Looking for a worldsheet description of the Nekrasov partition function

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We want to study some theory on a d-dim space M. To this end we consider the path integral over the d+1-dim fibration $M \to S^1$

$$\int_{p(0)=\mathcal{O}p(\beta)} \mathcal{D}p \;\; exp\left[-\frac{1}{\hbar}\int_0^\beta dt \;\; \mathcal{L}(p)\right] = Tr(e^{-\beta H}(-1)^F \mathcal{O})$$

when $\beta \to 0$ it localizes around paths such that $p(0) = p(\beta)$
fixed points of the operator $\mathcal{O} = \mathcal{O}(\epsilon_1, \epsilon_2)$

Moreover,

$$H = \sum_i \{Q_i, Q_i^\dagger\} \quad [(-1)^F, H] = \{(-1)^F, Q_i\} = \{(-1)^F, Q_i^\dagger\} = 0$$

Then $\quad Tr(e^{-\beta H}(-1)^F) = Tr_{H=0}(-1)^F$

if for some $Q \qquad [\mathcal{O}, Q] = [\mathcal{O}, Q^\dagger] = 0$

$$Tr(e^{-\beta H}(-1)^F \mathcal{O}) \longrightarrow Tr_{Q=0}(e^{-\beta H}(-1)^F \mathcal{O})$$

More specifically we have an $SU(N)$ gauge theory on $M = \mathbb{R}^4$, with $\mathcal{N} = 2$ supersymmetry that in generic points of moduli space reduces to $U(1)^{N-1}$ with vevs $a_1, \ldots . a_{N-1}$

The low energy Lagrangian depends on a single holomorphic function called the prepotential

$$\mathcal{F}(a)$$

we also chose $\mathcal{O}(\epsilon_1, \epsilon_2) = e^{-2\epsilon_- J_l^3} e^{-2\epsilon_+ (J_r^3 + J_I^3)}$

Then we define

$$Z(a, \epsilon_1, \epsilon_2) = Tr_{\mathcal{H}_a} (-1)^F e^{-\beta H} e^{-2\epsilon_- J_l^3} e^{-2\epsilon_+ (J_r^3 + J_I^3)}$$

and

$$\lim_{\epsilon_1, \epsilon_2 \to 0} \log Z(a, \epsilon_1, \epsilon_2) = -\frac{\mathcal{F}(a)}{\epsilon_1 \epsilon_2}$$

Geometric engineering:

A "physical" supersymmetric string theory living in 10-dim can be compactified on an appropriate 6-dim Calabi Yau K such that the low energy 4-dim theory is a certain gauge theory

$$\int \mathcal{D}X\mathcal{D}\psi \; exp\left[-\frac{1}{\alpha'}\int_{\Sigma}\mathcal{L}_{string}\right] \to \int \mathcal{D}\phi \; exp\left[-\frac{1}{g^2}\int_{\mathbb{R}^4}\mathcal{L}_{gauge}\right]$$

$$X : \Sigma(z,\bar{z}) \to K \times \mathbb{R}^4(x^i) \qquad X = X(z,\bar{z}), \;\; \phi = \phi(x^i)$$

In the limit $\epsilon_1 = -\epsilon_2 = \lambda$

$$\log Z(\lambda, a) = \sum_{g=0}^{\infty} \lambda^{2g-2} F^g(a,\bar{a})$$

They are the genus g topological string amplitudes on the same Calabi Yau K

Is it possible a generalization?

The Nekrasov partition function can be defined as a 5d index:

$$Z(a,\epsilon_1,\epsilon_2) = Tr_{\mathcal{H}_a}(-1)^F e^{-\beta H} e^{-2\epsilon_- J_l^3} e^{-2\epsilon_+(J_r^3+J_I^3)}$$

It is a regularization by ϵ_1,ϵ_2 of $Z(a)$

- In the limit $\epsilon_1,\epsilon_2 \to 0$ $\qquad \log Z(a,\epsilon_1,\epsilon_2) \to -\dfrac{\mathcal{F}(a)}{\epsilon_1\epsilon_2}$

$\mathcal{F}(a)$ is called the prepotential

- $$\log Z(a,\epsilon_1,\epsilon_2) \to_{(\epsilon_1=-\epsilon_2=\lambda)} \to \sum_{g=0}^{\infty} \lambda^{2g-2} F^g(a)$$

$$F^g(a) \to F^g(a,\bar{a})$$

that are exactly the genus g topological string amplitudes on the same Calabi Yau that geometrically engineers the gauge theory

The dependence of $F^g(a, \bar{a})$ by $\bar{a}$
is fixed by the holomorphic anomaly equation

$$\bar{\partial}_{\bar{i}} F^g = \frac{1}{2} \bar{C}_{\bar{i}\bar{j}\bar{k}} g^{\bar{j}j} g^{\bar{k}k} \left(D_j D_k F^{g-1} + \sum_{s=1}^{g-1} D_j F^s D_k F^{g-s} \right)$$

$$\bar{\partial}_{\bar{i}} F^g = \int_{\mathcal{M}_g} \langle \prod_{a=1}^{3g-3} \int G^- \mu_a \int \bar{G}^- \bar{\mu}_{\bar{a}} \int_{\Sigma_g} dz^2 \oint_{C_z} G^+ \oint_{C'_z} \bar{G}^+ \bar{\mathcal{O}}_{\bar{i}}(z, \bar{z}) \rangle_{\Sigma_g} =$$

$$= \int_{\mathcal{M}_g} \sum_{b,\bar{b}=1}^{3g-3} \partial_b \bar{\partial}_{\bar{b}} \langle \prod_{a \neq b} \int G^- \mu_a \int \bar{G}^- \bar{\mu}_{\bar{a}} \int_{\Sigma_g} \bar{\mathcal{O}}_{\bar{i}} \rangle_{\Sigma_g}$$

Thus there are contributions only from the boundary of the moduli space of the Riemann surface

It can be shown by direct computations that the Nekrasov partition function obeys a refined version of the holomorphic anomaly equation

$$\log Z = \sum_{g=0}^{\infty} \frac{\epsilon_-^{2g}\epsilon_+^{2n}}{-\epsilon_1\epsilon_2} F^{g,n}(a,\bar{a}) \qquad \epsilon_\pm = \frac{\epsilon_1 \pm \epsilon_2}{2}$$

$$\bar{\partial}_{\bar{i}} F^{g,n} =$$

$$= \frac{1}{2}\bar{C}_{\bar{i}\bar{j}\bar{k}} g^{\bar{j}j} g^{\bar{k}k} \left(D_j D_k F^{g-1,n} - D_j D_k F^{g,n-1} + \sum_{(s,r)\neq(0,0)\neq(g,n)} D_j F^{s,r} D_k F^{g-s,n-r} \right)$$

What is the reasonable most general worldsheet ansatz for that generates this equation?

- Correct definition of the worldsheet amplitude

$$\log Z = \sum_{g=0}^{\infty} \frac{\epsilon_-^{2g}(\epsilon_+/i)^{2n}}{-\epsilon_1\epsilon_2} \left[(i)^{2n} F^{g,n}(a,\bar{a})\right]$$

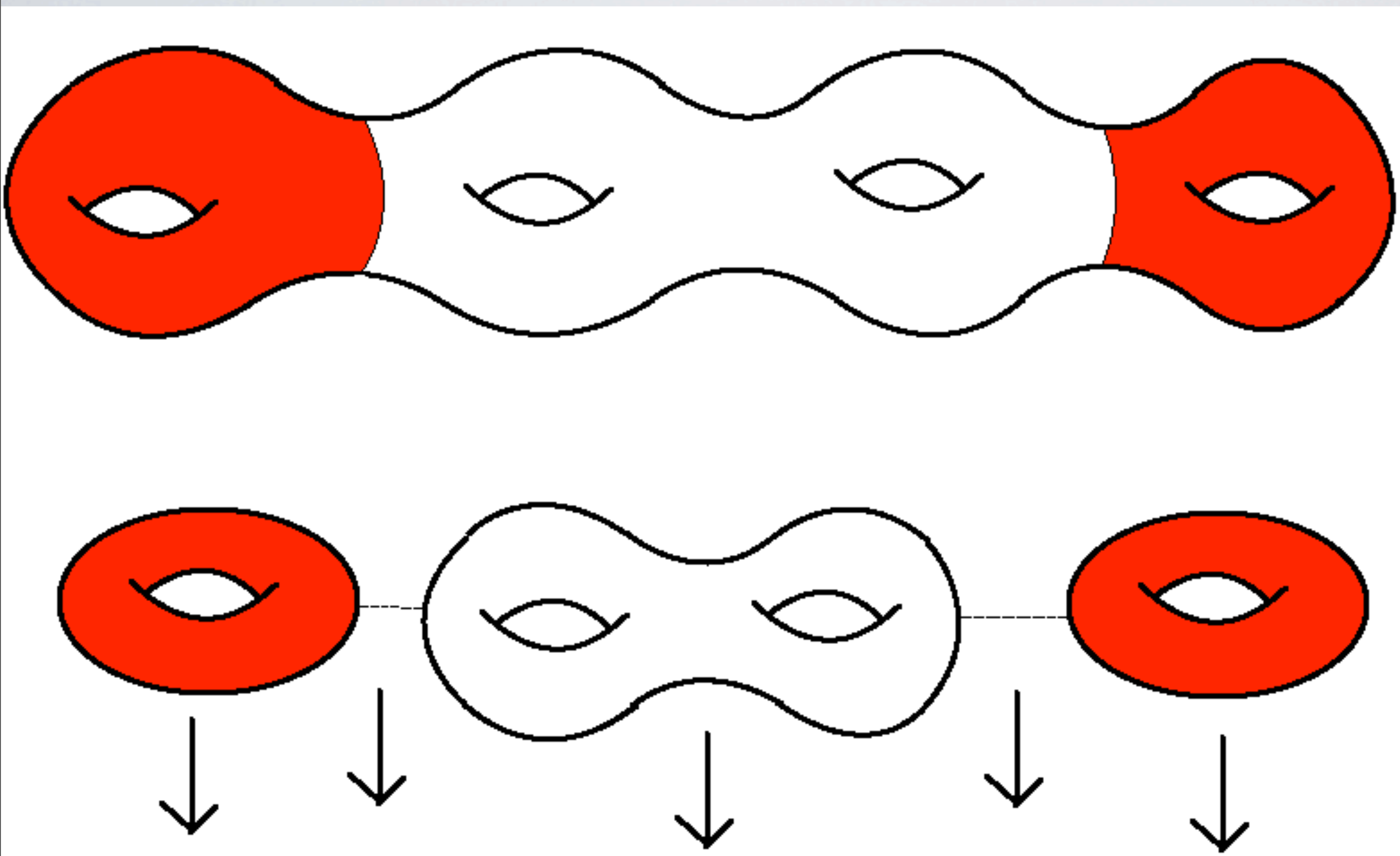
- We consider a Riemann surface of genus g+n and divide it into two types of domain and define:

$$(i)^{2n} F^{g,n} = \int D\phi \int_{\mathcal{M}_{g+n}} \prod_{a\bar{a}=1}^{3(g+n)-3} \int G^- \mu_a \int \bar{G}^- \bar{\mu}_{\bar{a}}\, e^{-\int_{\Sigma_{g+n}} \mathcal{L}_{top} - \sum_i \int_{\Sigma_{n_i}} \delta\mathcal{L}}$$

- Summing over all the domain splittings satisfying

$$\sum_i g_i = g \quad \sum_j n_j = n$$

The appearance of two coupling constants is understood from type II A point of view as turning on the vevs corresponding to ϵ_-, ϵ_+ generalizing the G-V relation $\lambda = g_s T_-$



$$\to \frac{(\epsilon_+/i)^4\epsilon_-^4}{-\epsilon_1\epsilon_2}$$

$$\frac{(\epsilon_+/i)^{2+1}}{-\epsilon_1\epsilon_2} \times \frac{-\epsilon_1\epsilon_2}{(\epsilon_+/i)\epsilon_-} \times \frac{\epsilon_-^{4+2}}{-\epsilon_1\epsilon_2} \times \frac{-\epsilon_1\epsilon_2}{(\epsilon_+/i)\epsilon_-} \times \frac{(\epsilon_+/i)^{2+1}}{-\epsilon_1\epsilon_2} = \frac{(\epsilon_+/i)^4\epsilon_-^4}{-\epsilon_1\epsilon_2}$$

$$\log Z(\epsilon_1 = -\epsilon_2 = \lambda) = \sum_{g=0}^{\infty} \lambda^{2g-2} F^g$$

Consider an F term in the type II 4d effective action like:

$$F_g(X) W^{2g}$$

where

$$X^I = X^I + \frac{1}{2} F_{\lambda\rho} \epsilon_{ij} \theta^i \sigma^{\lambda\rho} \theta^j + \dots$$

$$W^{ij}_{\mu\nu} = T^{ij}_{\mu\nu} - R_{\mu\nu\lambda\rho} \theta^i \sigma^{\lambda\rho} \theta^j + \dots$$

It is the superfield completion of the effective action

$$g F_g (T_-^{2g-2} R_-^2 + 2(g-1)(R_- T_-)^2 T_-^{2g-4})$$

whose coefficient is given by a genus g string amplitude

(Type II) $$\langle T_-^{2g-2} R_-^2 \rangle_{\Sigma_g} = (g!)^2 F_g$$ (Topological)

$$\log Z = \sum_{g=0}^{\infty} \frac{\epsilon_-^{2g}(\epsilon_+/i)^{2n}}{-\epsilon_1\epsilon_2}\left[(i)^{2n}F^{g,n}\right]$$

And we consider an F term in the heterotic 4d effective theory

$$\tilde{F}_{g,n}(X)[\Pi f(X\bar{X})]^n W^{2g}$$

which is the superfield completion of

$$\tilde{F}_{g,n}F_{dil}^{2n}g(T_-^{2g-2}R_-^2 + 2(g-1)(R_-T_-)^2T_-^{2g-4})$$

In the large dilaton limit they satisfy an holomorphic anomaly equation that leads to the identification

$$\tilde{F}_{g,n} = (i)^{2n}F^{g,n}$$

Linear approximation description:

$$Z(a, \epsilon_1, \epsilon_2) = Tr_{\mathcal{H}_a}(-1)^F e^{-\beta H} e^{-2\epsilon_- J_l^3} e^{-2\epsilon_+ (J_r^3 + J_I^3)}$$

- $\epsilon_+ \to 0$ $\qquad Z(a, \epsilon_-) = Tr_{\mathcal{H}_a}(-1)^F e^{-\beta H} e^{-2\epsilon_- J_l^3}$

It is equivalent to a vacuum to vacuum amplitude on the background

$$ds^2 = (dx^\mu + \Omega^\mu d\theta)^2 + d\theta^2$$

$$\downarrow (\beta \to 0)$$

$$d\Omega = T = \epsilon_1 dx^1 \wedge dx^2 + \epsilon_2 dx^3 \wedge dx^4$$

$$ds^2 = dx^2 + (d\theta + \Omega_\mu dx^\mu)^2$$

- In general

$$ds^2 = (dx^\mu + \Omega^\mu d\theta)^2 + g_{i\bar{j}}(dy^i + v^i d\theta)(dy^{\bar{j}} + v^{\bar{j}} d\theta) + d\theta^2$$

$$y^I \to y^I + v^I(y, \epsilon_+)$$

$$\downarrow (\beta \to 0)$$

$$ds^2 = dx^2 + dy^2 + (d\theta + \Omega_\mu dx^\mu + v_I dy^I)^2$$

Future developments

- Derive the Lagrangian deformation from one loop exact expressions

- Reproduce the behaviour around certain points in moduli space ($c = 1$ string)

- More difficult checks from higher genus amplitudes, in particular regarding the boundary conditions (conformal interface)

Thank you