Conformal Field Theory Associated to C_2-cofinite Vertex Operator Algebras (joint with Professor Akihiro Tsuchiya)

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References

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- Matsuo and Nagatomo, MSJ Memoirs vol.4, 1999
- Nagatomo and Tsuchiya, math/0206223, Duke Math.
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Contents

1. Flat structure on a disc

$$\mathbb{C}[[z]] = \left\{ \sum_{n=0}^{\infty} a_n z^n \mid a_n \in \mathbb{C} \right\} \qquad \text{local ring}$$

$$\mathbb{C}((z)) = \left\{ \sum_{n=-N}^{\infty} a_n z^n \mid a_n \in \mathbb{C},\ N = 0, 1, 2, \ldots \right\} \qquad \text{field}$$

$D_p \underset{\text{def}}{=} \operatorname{Spec} \mathbb{C}[[z]]$ disc centered at $p \in D$

$D_p^* \underset{\text{def}}{=} \operatorname{Spec} \mathbb{C}((z))$ punctured disc

Symmetry

$$K \underset{\text{def}}{=} \text{Aut}\, D_p \ni \varphi$$

$$\begin{aligned} \varphi^* : \ \mathbb{C}[[z]] &\longrightarrow \mathbb{C}[[z]] \\ z &\longmapsto a_1(\varphi)z + a_2(\varphi)z^2 + a_3(\varphi)z^3 + \cdots \\ & \qquad (a_1(\varphi) \neq 0) \end{aligned}$$

$$K \cong z\mathbb{C}[[z]] \setminus z^2\mathbb{C}[[z]]$$

$$\text{Lie}(K) = \mathbb{C}[[z]] \cdot z\partial_z \qquad (\partial_z = \frac{\partial}{\partial z})$$

Fact

$$K = \mathsf{Aut}\ D_p = \mathsf{Aut}\ D_p^* = \mathsf{Gal}(\mathbb{C}((z))/\mathbb{C})$$

$D_p = \mathsf{Spec}\,\mathbb{C}[[z]]$ has only one geometric point p

$\mathcal{O}_{D_p} = \mathbb{C}[[z]]$ local ring

Vector fields

$$\Theta_{D_p} = \mathbb{C}[[z]] \cdot \partial_z = \underline{\mathbb{C} \cdot \partial_z} \oplus \mathrm{Lie}(K)$$

$$\Theta_{D_p^*} = \mathbb{C}((z)) \cdot \partial_z = \underline{\left(\bigoplus_{n=0}^{\infty} \mathbb{C} \cdot z^{-n} \partial_z \right)} \oplus \mathrm{Lie}(K)$$

Hidden symmetry

Flat structure on the disc D_p

For $\varphi \in K$ with

$$\varphi^* z = f_\varphi(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots \qquad (a_1 \neq 0)$$

there is a canonical deformation φ_w of φ such that

$$\begin{aligned} {\varphi_w}^* z &= f_\varphi(w+z) - f_\varphi(w) \\ &= \sum_{n=1}^{\infty} a_n((w+z)^n - w^n) \\ &= \sum_{n=1}^{\infty} \left(\sum_{k=0}^{\infty} \binom{n+k}{n} a_{n+k} w^k \right) z^n \end{aligned}$$

$\underline{K \text{ connection on a smooth curve}}$

C smooth curve over $\mathbb{C}$

$\mathcal{O}_{C,\,p} \cong \mathbb{C}[[z]]$ completed local ring at $p \in C$

$\text{Coord}_p(C) \underset{\text{def}}{=} \left\{\mathcal{O}_{C,\,p} \cong \mathbb{C}[[z]]\right\}$ free, transitive K-action

$\text{Coord}(C) \underset{\text{def}}{=} \bigcup_{p \in C} \text{Coord}_p(C) \to C$ principal K-bundle

There is a canonical (flat) connection on it induced by the flat structure of D_p.

Virasoro algebra

Central extension of a Lie algebra

$$0 \to \mathbb{C} \to \mathrm{Vir}(c)^{\mathrm{f}} \to \mathbb{C}[z,\, z^{-1}]\partial_z \to 0$$

$$[L_n,\, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n,\,-m}c$$

$$L_n \longmapsto -z^{n+1}\partial_z$$

$c \in \mathbb{C}$ central charge

$$0 \to \mathbb{C} \to \mathrm{Vir}(c) \to \mathbb{C}((z))\partial_z \to 0$$

$\mathfrak{g} = \mathrm{Vir}(c)$ $K = \mathrm{Aut}\ D_p$

Harish-Chandra pairs

The **Harish-Chandra pair** $(\mathfrak{g}, K)$ is a pair of a complex Lie group K and a K-equivariant Lie algebra $\mathfrak{g}$ with a K-equivariant embedding $\mathrm{Lie}\, K \subset \mathfrak{g}$.

A $(\mathfrak{g}, K)$-**space** $P \to X$ is a principal K-bundle with a compatible infinitesimally transitive $\mathfrak{g}$-action on P.

A $(\mathfrak{g}, K)$-**module** is a $\mathbb{C}$ vector space with compatible actions of $\mathfrak{g}$, K.

localization functor $(\mathfrak{g}, K)$-mod $\to \mathcal{D}_X$-mod

2. Vertex operator algebra

(a) Locality of fields

(b) Definition of vertex algebra

(c) Current Lie algebras

2.(a) Locality of fields

Notations

$M \quad \mathbb{C}$-vector space

$M((z)) = M[[z]] \oplus z^{-1}M[z^{-1}]$

Fields

$$A(z) : M \to M((z)) \quad \textbf{field} \text{ on } \operatorname{Spec} \mathbb{C}[[z]]$$

$$A(z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-1}, \qquad A_n : M \to M$$

$$A_n u = 0 \quad \text{for any} \quad n > n_0(u)$$

In general, $\dim M = \infty$, $A(z) \notin \operatorname{End}(M)((z))$

Equation of motion

$$T : M \to M$$

$$\partial_z A(z) = [T, A(z)] \quad \text{equation of motion}$$

- Coordinate-free style $\to$ Language of D-modules

Fourier modes

$$\partial_z \sum A_{n-1} z^{-n} = [T, \sum A_n \, z^{-n-1}]$$

$$-nA_{n-1} = [T, \, A_n]$$

Essentially,

$$\begin{array}{ccccccccc} A_0 & \leftarrow & A_1 & \leftarrow & A_2 & \leftarrow & \cdots & \text{Lie} \\ A_{-1} & \rightarrow & A_{-2} & \rightarrow & A_{-3} & \rightarrow & \cdots & \text{Comm} \end{array}$$

- Poisson algebras

Composition of fields

For fields $A(z) = \sum A_n z^{-n-1}$, $B(z) = \sum B_n z^{-n-1}$,

$$A(z)B(z)u = \sum_k \Big(\sum_{n \le n_0} A_{k-n-1} B_n u \Big) z^{-k-1}$$

is not well-defined.

Regularization

$$A(z)B(w) : M \to M((z))((w))$$

$$A(z)B(w)u = \sum_{n \leq n_0} \sum_{m \leq m_0(n)} A_m B_n u \cdot z^{-m-1} w^{-n-1}$$

- limit $\quad z - w \to 0$

Remark

$$M((z))((w)) = \{ \sum_{n \le n_0} \sum_{m \le m_0(n)} u_{m,n} z^{-m-1} w^{-n-1} \}$$

$$M((z, w)) := \{ \sum_{n \le n_0} \sum_{m \le m_0} u_{m,n} z^{-m-1} w^{-n-1} \} = M[[z, w]][(zw)^{-1}]$$

$$M((z, w)) = M((z))((w)) \cap M((w))((z)) \subset M[[z^{\pm 1}, w^{\pm 1}]]$$

Locality

$$A(z),\, B(w) \quad \textbf{mutually local}$$

$$\underset{\text{def}}{\Longleftrightarrow} \quad \exists N \in \mathbb{Z}_{\geq 0}, \quad (z-w)^N [A(z),\, B(w)] = 0$$

- Fundamental notion of vertex algebras and CFT

Delta function

$$\delta(z-w) := \sum_{n\in\mathbb{Z}} z^n w^{-n-1} \in \mathbb{C}[[z^{\pm 1}, w^{\pm 1}]]$$

$$(z-w)\cdot\delta(z-w) = 0$$

$$(z-w)^{k+1}{\partial_w}^k \delta(z-w) = 0$$

Fock space (Kashiwara's category equivalence)

$$[z-w, \partial_w] = 1, \quad z-w : \text{annihilation}, \quad \partial_w : \text{creation}$$

$$|0\rangle = \delta(z-w), \qquad (z-w)|0\rangle = 0$$

Binomial expansion

For $n \in \mathbb{Z}$

$$[(z+w)^n]_{|z|<|w|} = \sum_{i \geq 0} \binom{n}{i} z^i w^{n-i} \in \mathbb{C}((w))((z)).$$

$$\delta(z-w) = \left[\frac{1}{z-w}\right]_{|w|<|z|} - \left[\frac{1}{z-w}\right]_{|z|<|w|}$$

Product

If $(z-w)^N[A(z),\, B(w)] = 0$,

$$(z-w)^N A(z)B(w)u = (z-w)^N B(w)A(z)u \in M((z,\, w)).$$

$\Rightarrow \exists A(z) \circ B(w) \in \mathrm{Hom}(M,\, M((z,\, w)))[(z-w)^{-1}]$ s. t.

$$[A(z) \circ B(w)]_{|z|>|w|} = A(z)B(w)$$
$$[A(z) \circ B(w)]_{|w|>|z|} = B(w)A(z)$$

Operator product expansion (**OPE**)

$$[A(z) \circ B(w)]_{|z-w|<|w|} = \sum_{n<N,\, m\in\mathbb{Z}} C_{m,\,n} w^{-m-1}(z-w)^{-n-1}$$
$$\in \mathrm{Hom}(M,\ M((w)))((z-w))$$

Borcherds' identity

Put $z = tw$ (blowing up at $z = w = 0$)

$$A(tw) \circ B(w) \in \mathrm{Hom}(M,\, M[t,\, t^{-1}]((w)))[(t-1)^{-1}]$$

The residue theorem for $A(tw) \circ B(w)\, t^p (t-1)^r dt$ implies

$$\sum_{i \geq 0} \binom{p}{i} C_{p+q-i,\, r+i}$$

$$= \sum_{j \geq 0} (-1)^j \binom{r}{j} (A_{p+r-j} B_{q+j} - (-1)^r B_{q+r-j} A_{p+j})$$

Definition of vertex algebra

Vertex algebra $\quad (V,\, Y_z,\, T,\, |0\rangle)$

V $\mathbb{C}$-vector space; $T : V \to V$; vacuum $|0\rangle \in V$

$$Y_z : V \otimes V \to V((z)), \quad Y_z(a \otimes b) = a(z)b = \sum_n a_{(n)}b \cdot z^{-n-1}$$

1. $a(z),\, b(z)$ mutually local fields

2. $\partial_z a(z) = [T,\, a(z)]$

3. $a(z)|0\rangle - a \in zV[[z]]$, $T|0\rangle = 0$

Example: commutative vertex algebra

- $V = \mathbb{C}[[t]], \quad T = \partial_t, \quad |0\rangle = 1, \quad Y_z(f(t) \otimes g(t)) = f(t+z)g(t)$
- $V = \mathbb{C}((t)), \quad T = \partial_t, \quad |0\rangle = 1, \quad Y_z(f(t) \otimes g(t)) = [f(t+z)g(t)]_{|z|<|t|}$

Example

Heisenberg Lie algebra

$$0 \to \mathbb{C}c \to \mathcal{H} \to \mathbb{C}((t)) \to 0, \quad [f, g] = \mathrm{Res}_{t=0}[f dg]c \quad (f, g \in \mathbb{C}((t)))$$

$$\widetilde{\mathcal{H}} = U(\mathcal{H})_{c=1}, \quad \pi = \widetilde{\mathcal{H}} \otimes_{\widetilde{\mathcal{H}}_+} \mathbb{C}$$

Examples

- Tensor product
- Enveloping algebra of a vertex Lie algebra
- Construction by screening operators

Proposition

1. $a = a(z)|0\rangle|_{z=0} = a_{(-1)}|0\rangle$. $\quad a \mapsto a(z)$ is injective.

2. $Ta = Ta(z)|0\rangle|_{z=0} = \partial_z a(z)|0\rangle|_{z=0} = a_{(-2)}|0\rangle$. $\quad T$ is unique.

3. $a(z)|0\rangle = e^{zT}a$

4. $[a(z-w)]_{|w|<|z|} = e^{-wT}a(z)e^{wT}$

Skew-symmetry

1. $a(z) \circ b(w)|0\rangle = e^{wT} a(z-w) b = e^{zT} b(w-z) a$

2. $a(z)b = e^{zT} b(-z) a,$

$$a_{(n)}b = \sum_{k \geq 0} (-1)^{n+k+1} \frac{T^k}{k!} b_{(n+k)} a,$$

$$a_{(n)}b + (-1)^n b_{(n)} a \in TV$$

Proof

$$[a(z) \circ b(w)|0\rangle]_{|w|<|z|} = \left[e^{wT} a(z-w) b \right]_{|w|<|z|}$$

Corollary

1. $Y_z(|0\rangle \otimes a) = a$. $|0\rangle$ is unique.

2. $(Ta)(z) = \partial_z a(z)$. In other words, $(Ta)_{(n)} = -na_{(n-1)}$.

Associativity

$$[a(z) \circ b(w)]_{|z-w|<|w|} = (a(z-w)b)(w)$$
$$= \sum_{n \in \mathbb{Z}} (a_{(n)}b)(w) \cdot (z-w)^{-n-1}$$

Proof

$$[(a(z) \circ b(w))c]_{|w|<|z|} = \left[e^{wT}(a(z-w) \circ c(-w))b\right]_{|w|<|z|}$$
$$\therefore \quad (a(z) \circ b(w))c = e^{wT}(a(z-w) \circ c(-w))b$$
$$\therefore \quad [(a(z) \circ b(w))c]_{|z-w|<|w|} = e^{wT}c(-w)a(z-w)b$$
$$= (a(z-w)b)(w)c$$

Borcherds identity

1. $\sum_{i \geq 0} \binom{p}{i} (a_{(r+i)} b)_{(p+q-i)}$
$= \sum_{j \geq 0} (-1)^j \binom{r}{j} (a_{(p+r-j)} b_{(q+j)} - (-1)^r b_{(q+r-j)} a_{(p+j)})$

2. $[a_{(p)}, b_{(q)}] = \sum_{j \geq 0} \binom{p}{j} (a_{(j)} b)_{(p+q-j)}$

3. $[a_{(0)}, b_{(0)}] = (a_{(0)} b)_{(0)}$ Jacobi identity

Remark

- vertex algebra
  = commutative algebra on $z - w \neq 0$, Lie algebra on $z - w = 0$

- If a binary operation satisfies $a(bc) = b(ac)$, $a1 = a$,

$$\Rightarrow \quad \begin{aligned} ba &= b(a1) = a(b1) = ab \\ a(bc) &= a(cb) = c(ab) = (ab)c. \end{aligned}$$

Virasoro algebra

$$0 \to \mathbb{C}c \to \mathsf{Vir}(c) \to \mathbb{C}((t))\partial_t \to 0$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3 - n}{12}\delta_{n,-m}c$$

$$L_n \longmapsto -t^{n+1}\partial_t$$

V-modules

V vertex algebra

$$Y_z^M : V \otimes M \to M((z)), \quad Y_z^M(a \otimes u) = a^M(z)u = \sum (a_{(n)}^M u) z^{-n-1}$$

- $|0\rangle^M(z) = \mathrm{id}_M$
- $a^M(z),\ b^M(z)$ mutually local for any $a, b \in V$
- Associativity $\left[a^M(z) \circ b^M(w)\right]_{|z-w|<|w|} = (a(z-w)b)^M(w)$

Vertex operator algebra

$$V = \bigoplus_{n \geq 0} V[n], \quad \dim V[n] < \infty, \quad V[0] = \mathbb{C}|0\rangle$$

$$\omega \in V[2] \quad \text{Virasoro element}, \quad \omega(z) = \sum L_n z^{-n-2}$$

$$L_{-1} = T, \quad L_0|V[n] = n$$

$$(\mathrm{Vir}(c),\ K) \curvearrowright V$$

3. The Lie algebra and the associative algebra associated to a VOA

$$V/TV \otimes V/TV \to V/TV, \quad [a] \otimes [b] \mapsto [\mathrm{Res}\, a(z) b\, dz] = [a_{(0)} b]$$

- well-definedness $\quad \partial_z a(z) = [T, a(z)] = (Ta)(z)$
- $[a_{(0)} b] = -[b_{(0)} a] \quad (\Leftarrow\ a(z)b = e^{zT} b(-z) a)$
- Jacobi identity $\quad [a_{(0)}, b_{(0)}] = (a_{(0)} b)_{(0)}$

Tensor product

V_1, V_2 vertex algebras $\Rightarrow$ $V_1 \otimes V_2$ vertex algebra

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$$Y_z^{12}((a_1 \otimes a_2) \otimes (b_1 \otimes b_2)) = Y_z^1(a_1 \otimes b_1) \otimes_{\mathbb{C}((z))} Y_z^2(a_2 \otimes b_2)$$

$$T_{12} = T_1 \otimes 1 + 1 \otimes T_2, \quad |0\rangle_{12} = |0\rangle_1 \otimes |0\rangle_2$$

Current Lie algebra

V vertex algebra $\Rightarrow$ $V((t)) = V \hat{\otimes} \mathbb{C}((t))$ vertex algebra

$$\mathrm{Lie}(V) = V((t))/(\partial_t + T)V((t))$$

$$[a\, f(t),\, b\, g(t)] = \mathrm{Res}_{z=0}[a(z)b \cdot f(t+z)g(t)\, dz]$$

$$[a\, t^p,\, b\, t^q] = \sum_{j \geq 0} \binom{p}{j} (a_{(j)}b)\, t^{p+q-j}$$

Anti-automorphism

For a VOA V, we have an anti-automorphism $\theta : \mathrm{Lie}(V) \to \mathrm{Lie}(V)$

$$\theta(J_n(a)) = J_{-n}(\mathrm{e}^{L_1}(-1)^{\Delta} a)$$

where

$$a \in V, \quad L_0 a = \Delta a, \quad J_n(a) = a_{(n+\Delta-1)}$$

Current algebra

$$\tilde{\mathcal{U}} = \frac{U(\mathrm{Lie}(V))}{[|0\rangle \otimes t^n] = \delta_{n,-1}}$$

$$\mathcal{U}(V) \sim \frac{\tilde{\mathcal{U}}}{\text{Borcherds rel.}}$$

The RHS of Borcherds relation has infinite terms.

Filtration

$$F_pV = \bigoplus_{n\le p} V[n]$$

$F_p\mathrm{Lie}(V)$, $\quad \deg a_{[n]} = \deg a - n + 1$

$F_p\widetilde{U}$

$I_n = \widetilde{U} \cdot F_{-n-1}\widetilde{U}$

$$F_p\mathcal{U}(V) \underset{\text{def}}{=} \varprojlim F_p(\widetilde{U}/I_n)$$

Vertex Lie algebra $(L, (T : L \to L), Y_z^-)$

$Y_z^- : L \otimes L \to L((z))/L[[z]] = z^{-1}L[z^{-1}]$

$$Y_z^-(a \otimes b) = a[z]b = \sum_{n \geq 0} (a_{[n]}b) z^{-n-1}$$

- $(Ta)[z] = \partial_z a[z]$
- $a[z]b = e^{zT} b[-z]a \quad \mathrm{mod}\ L[[z]]$
- $[a_{[k]}, b_{[n]}] = \sum_{j=0}^{k} \binom{k}{j} (a_{[j]}b)_{[n+k-j]}$. In other words,

$$\mathrm{Res}_{z=0}[[a[z], b[w]] \cdot f(z)dz] = \mathrm{Res}_{z=w}[a[z-w]b \cdot f(z)dz]$$

Current Lie algebra functor

$$\text{vertex algebras} \underset{\text{forget}}{\longrightarrow} \text{vertex Lie algebras} \underset{\text{Lie}}{\longrightarrow} \text{Lie algebras}$$

Enveloping vertex algebra

L vertex Lie algebra

$$\mathsf{Lie}(L) = L((t))/(\partial_t + T)L((t)), \quad a_{(n)} := a\,t^n$$

$$\mathsf{Lie}_+(L) = L[[t]]/(\partial_t + T)L[[t]]$$

$$\mathsf{Vac}(L) = U(\mathsf{Lie}(L)) \otimes_{U(\mathsf{Lie}_+(L))} \mathbb{C}$$

$$|0\rangle = 1 \otimes 1, \quad (a_{(-1)}|0\rangle)(z) = \sum a_{(n)} z^{-n-1}$$

4. C_2-cofiniteness

V vertex algebra $n \geq 2$

$$C_n(V) := \mathrm{span}\{a_{(-n)}b \mid a,\, b \in V\}$$

$$V \quad C_n\text{-finite} \quad \underset{\text{def}}{\Longleftrightarrow} \quad \dim V/C_n(V) < \infty$$

M V-module

$$C_n(M) := \mathrm{span}\{a_{(-n)}u \mid a \in V,\ u \in M\}$$

$$M \quad C_n\text{-cofinite} \quad \underset{\text{def}}{\Longleftrightarrow} \quad \dim M/C_n(M) < \infty$$

Remark

$$(Ta)_{(-n)} = na_{(-n-1)} \quad \Rightarrow \quad C_n(M) \supset C_{n+1}(M)$$

$$|0\rangle_{(-1)} = \mathrm{id} \quad \Rightarrow \quad C_1(M) = M$$

Fermionic property

$$a_{(-n)}(b_{(-n)}c),\ (a_{(-n)}b)_{(-n)}c \in C_{n+1}(V)$$

$$*_{(-n)}* : V/C_n(V) \otimes V/C_n(V) \to C_n(V)/C_{n+1}(V)$$

Corollary M C_2-cofinite $\Longleftrightarrow$ M C_n-cofinite

Zhu's theorem

$V/C_2(V)$ is a commutative Poisson algebra.

$$[a][b] := [a_{(-1)}b], \qquad \{[a], [b]\} := [a_{(0)}b]$$

Poisson filtrations

$(U, G_pU \subset G_{p+1}U)$: filtered algebra with

$$G_pU \cdot G_qU \subset G_{p+q}U, \quad [G_pU, G_qU] \subset G_{p+q-1}U$$

$\Rightarrow \quad \mathrm{gr}_G U$ is a Poisson algebra.

$\mathrm{Sym}(V/C_2(V)) \to \mathrm{gr}_G \mathcal{U}(V)$ surj

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- Symbol calculus
- Finiteness theorems $\quad \mathrm{Lie}(V) \subset \mathcal{U}(V)$ dense w. r. to $\mathcal{I}_n$

5. Coinvariants and conformal blocks

1. Broken symmetry of a formal punctured disk

2. Vertex operator algebras

3. Vertex algebra bundles

4. Conformal blocks

Broken symmetry of a formal punctured disk

A formal disk and a formal punctured disk

$$\mathcal{O} = \mathbb{C}[[z]], \quad \mathcal{K} = \mathbb{C}((z)) : \text{fraction field of } \mathcal{O}$$

$$\mathbb{C} \leftarrow \mathcal{O} \rightarrow \mathcal{K} \quad \sim \quad \mathbb{F}_p \leftarrow \mathbb{Z}_p \rightarrow \mathbb{Q}_p$$

Broken symmetry of a formal punctured disk

Aut $\mathcal{O}$ = Aut $\mathcal{K} = z\mathcal{O} \setminus z^2\mathcal{O} \curvearrowright \mathcal{O}, \mathcal{K}$ by

$$(z\mathcal{O} \setminus z^2\mathcal{O}) \times \mathcal{K} \to \mathcal{K}, \quad (f(z), g(z)) \mapsto f * g(z) = g(f(z))$$

$$\mathrm{Lie}(\mathrm{Aut}\ \mathcal{K}) = \mathrm{Der}_0\ \mathcal{O} := \mathcal{O}\, z\partial_z \subsetneq \mathrm{Der}\ \mathcal{O} = \mathcal{O}\partial_z \subsetneq \mathrm{Der}\ \mathcal{K} = \mathcal{K}\partial_z$$

Central extension

$$0 \to \mathbb{C} \to \mathrm{Vir}_c \to \mathrm{Der}\ \mathcal{K} \to 0$$

Harish-Chandra pairs

The **Harish-Chandra pair** $(\mathfrak{g}, K)$ is a pair of a complex Lie group K and a K-equivariant Lie algebra $\mathfrak{g}$ with a K-equivariant embedding $\operatorname{Lie} K \subset \mathfrak{g}$.

A $(\mathfrak{g}, K)$-**space** $P \to X$ is a principal K-bundle with a compatible infinitesimally transitive $\mathfrak{g}$-action on P.

A $(\mathfrak{g}, K)$-**module** is a $\mathbb{C}$ vector space with compatible actions of $\mathfrak{g}$, K.

localization functor $(\mathfrak{g}, K)\text{-mod} \to \mathcal{D}_X\text{-mod}$

Example: solution of the Kazhdan-Lusztig conjecture

G simple algebraic group $/\mathbb{C}$

$\mathfrak{g} = \mathrm{Lie}\, G$, $B \subset G$ Borel subgroup

$P = G \to X = G/B$

$$(\mathfrak{g}, B)\text{-mod} \to \mathcal{D}_X\text{-mod}$$

- Repr of Lie(G) without lift to repr of G
- ∞-dim. version?

Vertex algebra bundles

Curves with formal coordinates

$X \to S$ smooth (= submersion)

Coord$(X/S) \to X$ formal coordinates of fibers

$$\mathcal{V}_X := \text{Coord}(X/S) \times_K V \to X,$$
$$\mathcal{A}_X := \mathcal{V}_X \otimes \omega_{X/S} \quad \text{chiral algebra}$$

Conformal blocks

<u>V-modules</u>

V vertex algebra

$$Y_z^M : V \otimes M \to M((z)), \quad Y_z^M(a \otimes u) = a^M(z)u = \sum (a^M_{(n)}u) z^{-n-1}$$

- $|0\rangle^M(z) = \mathrm{id}_M$

- $a^M(z),\ b^M(z)$ mutually local for any $a, b \in V$

- Associativity $\left[a^M(z) \circ b^M(w)\right]_{|z-w|<|w|} = (a(z-w)b)^M(w)$

Modules over chiral algebras

$\pi : X \to S$ proper, smooth curves

$s : S \to X$ section

$\mathcal{M}_X = s_+(\mathcal{O}_S \otimes M)$ module over $\mathcal{A}_X$

de Rham functor

$$h : \mathcal{D}_X\text{-mod}^{\mathsf{right}} \to \mathbb{C}_X\text{-mod}, \quad h(\mathcal{M}) = \mathcal{M}/\mathcal{M} \cdot \Theta_X$$

$$DR = Lh \quad \text{left derived functor}$$

Resolutions

$$0 \to \mathcal{D}_X \otimes \bigwedge^{n} \Theta_X \to \cdots \to \mathcal{D}_X \otimes \bigwedge^{0} \Theta_X \to \mathcal{O}_X \to 0,$$

$$0 \to \mathcal{D}_X \otimes \Omega_X^0 \to \cdots \to \mathcal{D}_X \otimes \Omega_X^n \to \omega_X \to 0,$$

Coinvariants

$\pi : X \to S$ proper, smooth curve

$$R\pi \circ Lh(\mathcal{A}_X \boxtimes \mathcal{M}_X \to \mathcal{M}_X) \in D^b(\mathcal{D}_S\text{-mod})$$

- dual = conformal block

6. Factorization

1. Stable curves

2. Log D-modules

3. Factorization

Stable curves

- $\pi : X \to S$ proper curve with nodes $\Sigma \subset X$
- $\pi(\Sigma) = D$ normal crossing divisor
- discrete symmetry

Tower of stable curves

$$\begin{array}{ccccccccc}
\ldots & \to & X^{(3)} & \to & X^{(2)} & \to & X^{(1)} & = X \\
 & & \downarrow & \Box' & \downarrow & \Box' & \downarrow & \\
\ldots & \to & X^{(2)} & \to & X^{(1)} & \to & X^{(0)} & = S
\end{array}$$

- Deligne-Mumford stacks (= orbifold)

log D-modules

Algebras

- D-modules $[\partial, x] = 1$
  Fock space (Kashiwara's category equivalence)

- log D-modules $[x\partial, x] = x$

Coinvariants

$$R\pi \circ Lh(\mathcal{A}_X \boxtimes \mathcal{M}_X \to \mathcal{M}_X) \in D^b(\mathcal{D}_S(-\log D)\text{-mod})$$

Factorization

<u>Normalization of stable curves</u>

$\pi : (X, \Sigma) \to (S, D)$ stable curve

$D' \to D$ normalization

X' normalization of $\pi^{-1}(D) \times_D D'$

Factorization theorem

$$(R\pi \circ Lh(\mathcal{A}_X \boxtimes \mathcal{M}_X \to \mathcal{M}_X))|D'$$

$$\sim R\pi \circ Lh(\mathcal{A}_{X'} \boxtimes \mathcal{M}_{X'} \boxtimes \mathcal{T} \to \mathcal{M}_{X'} \boxtimes \mathcal{T})$$

Regular bimodule

$\mathcal{T} \subset \mathcal{U}(V)$: the sum of $\mathbb{C}[L_0\cdot, \cdot L_0]$-submodules of finite dim

dense in $(\mathcal{U}(V), \mathcal{I}_n)$