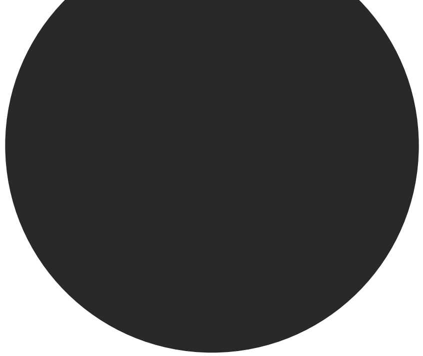
Recent reviews

Eds. K.-I. Maeda, T. Shiromizu, T. Tanaka
Higher dimensional black holes,
Progress of Theoretical Physics Supplement No. 189, 2011

Ed. G. Horowitz,
Black holes in higher dimensions,
Cambridge Univ. Press 2012

Schwarzschild in higher dimensions

Tangherlini 1963

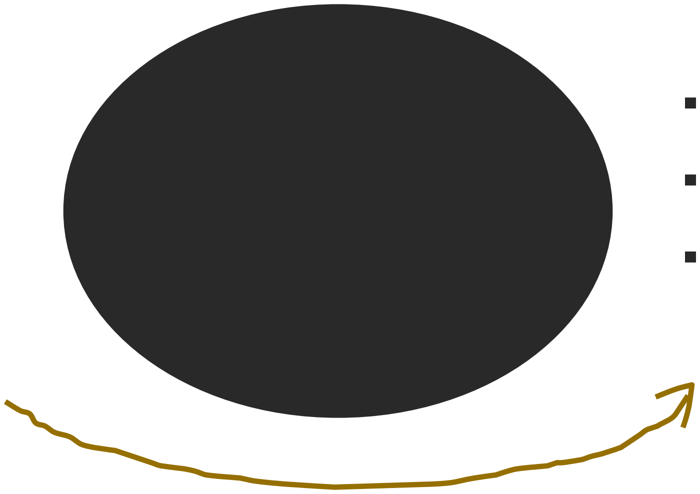


- spherical symmetric
- topology: (D-2)-sphere
- any dimensions

Kerr in higher dimensions

Myers&Perry 1986

- Myers-Perry black holes



- rotating
- topology: (D-2)-sphere
- any dimensions

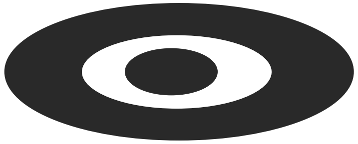
Black rings

Discovered by Emparan & Reall 2001

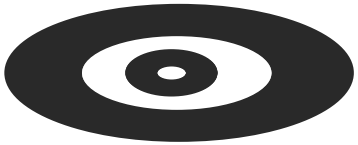
- D=5
- $S^1 \times S^2$
- rotation is important

Black object zoo

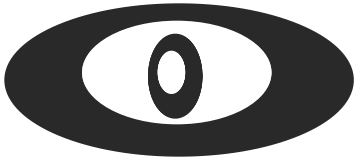
- Black Saturn Elvang and Figueras, 2007



- Di-Ring Iguchi and Mishima, 2007



- Bi-Cycling Ring Izumi, 2008, Elvang and Rodriguez, 2008



and so on

Rigidity theorem

- Hawking 1972

4D, stationary, rotating, asymptotically flat black holes are axisymmetric

- Hollands, Ishibashi & Wald 2006

any dimensions, stationary, rotating, asymptotically flat black object spacetimes are axisymmetric.

Topology

- Hawking 1972

The topology of 4D, stationary, rotating, asymptotically flat black hole is 2-sphere

- Galloway & Schoen 2005

In higher dimensions,

$$"\int_S {}^{(D-2)}R dS > 0"$$

${}^{(D-2)}R$:Ricci scalar of BH cross section

$\Rightarrow$ $D = 5: \quad S^3, \; S^1 \times S^2, \;$ connected sum

"Uniqueness" ≈ classification

- Gibbons, Ida & Shiromizu 2002

Higher dimensional Schwarzschild spacetime is unique static vacuum BH.

- Morisawa & Ida 2004

In D=5, the Myers-Perry solution is unique stationary vacuum BH if the topology is 3-sphere and 2 rotational symmetries are.

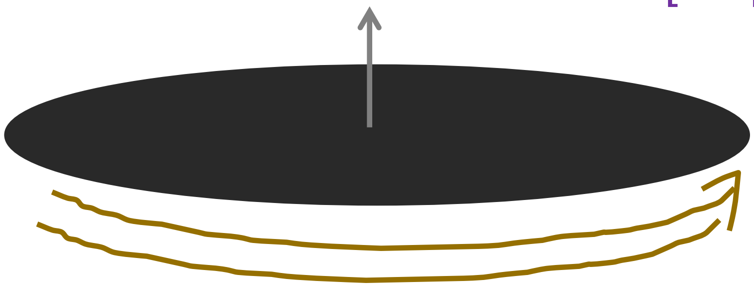
- …, Hollands & Yazadjiev 2007

In D=5, stationary BH with 2-rotational symmetries can be classified by mass, angular momentum and rod structure(~topology)

Stability

- Schwarzschild BH is stable[Ishibashi & Kodama 2003]

- Ultraspinning Myers-Perry(MP) solution (D>5) was conjecture to be unstable [Emparan & Myers 2003]



confirmed by a numerical study[Shibata & Yoshino 2009-2010]

settle down to BH with lower angular momentum.

Remaining issues

- Stability

  Although there are many efforts…

Asymptotic structure

- Complete classification

  requirement of rod structure(≈topology,…) is too strong…

- Construction of exact solution in D>5

  need a new way to find (exact) solution…

  blackfold approach…? Emparan et al 2009

3. Asymptotics

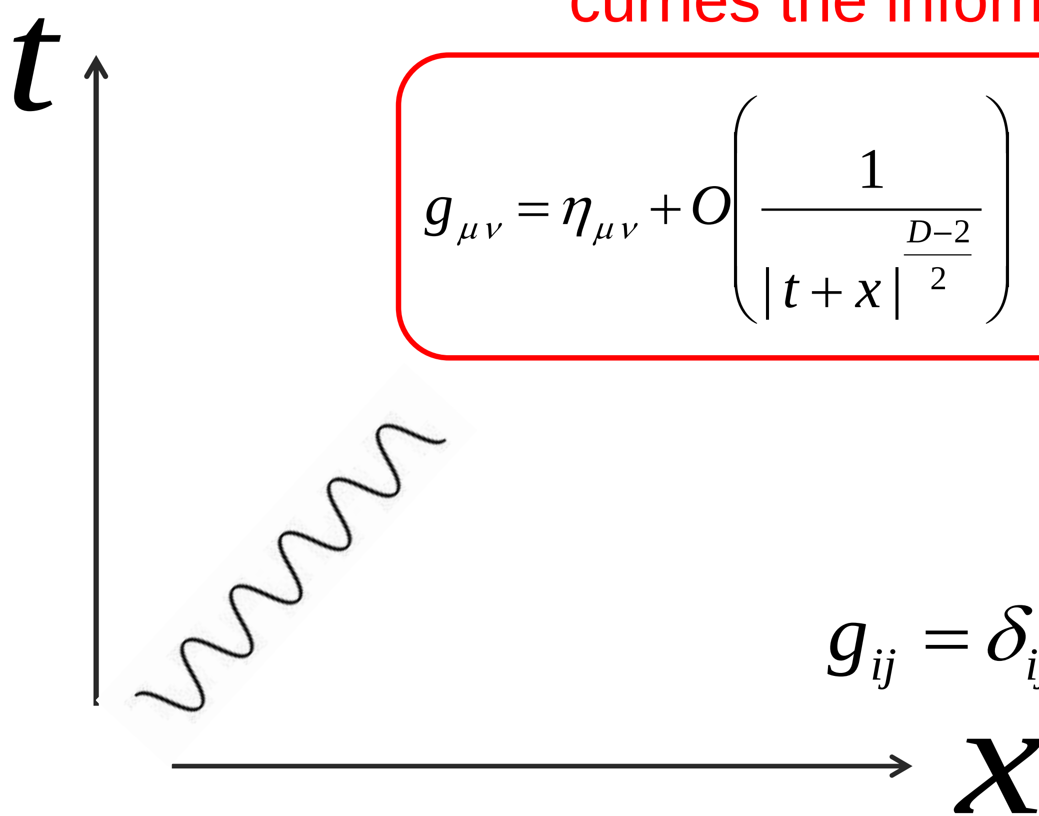
Asymptoics

- Naïve concept

curries the information of dynamics

$$g_{\mu\nu} = \eta_{\mu\nu} + O\left(\frac{1}{|t+x|^{\frac{D-2}{2}}} \right)$$

null infinity



spatial infinity

$$g_{ij} = \delta_{ij} + O(1/|x|^{D-3})$$

How to analyze

- …, Ashtekar & Hansen 1978, Hollands & Ishibashi 2005

conformal completion method

$$\boxed{0 \times \infty = \text{finite}}$$

$$\Omega^2 \times \eta_{\mu\nu} = \tilde{g}_{\mu\nu}$$

$$\text{infinity} \Leftrightarrow \Omega = 0$$

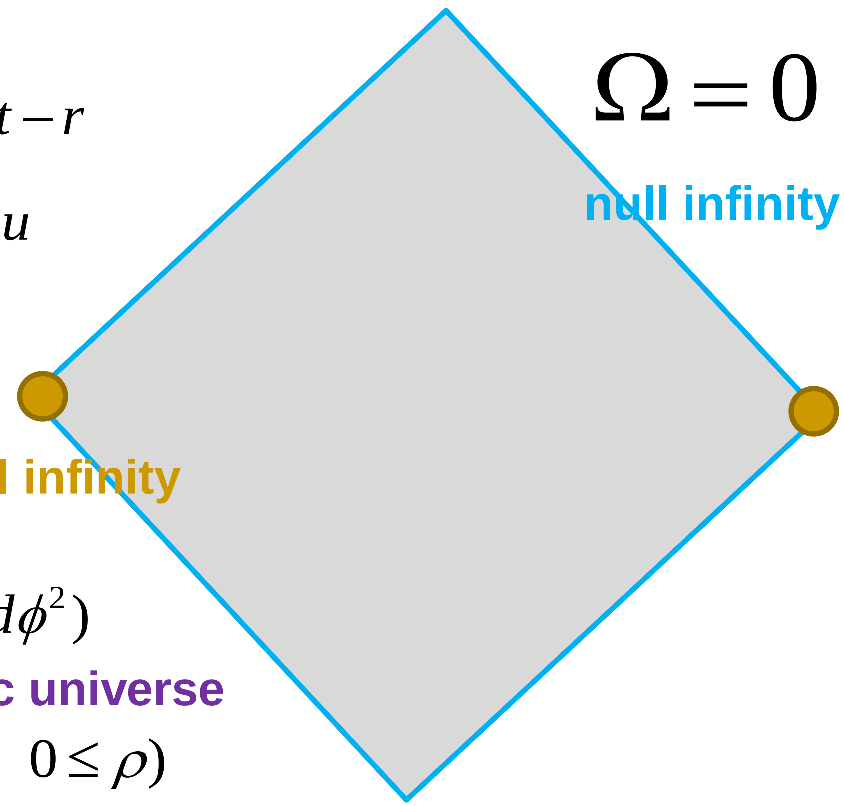
Minkowski spacetime

embemded into the Einstein static universe

$$\tilde{g}_{\mu\nu} = \Omega^2 \eta_{\mu\nu}$$

$$\Omega = 4(1+v^2)^{-1}(1+u^2)^{-1}, \quad v = t + r, u = t - r$$

$$\tau = \tan^{-1} v + \tan^{-1} u, \ \ \rho = \tan^{-1} v - \tan^{-1} u$$



$$\tilde{g}_{\mu\nu} dx^\mu dx^\nu = -d\tau^2 + d\rho^2 + \rho^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

Einstein static universe

$$(-\pi < \tau + \rho < \pi, \quad -\pi < \tau - \rho < \pi, \quad 0 \le \rho)$$

Conformal treatment in general

Near null infinity $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu} = O(1/r^{D/2-1})$

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \Omega \sim 1/r$$

Ω is used to be a coordinate in $\tilde{g}_{\mu\nu}$

$$h_{\mu\nu} \sim \Omega^{D/2-1}$$

-The half-integer for odd dimensions.

-It is supposed that the conformally transformed spacetime is regular.

-Without solving the Einstein equation, one discusses the asymptotics in the physical spacetime.

-As a result, we encounter **the bad behavior** because of the half-integer for odd dimensions.

Conformal treatment in general

The conformal treatment does not work for odd dimensions…

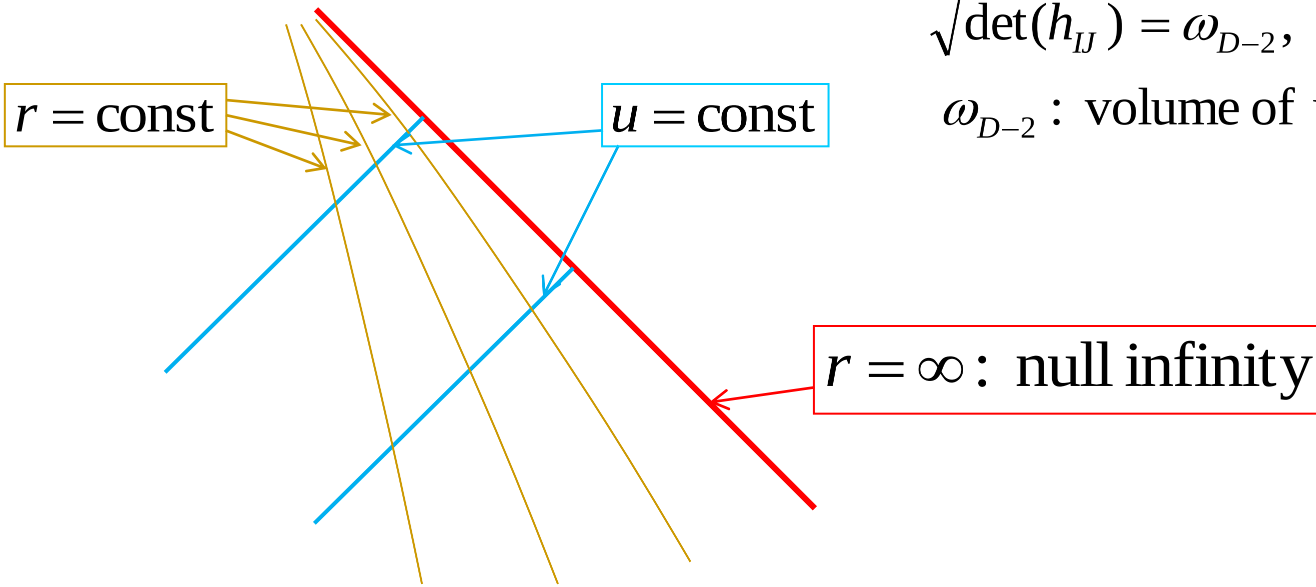
➡ Introduce the Bondi coordinate and then solve the Einstein equation near the null infinity.

Bondi coordinate

$$ds^2 = -Ae^B du^2 - 2e^B du dr + r^2 h_{IJ}(dx^I + C^I du)(dx^J + C^J du)$$

$$\sqrt{\det(h_{IJ})} = \omega_{D-2},$$

ω_{D-2} : volume of unit $(D-2)$-sphere



Vacuum Einstein equation

$$\Rightarrow \quad A = 1 + O(r^{-(D/2-1)}), B = O(r^{-(D-2)}), C^I = O(r^{-D/2})$$

$$h_{IJ} = \omega_{IJ} + O(r^{-(D/2-1)})$$

Definition of asymptotic flatness

The spacetime is said to be asymptotically flat if the metric in the Bondi coordinate behaves like

$$A = 1 + O(r^{-(D/2-1)}),\ B = O(r^{-(D-2)}),\ C^I = O(r^{-D/2})$$

$$h_{IJ} = \omega_{IJ} + O(r^{-(D/2-1)})$$

Bondi coordinate

$$ds^2 = -Ae^B du^2 - 2e^B du dr + r^2 h_{IJ}(dx^I + C^I du)(dx^J + C^J du)$$

$$\sqrt{\det(h_{IJ})} = \omega_{D-2}$$

Bondi mass

$$ds^2 = -Ae^B du^2 - 2e^B du dr + r^2 h_{IJ}(dx^I + C^I du)(dx^J + C^J du)$$

$$g_{uu} = -Ae^B = -1 - \sum_{k=0}^{k<D/2-2} \frac{A^{(k+1)}}{r^{D/2+k-1}} + \frac{m(u,x^I)}{r^{D-3}} + O(1/r^{D-5/2})$$

$$\Rightarrow \quad \boxed{M_{Bondi}(u) = \frac{D-2}{16\pi}\int_{S^{D-2}} m\, d\omega}$$

$A^{(k+1)}$ does not contribute to the surface integral because of $A^{(k+1)} \propto D^I D^J h_{IJ}^{(k+1)}$

Bondi mass loss law

$$\left\{\begin{aligned} & ds^2 = -Ae^B du^2 - 2e^B du dr + r^2 h_{IJ}(dx^I + C^I du)(dx^J + C^J du) \\ & g_{uu} = -Ae^B = -1 - \sum_{k=0}^{k<D/2-2} \frac{A^{(k+1)}}{r^{D/2+k-1}} + \frac{m(u,x^I)}{r^{D-3}} + O(1/r^{D-5/2}) \\ & M_{Bondi}(u) = \frac{D-2}{16\pi}\int_{S^{D-2}} m\, d\omega \end{aligned}\right.$$

The Einstein equation implies

$$\partial_u m = -\frac{1}{2(D-2)} \partial_u h^{(1)}_{IJ} \partial_u h^{(1)IJ} + \text{total derivative}$$

Flux of gravitational waves

$$\Rightarrow \quad \frac{d}{du} M_{Bondi}(u) = -\frac{1}{32\pi}\int_{S^{D-2}} \partial_u h^{(1)}_{IJ} \partial_u h^{(1)IJ} d\omega \le 0$$

Bondi mass loss law

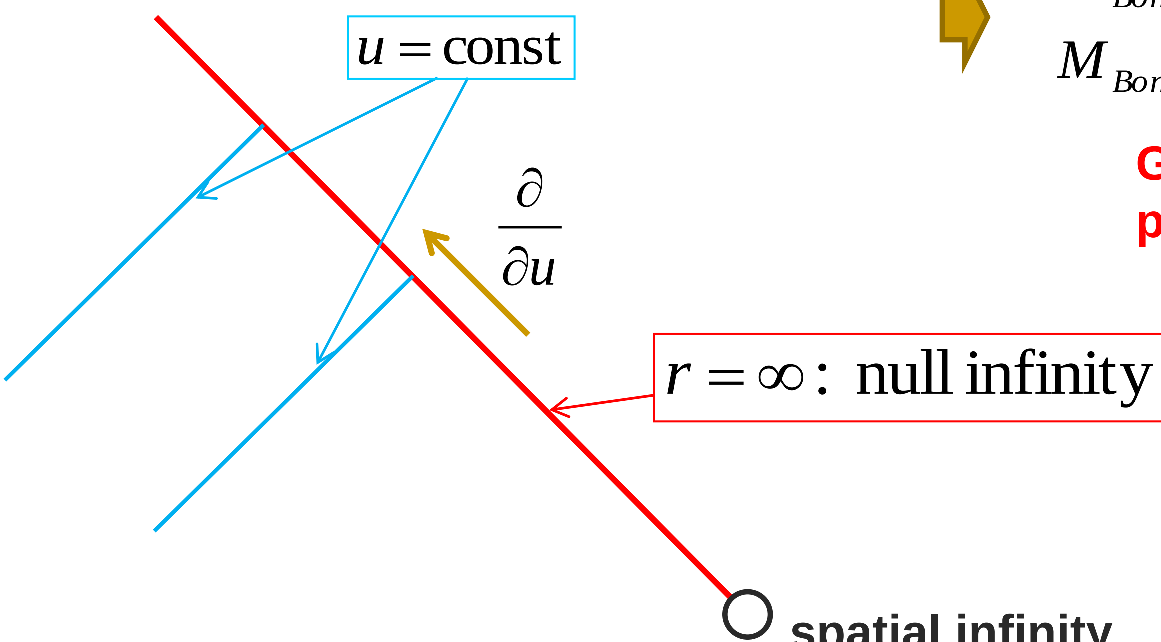
Flux of gravitational waves

$$\frac{d}{du}M_{Bondi}(u) = -\frac{1}{32\pi}\int_{S^{D-2}} \partial_u h_{IJ}^{(1)} \partial_u h^{(1)IJ} d\omega \le 0$$

$$M_{Bondi}(-\infty) = M_{initial} = M_{ADM} \ge 0$$

$$M_{Bondi}(\infty) = M_{final} < M_{initial}$$

Gravitational wave curries the positive energy



Asymptotic symmetry

Asymptotic symmetry is defined to be the transformation group which preserves the asymptotic structure at null infinity

$$ds^2 = -Ae^B du^2 - 2e^B du dr + r^2 h_{IJ}(dx^I + C^I du)(dx^J + C^J du)$$

$$A = 1 + O(r^{-(D/2-1)}), B = O(r^{-(D-2)}), C^I = O(r^{-D/2})$$

$$h_{IJ} = \omega_{IJ} + O(r^{-(D/2-1)})$$

$$\sqrt{\det(h_{IJ})} = \omega_{D-2}$$

Bondi coordinate condition

$$\delta g_{rr} = 0,\ \delta g_{rI} = 0,\ g^{IJ}\delta g_{IJ} = 0$$

$$\delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$

$$\xi^u = f(u, x^I),$$

$$\xi^I = f^I(u, x^I) + \int dr \frac{e^B}{r^2} h^{IJ} D_J f,$$

$$\xi^r = -\frac{r}{D-2}(C^I D_I f + D_I f^I)$$

D_I : covariant derivative w.r.t. ω_{IJ}

Asymptotic symmetry

$$ds^2 = -Ae^B du^2 - 2e^B du dr + r^2 h_{IJ}(dx^I + C^I du)(dx^J + C^J du)$$

$$A = 1 + O(r^{-(D/2-1)}), B = O(r^{-(D-2)}), C^I = O(r^{-D/2}), h_{IJ} = \omega_{IJ} + O(r^{-(D/2-1)})$$

The asymptotic behavior

$$\delta g_{uu} = O(1/r^{D/2-1}),\ \delta g_{rI} = O(1/r^{D/2-2}),\ \delta g_{ur} = O(1/r^{D-2}),\ \delta g_{IJ} = O(1/r^{D/2-3})$$

$$\text{e.g.,}\quad \delta g_{IJ} = r^2\left[D_I f_J + D_J f_I - \frac{2D_K f^K}{D-2}\omega_{IJ}\right] - 2r\left[D_I D_J f - \frac{D_K D^K f}{D-2}\omega_{IJ}\right] + O(1/r^{D/2-3})$$

$$\partial_u f^I = 0,\ \ D_I f_J + D_J f_I = \frac{2D_K f^K}{D-2}\omega_{IJ}, D_I f^I = (D-2)\partial_u f,\ \ D_I D_J f = \frac{D_K D^K f}{D-2}\omega_{IJ}$$

Asymptotic symmetry is Poincare

The asymptotic symmetry is generated by f and f^I

$$\partial_u f^I = 0, \quad D_I f_J + D_J f_I = \frac{2 D_K f^K}{D-2}\omega_{IJ}, D_I f^I = (D-2)\partial_u f, \quad D_I D_J f = \frac{D_K D^K f}{D-2}\omega_{IJ}$$

$f \Rightarrow$ translation

$f^I \Rightarrow$ Lorentz group

D=4 is special

$$\cdots, \delta g_{IJ} = O(1/r^{D/2-3}) \Rightarrow \delta g_{IJ} = O(r)$$

$$\delta g_{IJ} = r^2\left[D_I f_J + D_J f_I - \frac{2D_K f^K}{D-2}\omega_{IJ}\right] - 2r\left[D_I D_J f - \frac{D_K D^K f}{D-2}\omega_{IJ}\right] + O(1/r^{D/2-3})$$

$O(1/r^{D/2-3}) \Rightarrow O(r)$

$$\Rightarrow \partial_u f^I = 0, \quad D_I f_J + D_J f_I = \frac{2D_K f^K}{D-2}\omega_{IJ}, D_I f^I = (D-2)\partial_u f,$$

$$\Rightarrow \begin{cases} f \Rightarrow \text{translation +supertranslation} \\ f^I \Rightarrow \text{Lorentz group} \end{cases}$$

This gives us troublesome when one defines angular momentum at null infinity

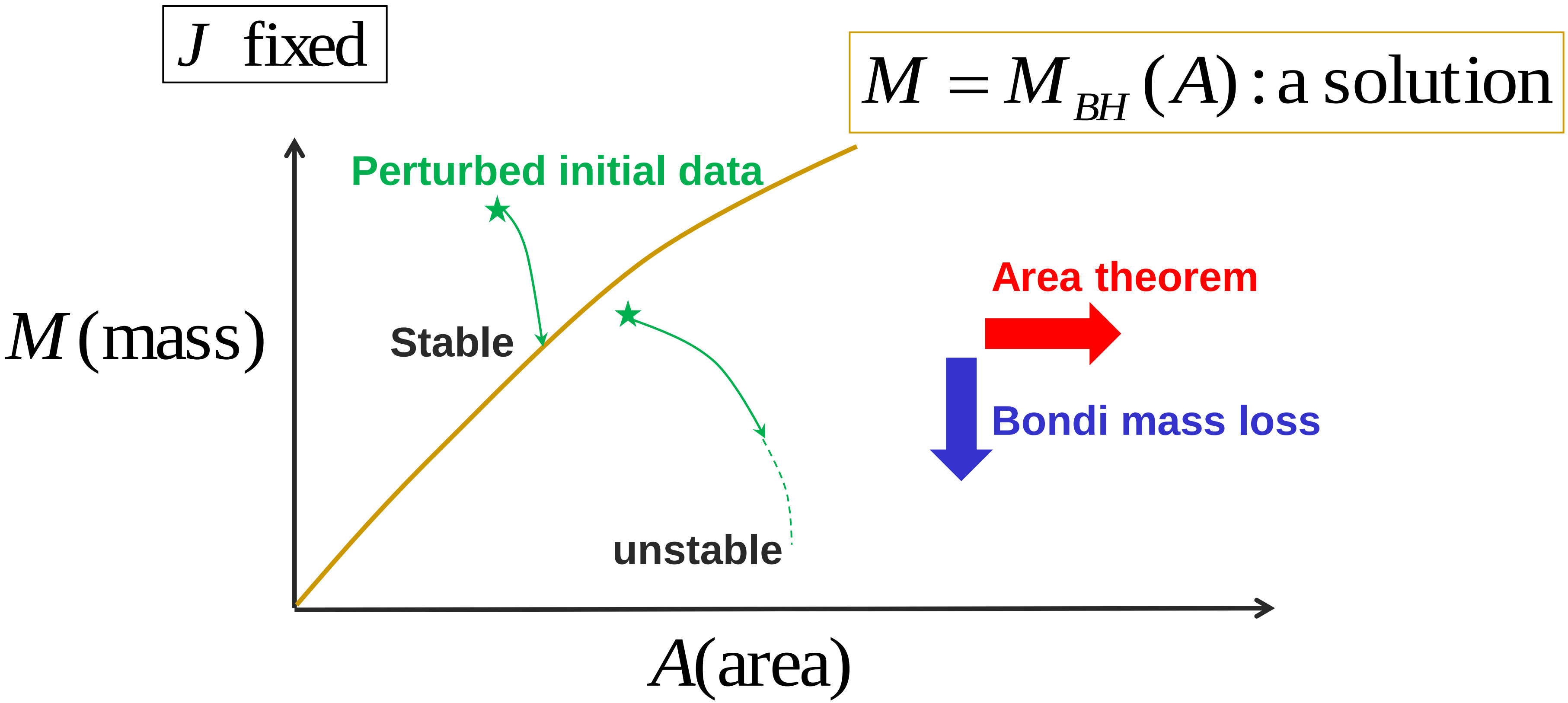
4. Application

Stability of BHs

- The symmetry which exact BH solutions have are not enough to analyse the stability.
- Together with the general properties of dynamical BHs, we may be able to have some indications.

Stability of BHs

Figueras, Murata & Reall 2011, Hollands & Wald 2012



Classification

Weyl tensor: $C_{\mu\nu\alpha\beta} = \text{trace free part of Riemann tensor } R_{\mu\nu\alpha\beta}$

contains information of spacetime(gravity)

Peeling property in 4D

$$C_{\mu\nu\alpha\beta} = \frac{C^{(N)}_{\mu\nu\alpha\beta}}{r} + \frac{C^{(III)}_{\mu\nu\alpha\beta}}{r^2} + \frac{C^{(II,D)}_{\mu\nu\alpha\beta}}{r^3} + \frac{C^{(I)}_{\mu\nu\alpha\beta}}{r^4} + O(1/r^5)$$

Algebraic classification of Weyl tensor (Petrov type) ⇔ asymptotic behavior

In Type D the equations will be significantly simplified and then solved completely. The Kerr solution, C-metric ,….

Using our result, a peeling properties has been discussed in higher dimensions [Godazgar & Reall 2012].

$(i)\ D>5$

$$C_{\mu\nu\alpha\beta} = \frac{C^{(N)}_{\mu\nu\alpha\beta}}{r^{D/2-1}} + \frac{C^{(II)}_{\mu\nu\alpha\beta}}{r^{D/2}} + \frac{C^{(G)}_{\mu\nu\alpha\beta}}{r^{D/2+1}} + O(1/r^{D/2+2})[\text{even } D] \text{ or } O(1/r^{D/2+3/2})[\text{odd } D]$$

$(ii)\ D=5$

$$C_{\mu\nu\alpha\beta} = \frac{C^{(N)}_{\mu\nu\alpha\beta}}{r^{3/2}} + \frac{C^{(II)}_{\mu\nu\alpha\beta}}{r^{5/2}} + \frac{C^{(N)'}_{\mu\nu\alpha\beta}}{r^{3}} + \frac{C^{(G)}_{\mu\nu\alpha\beta}}{r^{7/2}} + O(1/r^4)$$

Classification

Similar to four dimensions, the classification of the Weyl tensor and the peeling property may give us a hint to classify and/or construct exact solutions in higher dimensions.

5. Summary and future issues

Summary

- For odd dimensions, we had to solve the Einstein equation to formulate the asymptotics at null infinity.
- We could show the Bondi mass loss law and the presence of the Poincare symmetry at null infinity.
- We also defined the momentum and angular momentum.
- There are a few efforts to discuss the stability and classification of higher dimensional BHs/spacetimes.

Remaining issues

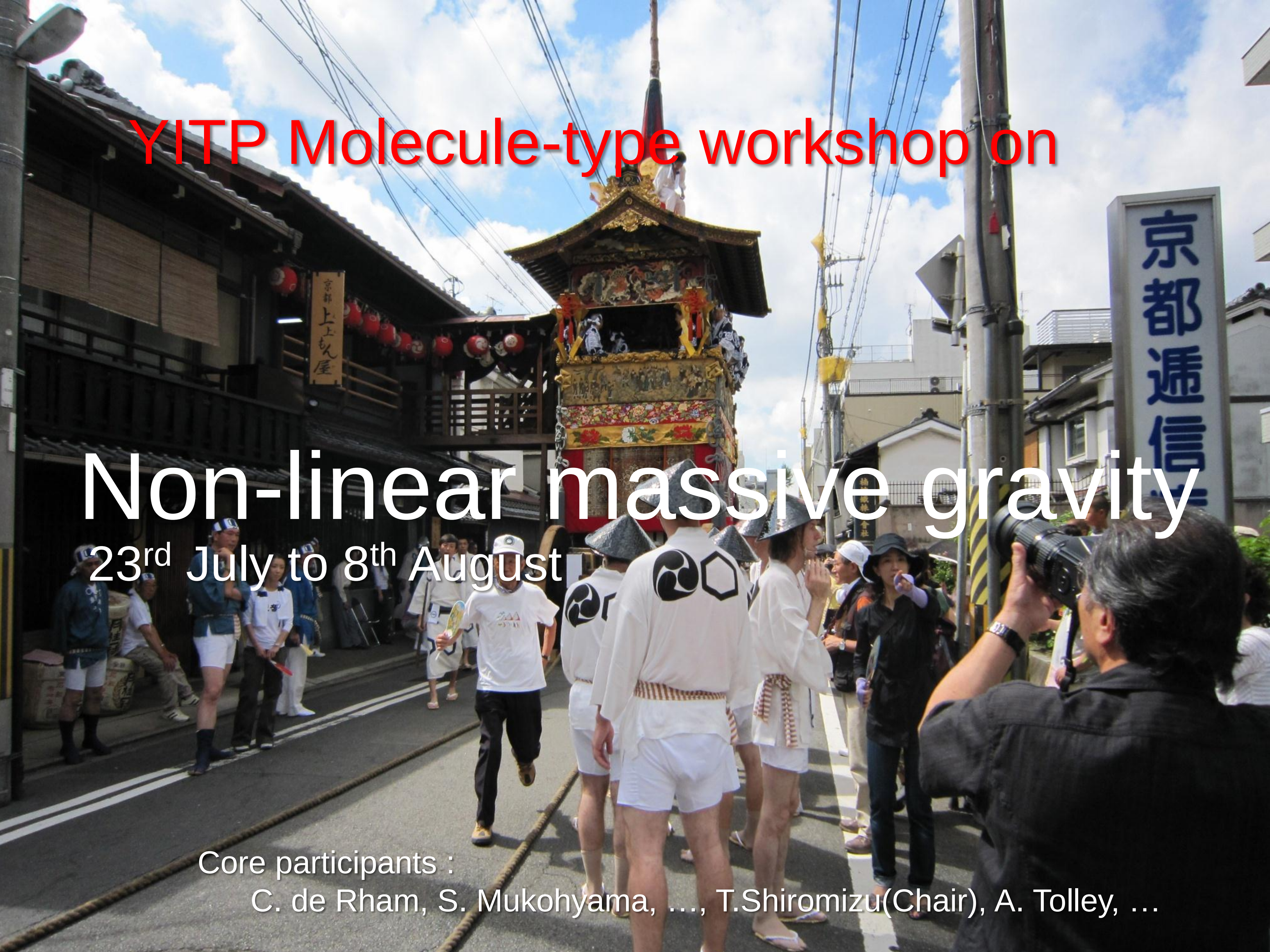
Still…

- Stability…
- Classification…
- Construction…

New issues?

- Asymptotically anti-deSitter spacetimes
  - ⇔ always unstable? Final fate? [Dias,Horowitz,Santos,2011]
- Asymptotics in spacetimes with compact space
  - ⇔ final fate of black string?

    [Latest numerical simulation:Lehner & Pretorius 2010]



YITP Molecule-type workshop on

Non-linear massive gravity

23rd July to 8th August

Core participants :

C. de Rham, S. Mukohyama, …, T.Shiromizu(Chair), A. Tolley, …

Appendix

Constraint and evolution eqs.

In Bondi coordinate,

$$ds^2 = -Ae^B du^2 - 2e^B du dr + \gamma_{IJ}(dx^I + C^I du)(dx^J + C^J du)$$

$$u \Leftrightarrow \text{time}$$

$$R_{r\mu} = R_{\mu\nu}\gamma^{\mu\nu} = 0 \;\Rightarrow\; \text{constraint equations}$$

$$R_{uu} = 0, R_{IJ} = 0 \;\Rightarrow\; \text{evolution equations}$$

Constraint equations

$$ds^2 = -Ae^B du^2 - 2e^B du dr + r^2 h_{IJ}(dx^I + C^I du)(dx^J + C^J du)$$

$$\begin{cases} R_{rr} = 0 \;\Rightarrow\; B' = \dfrac{r}{4(D-2)} h'_{IJ} h'_{KL} h^{IK} h^{JL} \\[2ex] R_{\mu\nu}\gamma^{\mu\nu} = 0 \;\Rightarrow\; (D-2)\dfrac{(r^{D-3}A)'}{r^{D-2}} = -{}^{(\omega)}\nabla_I\, C^{I'} - \dfrac{2(D-2)}{r}{}^{(\omega)}\nabla_I\, C^I - \dfrac{r^2 e^{-B}}{2} h_{IJ} C^{I'} C^{J'} \\ \qquad - \dfrac{e^B}{2r^2} h^{IJ\,(\omega)}\nabla_I B^{(\omega)}\nabla_J B - \dfrac{e^B}{r^2}{}^{(\omega)}\nabla_I (h^{IJ\,(\omega)}\nabla_J B) + \dfrac{e^B}{r^2}{}^{(h)}R \\[2ex] R_{rI} = 0 \;\Rightarrow\; \dfrac{1}{r^{D-2}}(r^D e^{-B} h_{IJ} C^{J'})' = -{}^{(\omega)}\nabla_I B' + \dfrac{D-2}{r}{}^{(\omega)}\nabla_I B + {}^{(h)}\nabla^J h'_{IJ} \end{cases}$$

Once h_{IJ} are given on the initial u-const surface, A, B, C^I are determined

Constraint equations

$$ds^2 = -Ae^B du^2 - 2e^B du dr + r^2 h_{IJ}(dx^I + C^I du)(dx^J + C^J du)$$

$$h_{IJ} = \omega_{IJ} + \sum_{k\geq 0} h_{IJ}^{(k+1)} r^{-(D/2+k-1)} \quad (k \in Z \ \text{ for even, } \ 2k \in Z \ \text{ for odd dimensions})$$

$$\sqrt{\det(h_{IJ})} = \omega_{D-2} \Rightarrow h_{IJ}^{(k+1)} (k < D/2-1) \ \text{ is traceless}$$

$$\begin{cases} B = B^{(1)} r^{-(D-2)} + O(1/r^{D-3/2}), \quad B^{(1)} = -\frac{1}{16}\omega^{IK}\omega^{JL} h_{IJ}^{(1)} h_{KL}^{(1)} \\ C^I = \sum_{k=0}^{k<D/2-1} \frac{C^{(k+1)I}}{r^{D/2+k}} + \frac{J^I(u,x^I)}{r^{D-1}} + O(1/r^{D-1/2}), \quad C^{(k+1)I} = \frac{2(D+2k-2)}{(D+2k)(D-2k-2)} {}^{(\omega)}\nabla_J h^{(k+1)IJ} \\ A = 1 + \sum_{k=0}^{k<D/2-2} \frac{A^{(k+1)}}{r^{D/2+k-1}} - \frac{m(u,x^I)}{r^{D-3}} + O(1/r^{D-5/2}), \quad A^{(k+1)} = -\frac{4(D+2k-4)}{(D+2k)(D-2k-2)(D-2k-4)} {}^{(\omega)}\nabla_I {}^{(\omega)}\nabla_J h^{(k+1)IJ} \end{cases}$$

Evolution equations

$$ds^2 = -Ae^B du^2 - 2e^B dudr + r^2 h_{IJ}(dx^I + C^I du)(dx^J + C^J du)$$

$$h_{IJ} = \omega_{IJ} + \sum_{k\geq 0} h_{IJ}^{(k+1)} r^{-(D/2+k-1)} \quad (k \in Z \ \text{ for even, } \ 2k \in Z \ \text{ for odd dimensions})$$

$$\sqrt{\det(h_{IJ})} = \omega_{D-2} \Rightarrow h_{IJ}^{(k+1)} (k < D/2-1) \ \text{ is traceless}$$

$$R_{IJ} = 0 \ \Rightarrow \ (k+1)\dot{h}_{IJ}^{(k+2)} = -\frac{1}{2}(D-2k-4)A^{(k+1)}\omega_{IJ} + \frac{1}{8}[D^2 - 6D - 4(k^2+k-4)]h_{IJ}^{(k+1)}$$
$$+\frac{1}{2}(-{}^{(\omega)}\nabla^2 h_{IJ}^{(k+1)} + 2{}^{(\omega)}\nabla_{(I}{}^{(\omega)}\nabla^K h_{J)K}^{(k+1)}) - \frac{1}{2}(D-2k-4){}^{(\omega)}\nabla_{(I} C_{J)}^{(k+1)} - {}^{(\omega)}\nabla^K C_K^{(k+1)}\omega_{IJ}$$

$$R_{uu} = 0 \ \Rightarrow \ \dot{m} = -\frac{1}{2(D-2)}\dot{h}_{IJ}^{(1)}\dot{h}^{(1)IJ} + \frac{D-5}{D-2}{}^{(\omega)}\nabla^I C_I^{(D/2-2)} + \frac{1}{D-2}{}^{(\omega)}\nabla^2 A^{(D/2-2)}$$

Asymptotic symmetry

Asymptotic symmetry

Asymptotic symmetry is defined to be the transformation group which preserves the asymptotic structure at null infinity

$$ds^2 = -Ae^B du^2 - 2e^B du dr + r^2 h_{IJ}(dx^I + C^I du)(dx^J + C^J du)$$

$$A = 1 + O(r^{-(D/2-1)}), B = O(r^{-(D-2)}), C^I = O(r^{-D/2})$$

$$h_{IJ} = \omega_{IJ} + O(r^{-(D/2-1)})$$

$$\sqrt{\det(h_{IJ})} = \omega_{D-2}$$

Bondi coordinate condition

$$\delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$

$$\delta g_{rr} = -2e^B \xi^{u\prime} = 0$$

$$\delta g_{rI} = -e^{B\,(\gamma)}\nabla_I \xi^u + \gamma_{IJ} C^J \xi^{u\prime} + \gamma_{IJ}\xi^{J\prime}$$

$$g^{IJ}\delta g_{IJ} = \xi^r (\log\gamma)' + \xi^u (\log\gamma) + 2^{(\gamma)}\nabla_I \xi^I + 2C^{I\,(\gamma)}\nabla_I \xi^u$$

$$\xi^u = f(u, x^I),$$

$$\xi^I = f^I(u, x^I) + \int dr \frac{e^B}{r^2} h^{IJ} D_J f,$$

$$\xi^r = -\frac{r}{D-2}(C^I D_I f + D_I f^I)$$

D_I : covariant derivative w.r.t. ω_{IJ}

Asymptotic symmetry

$$ds^2 = -Ae^B du^2 - 2e^B du dr + r^2 h_{IJ}(dx^I + C^I du)(dx^J + C^J du)$$

$$A = 1 + O(r^{-(D/2-1)}), B = O(r^{-(D-2)}), C^I = O(r^{-D/2})$$

$$\sqrt{\det(h_{IJ})} = \omega_{D-2}$$

$$h_{IJ} = \omega_{IJ} + O(r^{-(D/2-1)})$$

$$\delta g_{uu} = \frac{2r}{D-2}\partial_u {}^{(\omega)}\nabla_I f^I - \frac{2}{D-2}\partial_u({}^{(\omega)}\nabla^2 + (n-2)f) + \frac{2C_I^{(1)}\partial_u f^I}{r^{D/2-2}} + O(1/r^{D/2-1})$$

$$\delta g_{ur} = \frac{1}{D-2}[{}^{(\omega)}\nabla_I f^I - (D-2)\partial_u f] - \sum_{k=0}^{k<D/2-2} \frac{D+2k-2}{(D-2)(D+2k)} h_{IJ}^{(k+1)} \frac{{}^{(\omega)}\nabla^I {}^{(\omega)}\nabla^J f}{r^{D/2+k}} + O(1/r^{D-2})$$

$$\delta g_{uI} = r^2\partial_u f_I + \frac{r}{D-2}\partial_I[{}^{(\omega)}\nabla_I f^I - (D-2)\partial_u f] - \frac{1}{D-2}\partial_I[{}^{(\omega)}\nabla^2 f + (D-2)f] + \frac{h_{IJ}^{(1)}\partial_u f^J}{r^{D/2-3}} + O(1/r^{D/2-2})$$

$$\delta g_{IJ} = 2r^2\left[{}^{(\omega)}\nabla_{(I} f_{J)} - \frac{{}^{(\omega)}\nabla_K f^K}{D-2}\omega_{IJ}\right] - 2r\left[{}^{(\omega)}\nabla_I {}^{(\omega)}\nabla_J f - \frac{{}^{(\omega)}\nabla^2 f}{D-2}\omega_{IJ}\right] + O(1/r^{D/2-3})$$

$$\partial_u f^I = 0, \quad {}^{(\omega)}\nabla_I f_J + {}^{(\omega)}\nabla_J f_I = \frac{2\,{}^{(\omega)}\nabla_K f^K}{D-2}\omega_{IJ}, \; {}^{(\omega)}\nabla_I f^I = (D-2)\partial_u f, \quad {}^{(\omega)}\nabla_I {}^{(\omega)}\nabla_J f = \frac{{}^{(\omega)}\nabla_K {}^{(\omega)}\nabla^K f}{D-2}\omega_{IJ}$$

Asymptotic symmetry

$$\partial_u f^I = 0,\quad {}^{(\omega)}\nabla_I f_J + {}^{(\omega)}\nabla_J f_I = \frac{2\,{}^{(\omega)}\nabla_K f^K}{D-2}\omega_{IJ},\ {}^{(\omega)}\nabla_I f^I = (D-2)\partial_u f,\quad {}^{(\omega)}\nabla_I {}^{(\omega)}\nabla_J f = \frac{{}^{(\omega)}\nabla_K {}^{(\omega)}\nabla^K f}{D-2}\omega_{IJ}$$

$$\partial_u f^I = 0,\ F := {}^{(\omega)}\nabla_I f^I = (D-2)\partial_u f \ \Rightarrow\ f = \frac{F(x^I)}{D-2}u + \alpha(x^I)$$

$${}^{(\omega)}\nabla_I {}^{(\omega)}\nabla_J f = \frac{{}^{(\omega)}\nabla_K {}^{(\omega)}\nabla^K f}{D-2}\omega_{IJ} \ \Rightarrow\ {}^{(\omega)}\nabla_I {}^{(\omega)}\nabla_J F = \frac{{}^{(\omega)}\nabla_K {}^{(\omega)}\nabla^K F}{D-2}\omega_{IJ}$$

$${}^{(\omega)}\nabla^I {}^{(\omega)}\nabla^J \times \left({}^{(\omega)}\nabla_I f_J + {}^{(\omega)}\nabla_J f_I = \frac{2\,{}^{(\omega)}\nabla_K f^K}{D-2}\omega_{IJ} \right) \Rightarrow ({}^{(\omega)}\nabla^2 + (D-2))F = 0$$

→ $l=1$ modes of the scalar harmonics on S^{D-2}

time/space translations

$${}^{(\omega)}\nabla_I {}^{(\omega)}\nabla_J f = \frac{{}^{(\omega)}\nabla_K {}^{(\omega)}\nabla^K f}{D-2}\omega_{IJ} \ \Rightarrow\ {}^{(\omega)}\nabla_I {}^{(\omega)}\nabla_J \alpha = \frac{{}^{(\omega)}\nabla_K {}^{(\omega)}\nabla^K \alpha}{D-2}\omega_{IJ}$$

→ $l=0,1$ modes of the scalar harmonics on S^{D-2}

${}^{(trans)}f^I$: transvere part of f, *i.e.*, ${}^{(\omega)}\nabla_I {}^{(trans)}f^I = 0,\ {}^{(\omega)}\nabla_I {}^{(trans)}f_J + {}^{(\omega)}\nabla_J {}^{(trans)}f_I = 0$ → **Lorentz group**

Petrov type

Principal null direction

null tetrad

$$g_{\mu\nu} = -l_\mu n_\nu - n_\mu l_\nu + m_\mu m_\nu^* + m_\mu^* m_\nu,\ \ l_\mu n^\mu = -1, m_\mu m^{*\mu} = 1, l_\mu l^\mu = 0, n_\mu n^\mu = 0$$

Five complex functions

$$\psi_0 = C_{\mu\nu\alpha\beta} l^\mu m^\nu l^\alpha m^\beta, \psi_1 = C_{\mu\nu\alpha\beta} l^\mu n^\nu l^\alpha m^\beta, \psi_2 = C_{\mu\nu\alpha\beta} l^\mu m^\nu n^\alpha m^{*\beta},$$

$$\psi_3 = C_{\mu\nu\alpha\beta} l^\mu n^\nu n^\alpha m^\beta, \psi_4 = C_{\mu\nu\alpha\beta} n^\mu m^\nu n^\alpha m^\beta$$

Transformation of null tetrad and principal null direction

$$n^\mu \to n^\mu, m^\mu \to m^\mu + a n^\mu, l^\mu \to l^\mu + a m^{*\mu} + a^* m^\mu + a a^* n^\mu$$

$$\psi_0 \to \psi_0 + 4a\psi_1 + 6a^2\psi_2 + 4a^3\psi_3 + a^4\psi_4$$

ψ_0 can be made zero by satisfying the equation for a

$$\psi_0 + 4a\psi_1 + 6a^2\psi_2 + 4a^3\psi_3 + a^4\psi_4 = 0$$

I (four distinct roots), II(two coincident), D(two distinct double), III(three coincident), N(four coincident)

Petrov classification

I (four distinct roots), II(two coincident), D(two distinct double), III(three coincident), N(four coincident)

I $\quad \psi_0 = \psi_1 = 0$

II $\quad \psi_0 = \psi_1 = \psi_4 = 0$

D $\quad \psi_0 = \psi_1 = \psi_3 = \psi_4 = 0$

III $\quad \psi_0 = \psi_1 = \psi_2 = \psi_4 = 0$

N $\quad \psi_0 = \psi_1 = \psi_2 = \psi_3 = 0$