THE *Secret Life* OF SCATTERING AMPLITUDES

based on work with Lionel Mason and with Mat Bullimore

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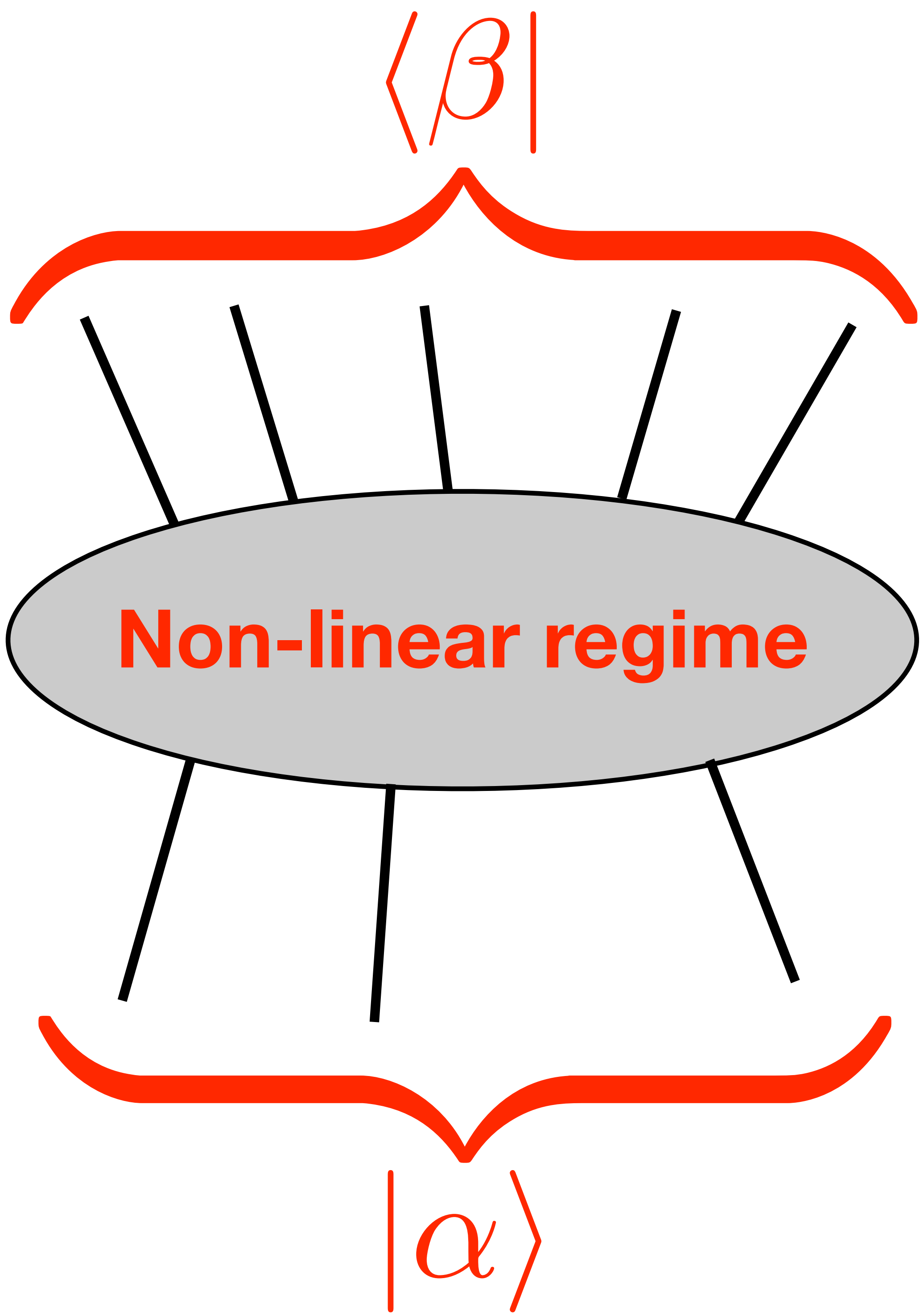
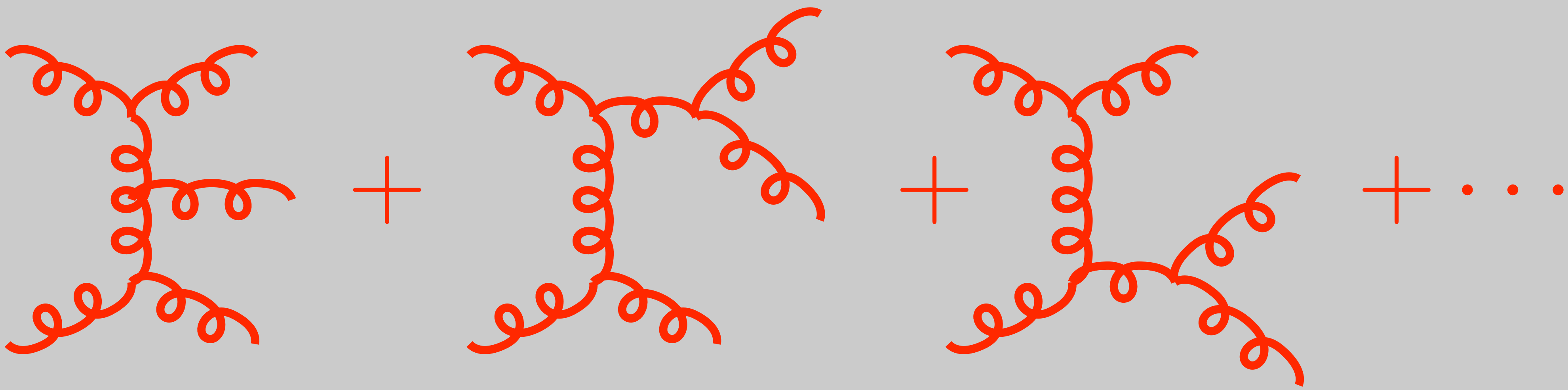
Scattering amplitudes are quantum mechanical overlap between states with prescribed asymptotic behaviour



- Point of contact between theorists and experimentalists
- Encode the dynamics of quantum systems
- Live at boundary of space-time - “holographic”

Perturbatively, scattering amplitudes are usually described in terms of Feynman diagrams

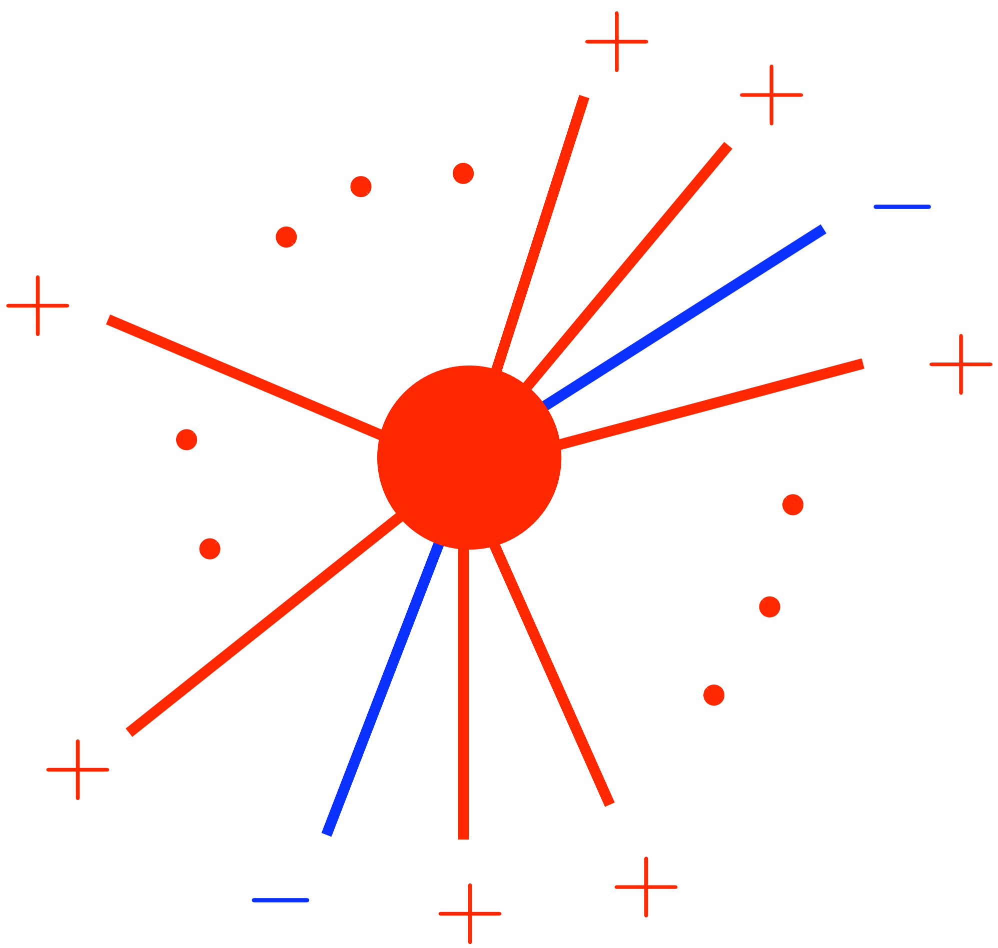
In Yang-Mills & gravity - theories we care most about - Feynman diagrams rapidly become very complicated



- Complexity directly attributable to ***gauge redundancy***
- Necessary consequence of bulk space-time description of massless spin 1 fields

Quite remarkably, the ***amplitude itself*** was sometimes found to remain very simple

[Parke,Taylor]



$$\mathcal{M}_n^{\mathrm{MHV}} = \frac{\langle ij\rangle^4\,\delta^4(\sum p_i)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$$

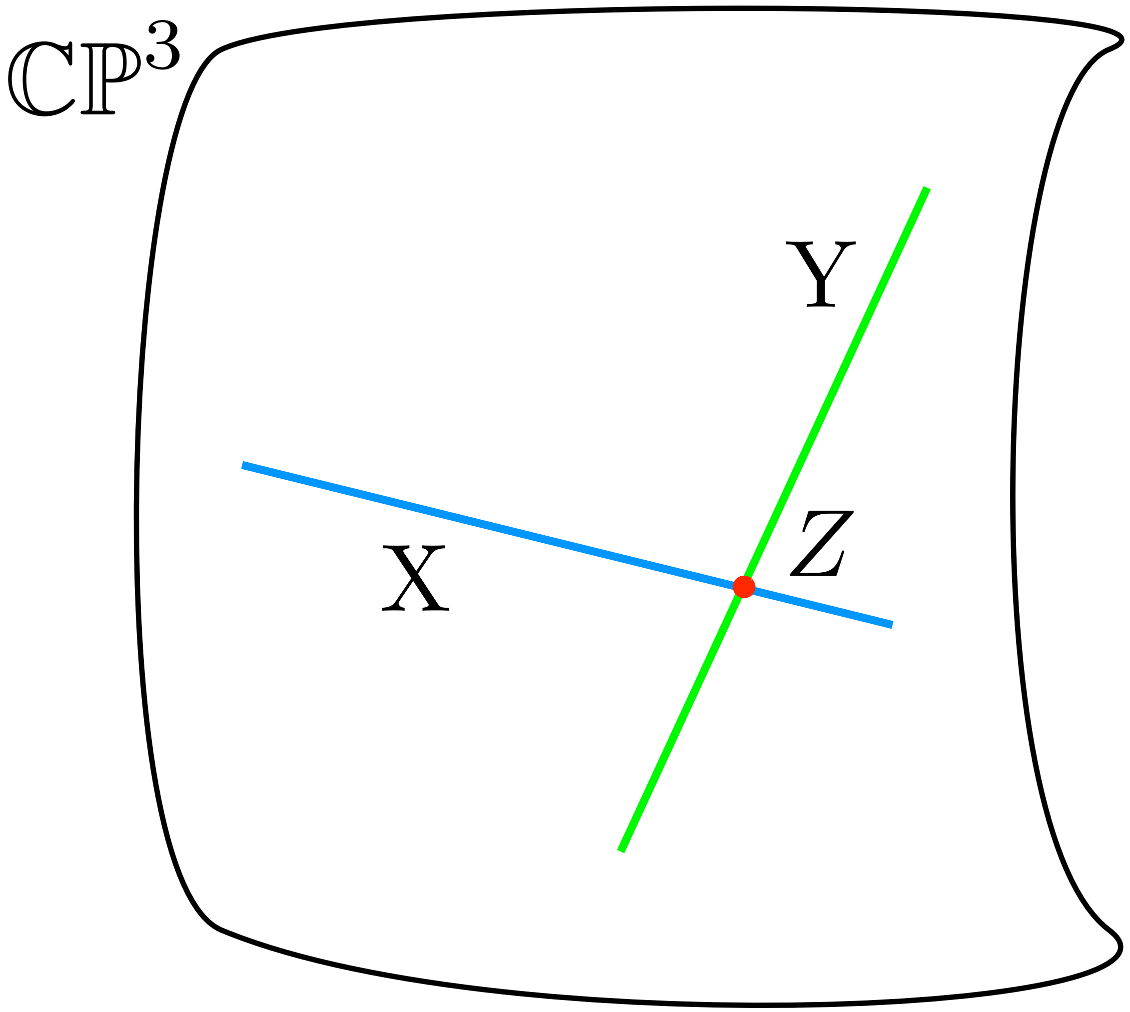
$$p^2 = 0 \;\Leftrightarrow\; p_{\alpha\dot\alpha} := p_\mu \sigma^\mu_{\alpha\dot\alpha} = \lambda_\alpha \bar\lambda_{\dot\alpha}$$

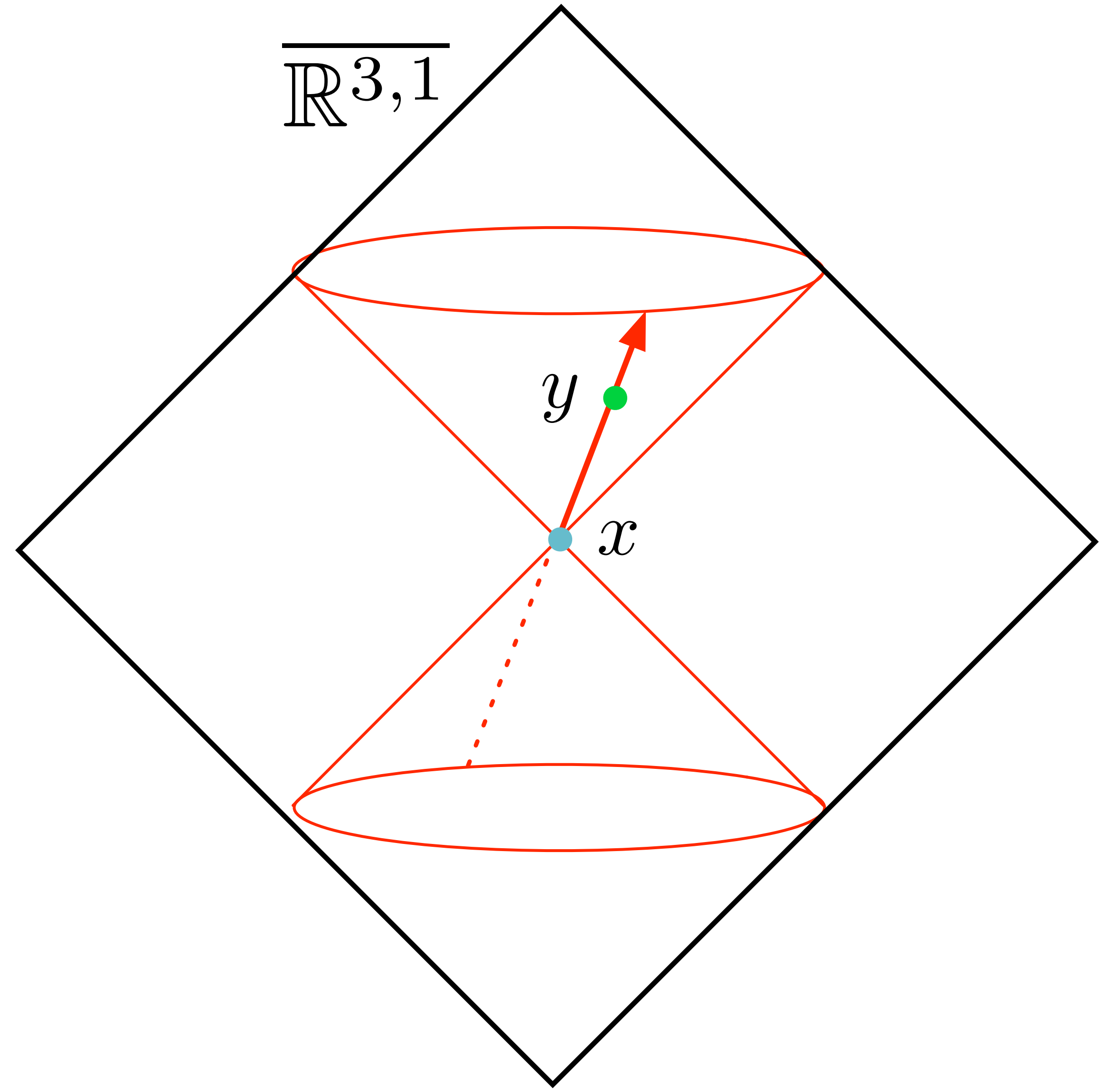
$$\langle ij\rangle := \epsilon^{\alpha\beta}\lambda_\alpha^{(i)}\lambda_\beta^{(j)}$$

Standard techniques obscure the true nature of scattering amplitudes

Twistors & MHV Diagrams

Twistor space is $\mathbb{CP}^3$, described by co-ords $Z^A \sim rZ^A$





$\mathbb{CP}^1$ in twistor space $\longleftrightarrow$ Point in space-time

Two lines intersect $\longleftrightarrow$ Separation is null

$\mathrm{X}^{AB} = Z_1^{[A} Z_2^{B]}$ $\qquad$ $\mathrm{X} \cdot \mathrm{X} = 0$ $\qquad$ $\mathrm{X} \sim r'\mathrm{X}$

One reason twistors provide a good way to describe scattering amplitudes is because they trivialize the external massless field equations

[Penrose]

$$\left\{\begin{array}{l}\text{Analytic sol}^{\text{n}}\text{ of wave}\\ \text{eq}^{\text{n}}\text{ for massless}\\ \text{free field, helicity } h\end{array}\right\} \cong \left\{\begin{array}{l}\textbf{\textit{Arbitrary}}\text{ holomor}^{\text{phc}}\\ \text{function of twistors,}\\ \text{homogeneity } 2h-2\end{array}\right\}$$

- Depends on three variables - field equations accounted for automatically

$$\Phi(x) = \oint \langle\lambda \mathrm{d}\lambda\rangle\, \phi(Z)|_{\mathrm{X}} \qquad Z^A\big|_{\mathrm{X}} = (\mu^{\dot\alpha}, \lambda_\alpha)\big|_{\mathrm{X}} = (x^{\beta\dot\alpha}\lambda_\beta, \lambda_\alpha)$$

$$\Box\Phi = \frac{\partial^2}{\partial x^\mu \partial x_\mu}\oint \langle\lambda \mathrm{d}\lambda\rangle\, \phi(Z)|_{\mathrm{X}} = \oint \langle\lambda \mathrm{d}\lambda\rangle\, \lambda^\alpha\lambda_\alpha\, \frac{\partial^2}{\partial\mu^{\dot\alpha}\partial\mu_{\dot\alpha}}\phi(Z)\Big|_{\mathrm{X}}$$

vanishes since antisymmetric

- Globally, $\phi \in H^1(\mathbb{CP}^3 - \mathrm{X}_\infty, \mathcal{O}(-2))$ on-shell

$$\left\{\begin{array}{c}\text{Self-dual sol}^{\text{ns}}\\ \text{of Yang-Mills eq}^{\text{ns}}\end{array}\right\} \xleftrightarrow{\text{[Penrose; Ward]}} \left\{\begin{array}{c}\text{Holomorphic vector}\\ \text{bun}^{\text{dls}}\text{ on twistor space}\end{array}\right\}$$

- Basis of ADHM construction of instantons; purely algebraic
- Related to many integrable systems by choices of symmetry reduction
- Similar construction for s.d. gravity - HK / QK manifolds

Just as holomorphic functions arise as field equations of $S = \int \mathrm{D}^3 Z \wedge \phi \wedge \bar{\partial}\phi$, so too holomorphic bundles arise as field equations of holomorphic Chern-Simons theory

$$S = \int \Omega \wedge \mathrm{Tr}\left(\mathcal{A} \wedge \bar{\partial}\mathcal{A} + \frac{2}{3}\mathcal{A}^3\right)$$

[Witten]

- $\Omega := \epsilon_{ABCD} Z^A \mathrm{d}Z^B \wedge \mathrm{d}Z^C \wedge \mathrm{d}Z^D\, \mathrm{d}^4\chi$ is top hol. form on $\mathbb{CP}^{3|4}$

$$\mathcal{A}(Z,\chi) = a(Z) + \chi^a \tilde{\gamma}_a(Z) + \frac{1}{2}\chi^a\chi^b\phi_{ab}(Z) + \frac{\epsilon_{abcd}}{3!}\chi^a\chi^b\chi^c\gamma^d(Z) + \frac{\epsilon_{abcd}}{4!}\chi^a\chi^b\chi^c\chi^d g(Z)$$

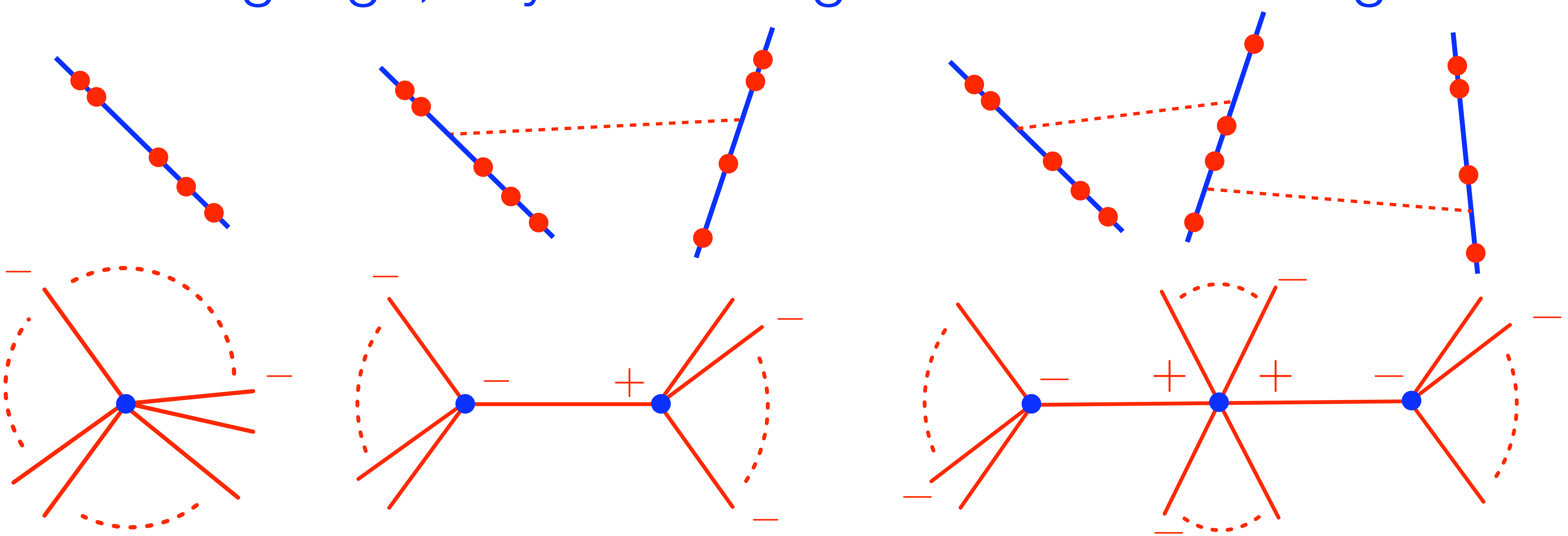
$\mathcal{N} = 4$ multiplet in twistor space [Ferber]

$\mathcal{N}=4$ SYM is described on twistor space by the action

$$S=\frac{1}{2}\int\Omega\wedge\mathrm{Tr}\left(\mathcal{A}\bar\partial\mathcal{A}+\frac{2}{3}\mathcal{A}^3\right)+\mathrm{g}^2\int\mathrm{d}^{4|8}x\,\log\det\left(\bar\partial+\mathcal{A}\right)_{\mathrm{X}}$$

[Witten; Nair; Boels,Mason,DS]

- In axial gauge, Feynman diagrams are MHV diagrams

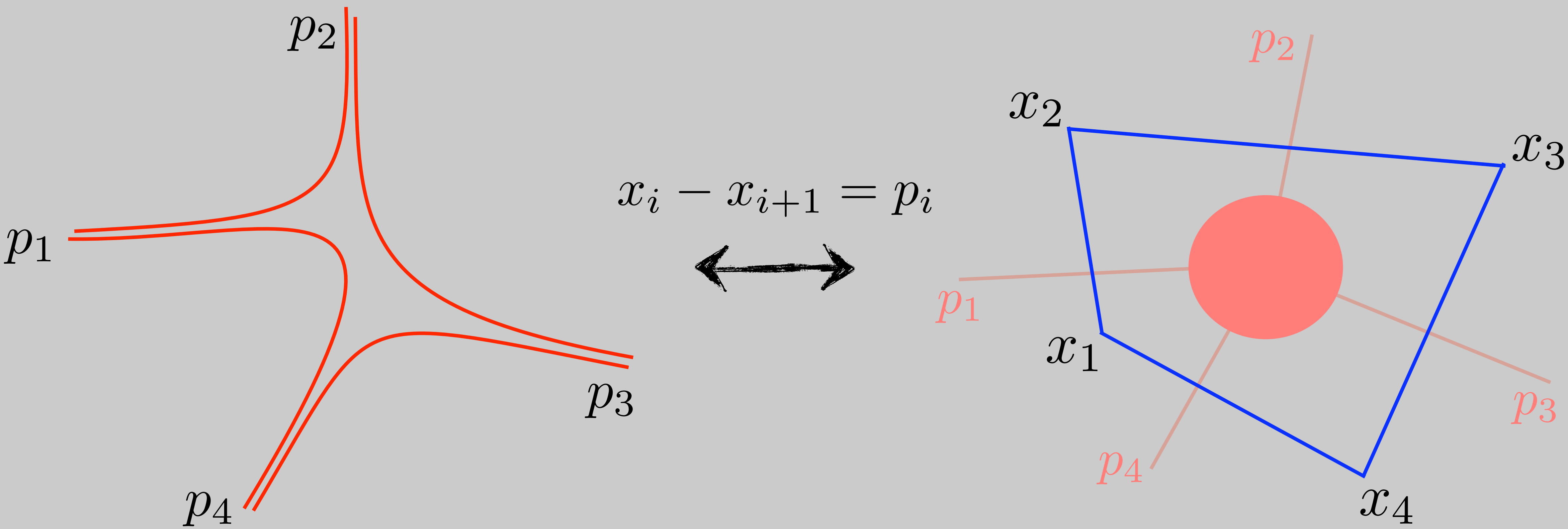


- Reduces to standard form if $\mathcal{A}$ is harmonic on each X
- Similar construction for gravity[Mason, DS]

The Amplitude / Wilson Loop Duality

MHV Amplitudes are Null Polygonal Wilson Loops

[Alday,Maldacena]



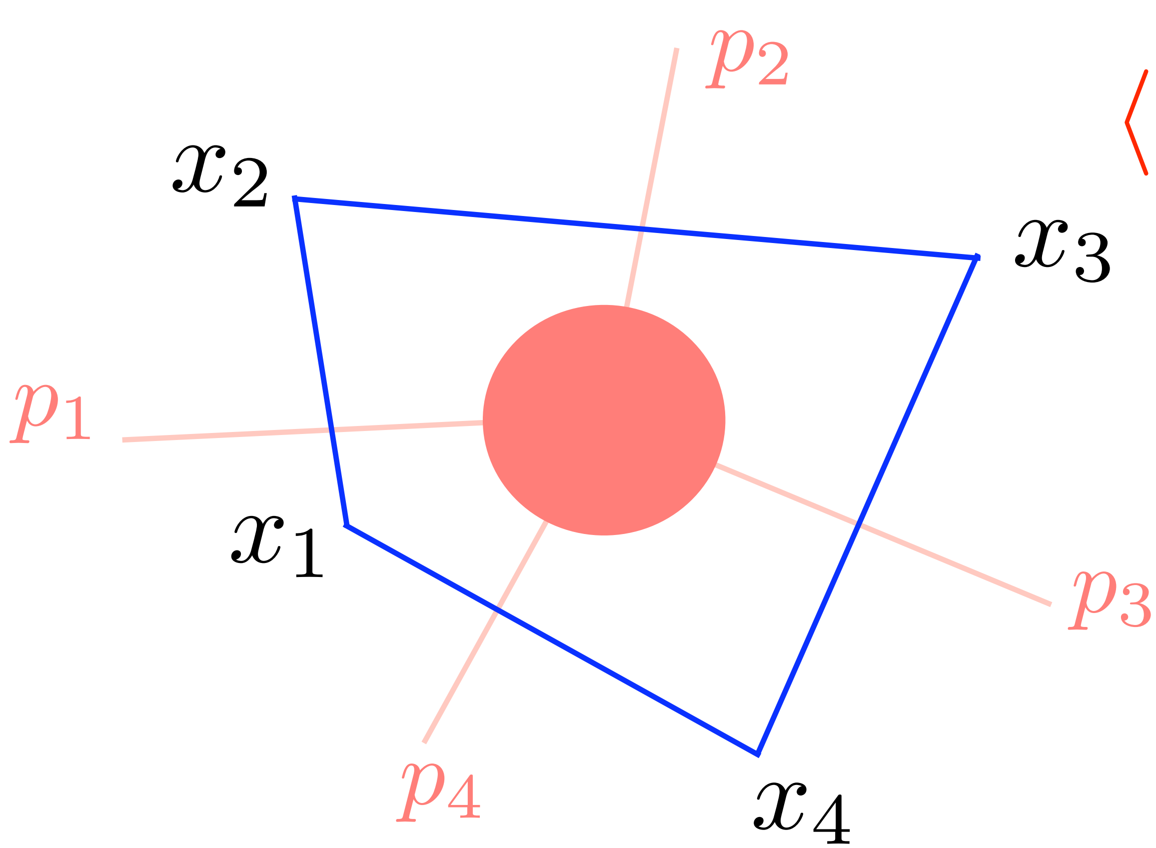
momentum conserved ⟷ closed polygon

massless particles ⟷ null edges

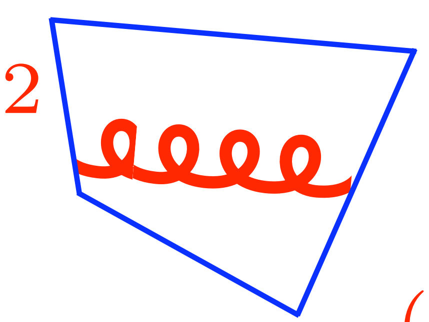
- Amplitude given by area of minimal surface in AdS_5
- For $n = 4, 5$ agrees with expectation from BDS ansatz

The duality between scattering amplitudes and Wilson Loops was also found to hold at weak coupling

[Drummond,Henn,Korchemsky,Sokatchev;Brandhuber,Heslop,Travaglini;
Bern,Dixon,Kosower,Roiban,Spradlin,Vergu,Volovich]



$$\langle \mathrm{W} \rangle = \frac{1}{N} \left\langle \mathrm{Tr\,P} \exp\left(-\mathrm{g} \oint A \right) \right\rangle$$

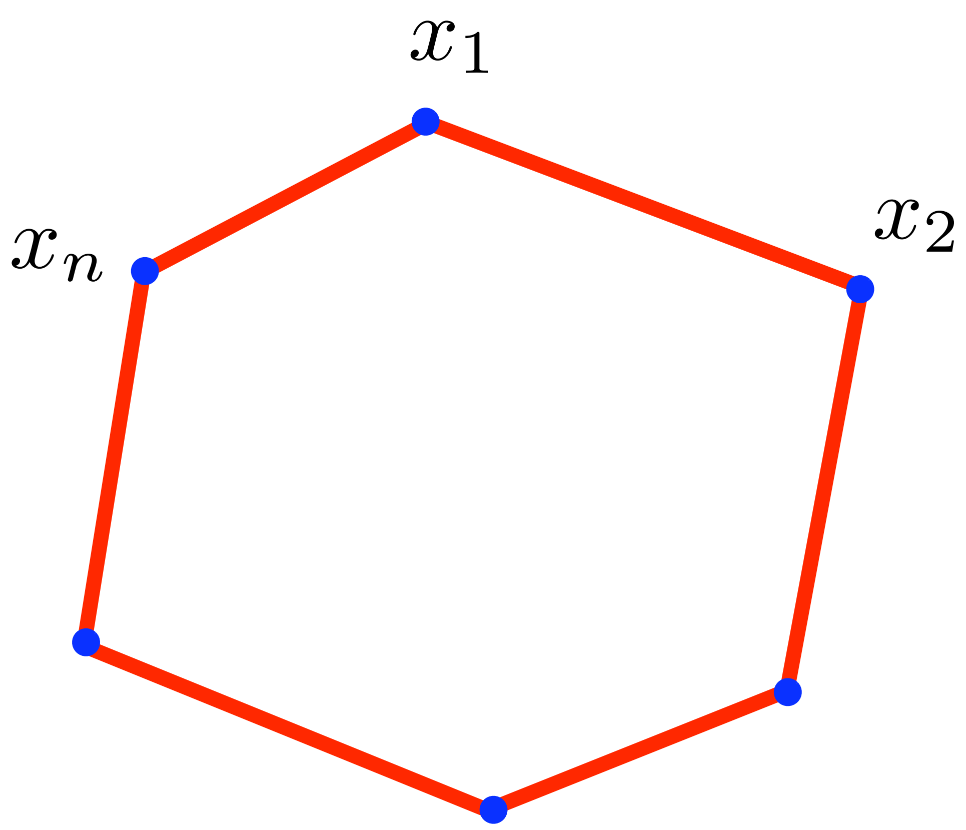
$$= 1 + \mathrm{g}^2$$  $$+ \mathcal{O}(\mathrm{g}^4)$$

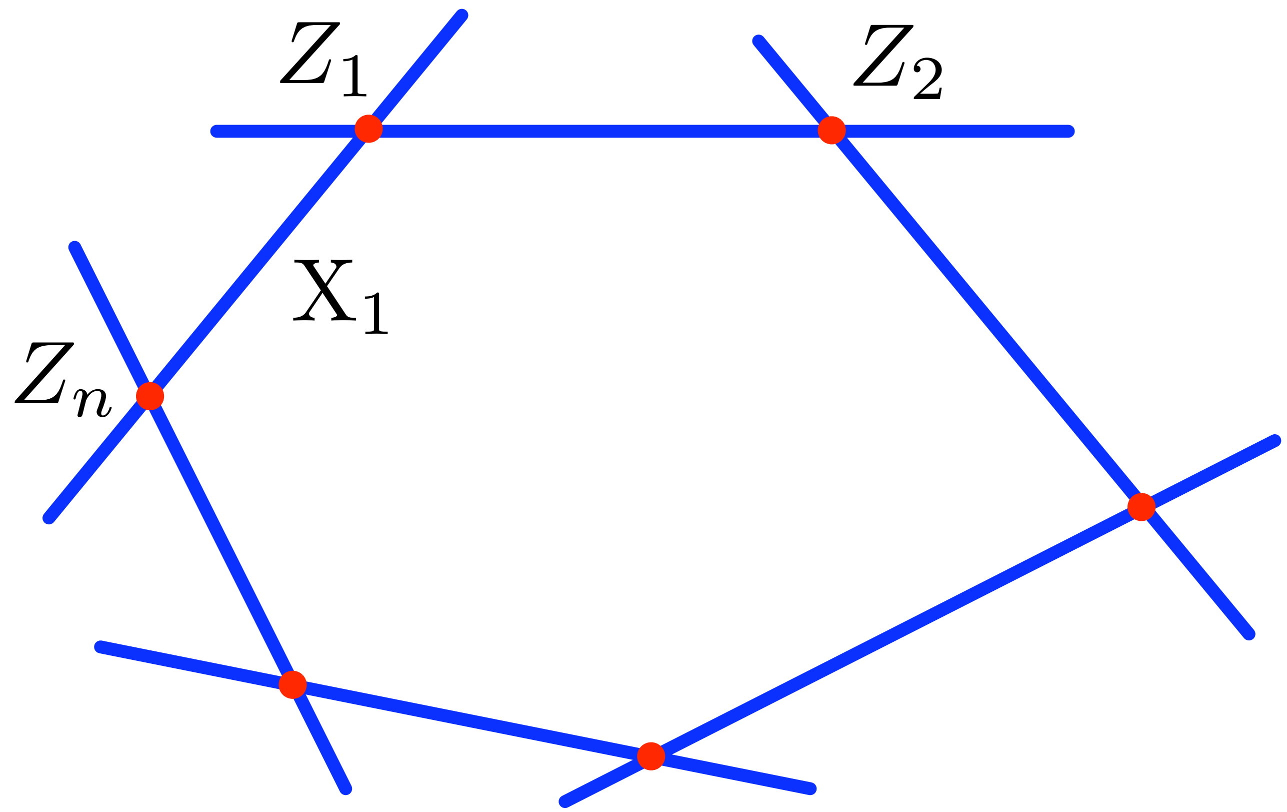
$$= \frac{\mathcal{M}^{(0)}_{\mathrm{MHV}}}{\mathcal{M}^{(0)}_{\mathrm{MHV}}} + \frac{\mathcal{M}^{(1)}_{\mathrm{MHV}}}{\mathcal{M}^{(0)}_{\mathrm{MHV}}} + \cdots$$

- Underlying reason for duality obscure
- Not clear how to extend to arbitrary helicity amplitudes

Space-time vertices $\longleftrightarrow$ Twistor lines

Null edges of polygon $\longleftrightarrow$ Twistor vertices



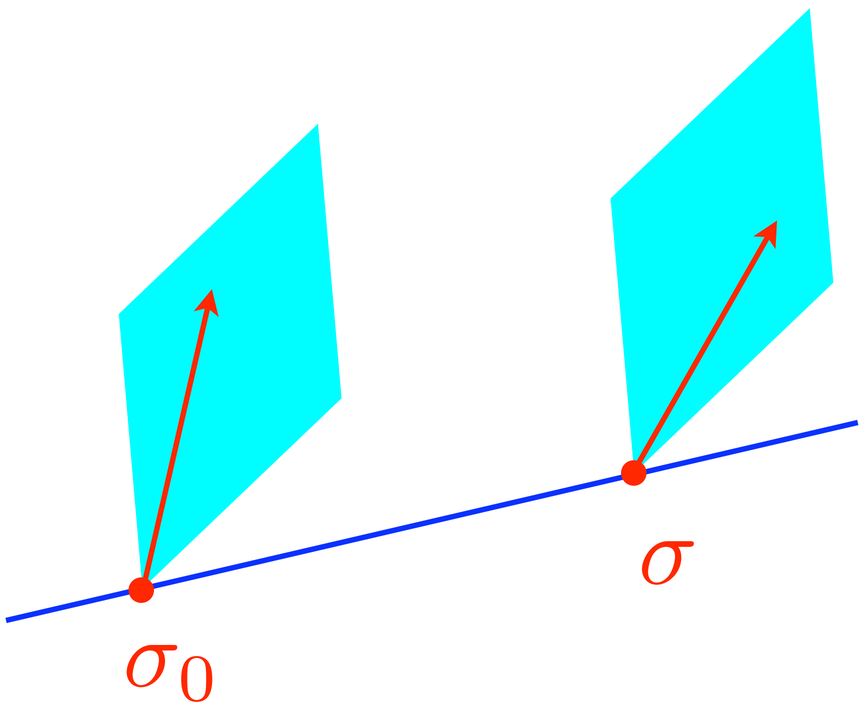


The twistor data is *unconstrained*: given arbitrary Z_i , the twistor lines intersect by construction, so the corresponding space-time vertices are inevitably null separated.

The duality extends to all helicities if one constructs a supersymmetric extension of the Wilson Loop [Mason,DS]

$$(\bar{\partial} + \mathcal{A})\big|_{\mathrm{X}} \, \mathrm{U}(\sigma_1, \sigma_2) = 0 \qquad \mathrm{U}(\sigma_1, \sigma_1) = \mathrm{id}$$

$\mathcal{N} = 4$ twistor superfield



Formally, we write

$$\mathrm{U}(\sigma, \sigma_0) = \mathrm{P}\exp\left(-\int \omega \wedge \mathcal{A}\right)$$

where ω is unique meromorphic 1-form with simple poles at $\{\sigma_0, \sigma\}$

$$\mathrm{U}(\sigma_1, \sigma_2)\mathrm{U}(\sigma_2, \sigma_3) = \mathrm{U}(\sigma_1, \sigma_3)$$ ***concatenation***

$$\mathrm{U}(\sigma_2, \sigma_1) = \mathrm{U}(\sigma_1, \sigma_2)^{-1}$$ ***inverse***

$$\mathrm{U}(\sigma_1, \sigma_2) \to g^{-1}(\sigma_1)\mathrm{U}(\sigma_1, \sigma_2)g(\sigma_2)$$ ***gauge transform***

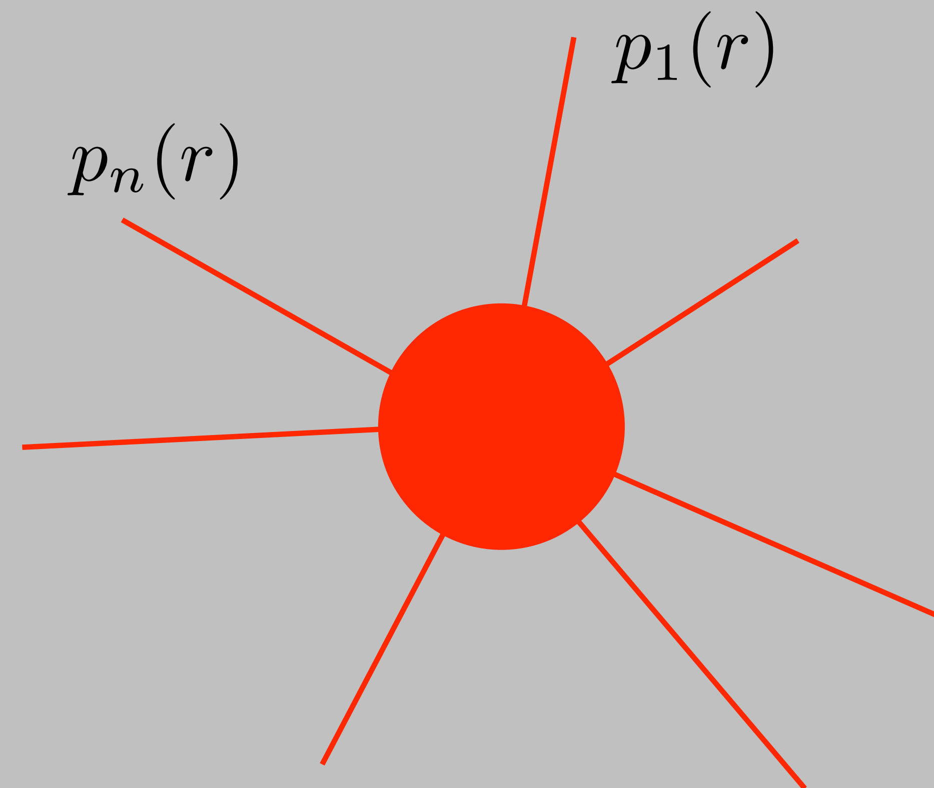
BCFW Recursion for null polygonal superloops

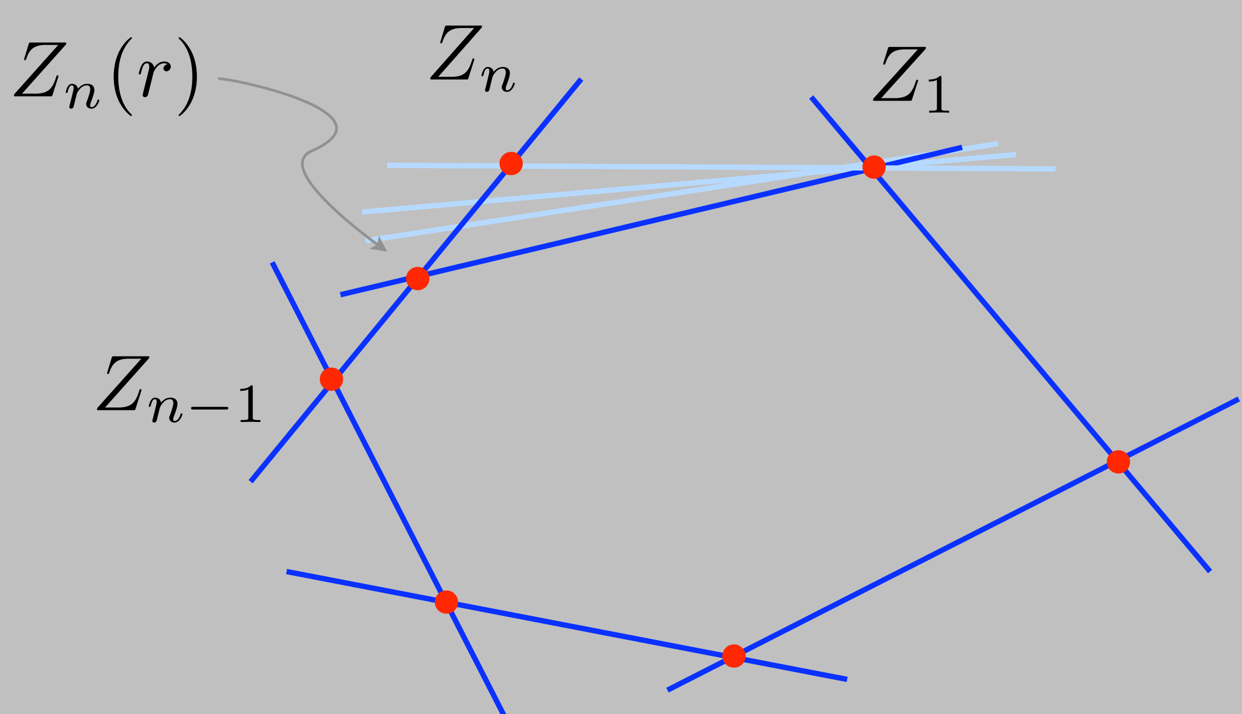
Deform external momenta

$$p_i \rightarrow p_i(r)$$

subject to the constraints

$$\sum p_i(r) = 0 \qquad p_i^2(r) = 0$$





In twistor space there are no constraints - we just vary the Z_i freely

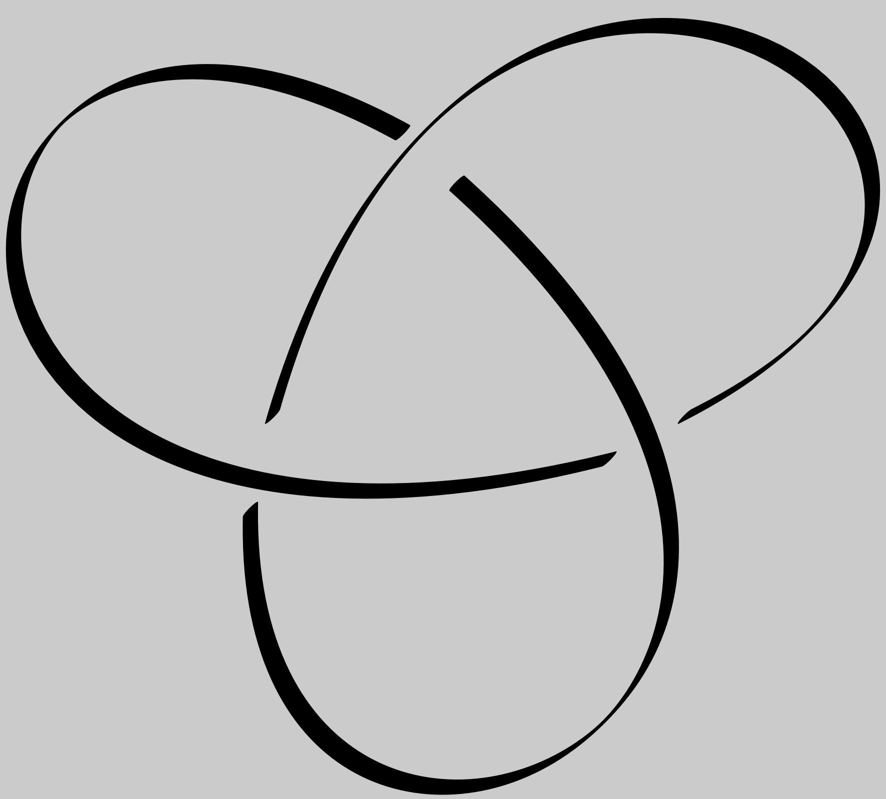
As the curve varies, the Wilson Loop obeys

$$\bar{\delta}\mathrm{W}[C] = -\int_C \omega \wedge \mathrm{d}\bar{Z}^{\bar{\imath}} \wedge \bar{\delta}\bar{Z}^{\bar{\jmath}}\,\mathrm{Tr}\left[\mathcal{F}_{\bar{\imath}\bar{\jmath}}(\mathcal{Z})\,\mathrm{P}\exp\left(-\int_C \omega \wedge \mathcal{A}\right)\right]$$

- Behaviour of correlator is controlled by loop equations

[Migdal,Makeenko; Polyakov]

Wilson Loops in real Chern-Simons compute knot invariants such as the HOMFLYPT polynomial [Witten]



Naively, varying the curve doesn't change this topological quantity

Loop equations give derivation of the skein relations - *i.e.* **recursion relations** for the knot polynomial

[Cotta-Ramusino,Guadagnini,Martellini,Mintchev]

In pure holomorphic Chern-Simons theory, the loop equations give[Bullimore,DS]

$$\langle \bar\delta \mathrm{W}[C(r)] \rangle = -\int_C \omega \wedge \left\langle \mathrm{Tr}\, \mathcal{F}^{(0,2)}(Z)\, \mathrm{P}\exp\left(-\int \omega\wedge\mathcal{A}\right)\right\rangle_{\mathrm{hCS}}$$

$$= -\int_C \omega \wedge \left\langle \mathrm{Tr}\, \frac{\delta S_{\mathrm{hCS}}}{\delta\mathcal{A}(Z)} \mathrm{P}\exp\left(-\int \omega\wedge\mathcal{A}\right)\right\rangle_{\mathrm{hCS}}$$

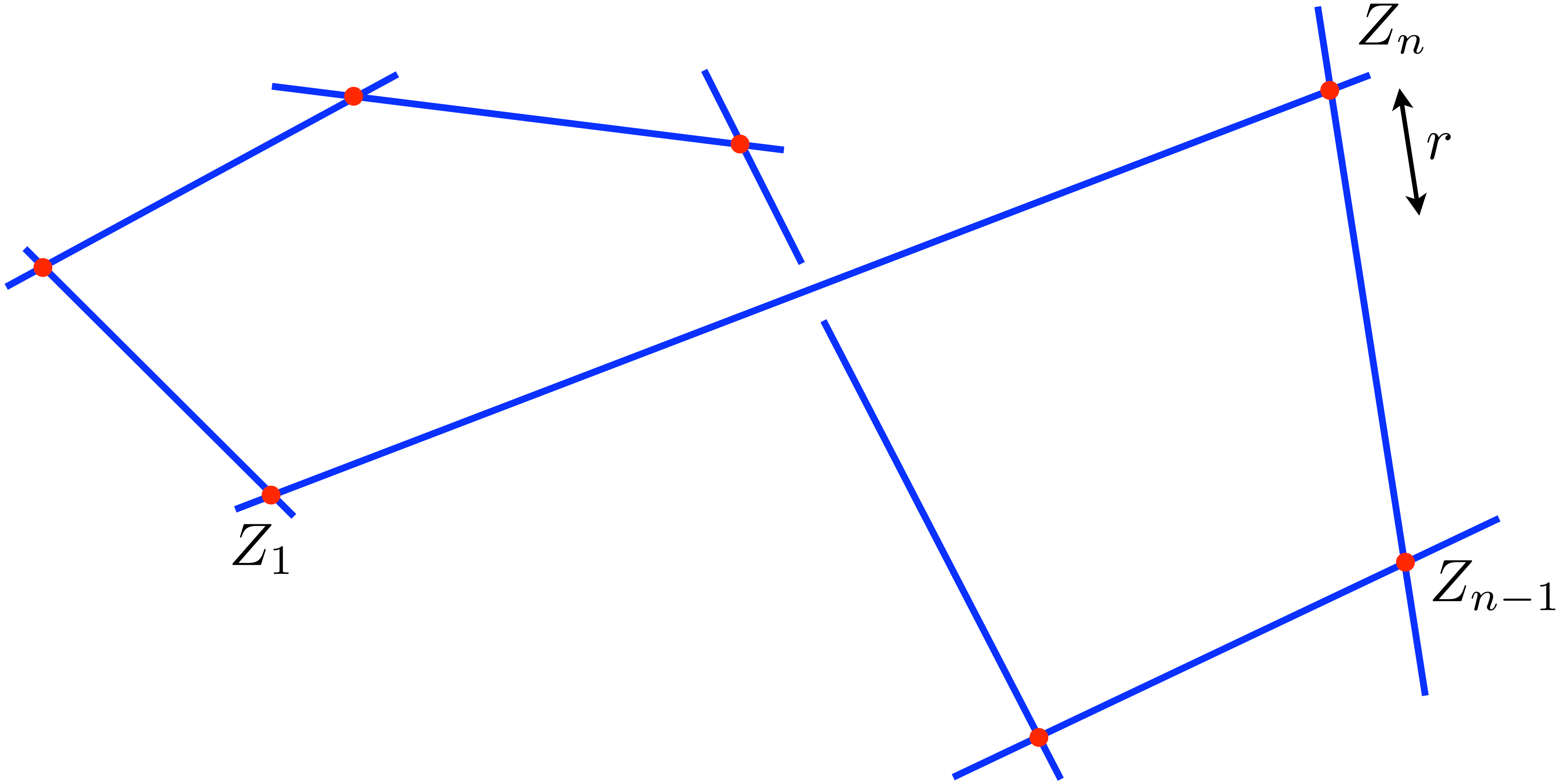
$$= \int_{C\times C} \omega\wedge\omega'\wedge\bar\delta^{3|4}(Z,Z')\ \langle \mathrm{W}[C_1]\,\mathrm{W}[C_2]\rangle_{\mathrm{hCS}}$$

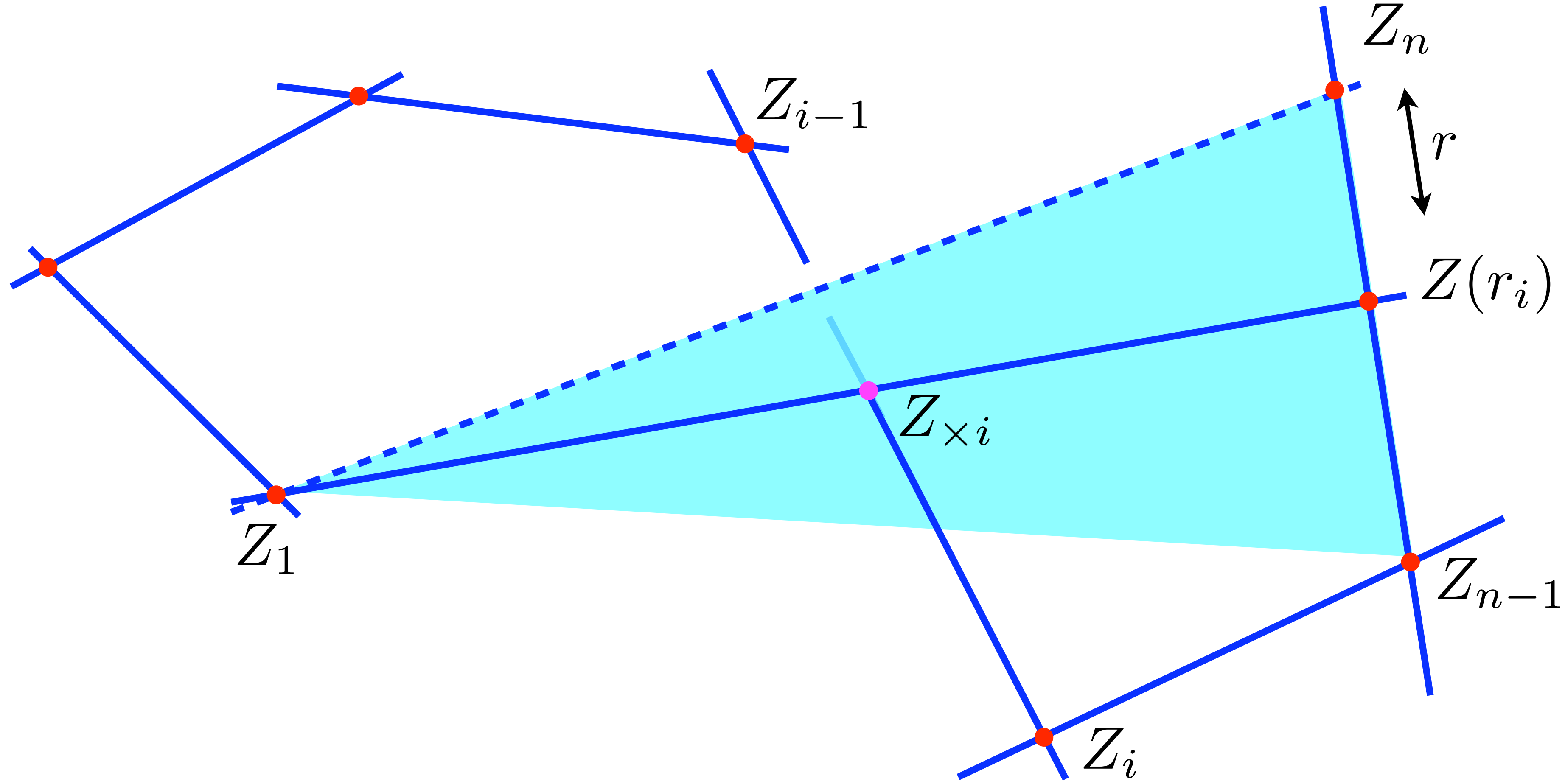
integrate by parts

only get contribution if $C(r)$ self-intersects as we deform

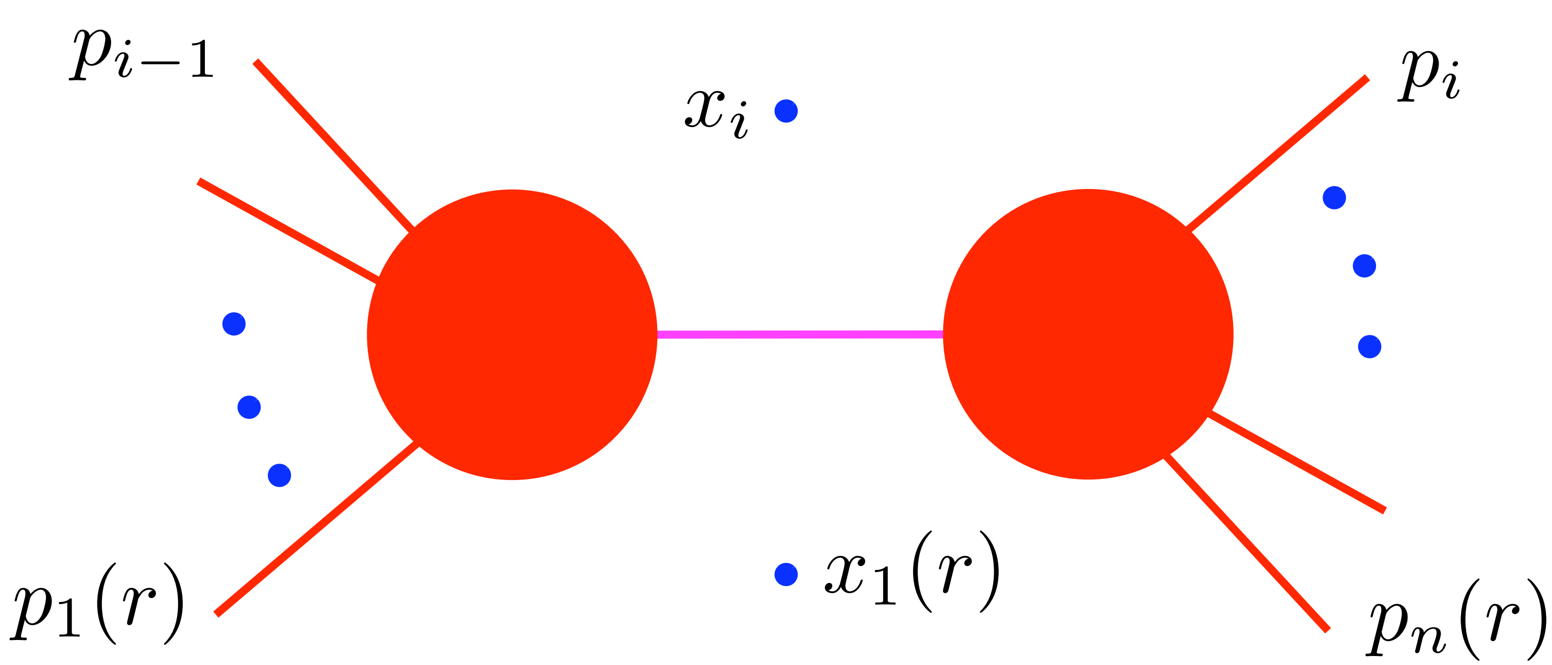
$$= \int_{C\times C} \omega\wedge\omega'\wedge\bar\delta^{3|4}(Z,Z')\ \langle \mathrm{W}[C_1]\rangle\,\langle \mathrm{W}[C_2]\rangle$$

in planar limit


Z_n
r
Z_1
Z_{n-1}



Intersecting twistor lines $\longleftrightarrow$ Null separation $\longleftrightarrow$ Factorization channel



For this deformation, the loop equations give

$$\langle \mathrm{W}[C_n]\rangle = \langle \mathrm{W}[C_{n-1}]\rangle + \sum_{i=3}^{n-1} [n-1, n, 1, i-1, i]\, \langle \mathrm{W}[C_i]\rangle\, \langle \mathrm{W}[C_i']\rangle$$

This is tree-level BCFW recursion.

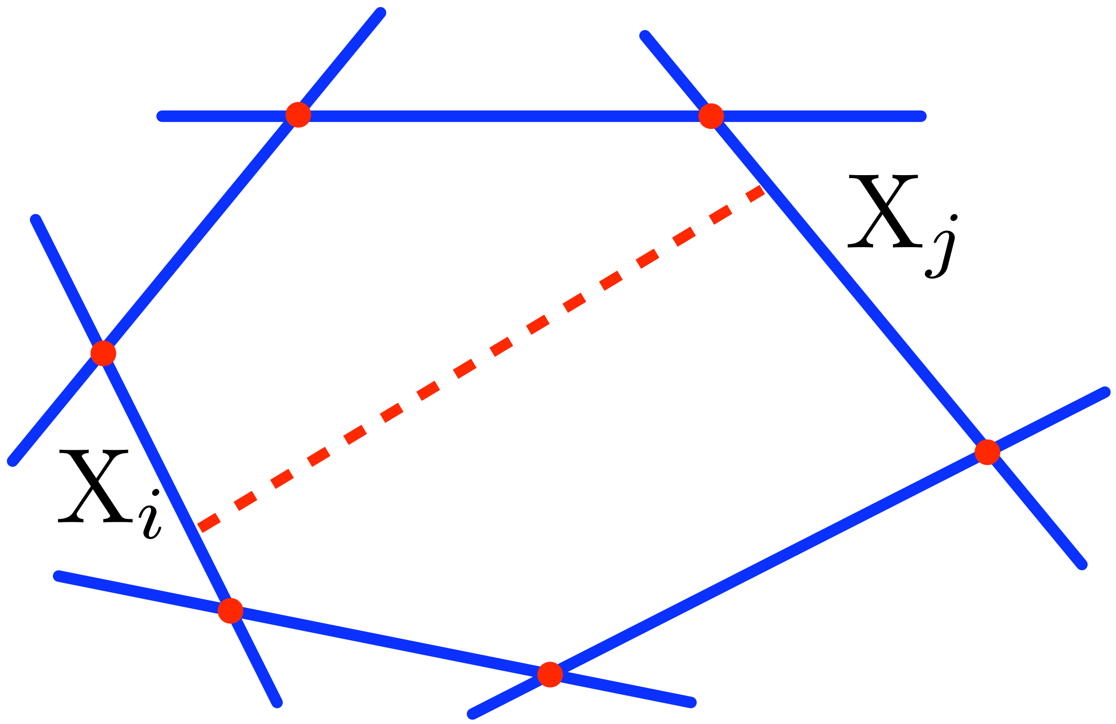
- The amplitudes are natural holomorphic analogues of knot invariants, with BCFW recursion as a skein relation!
- Repeating the derivation for the full twistor action (including MHV vertices) leads to all-loop generalization

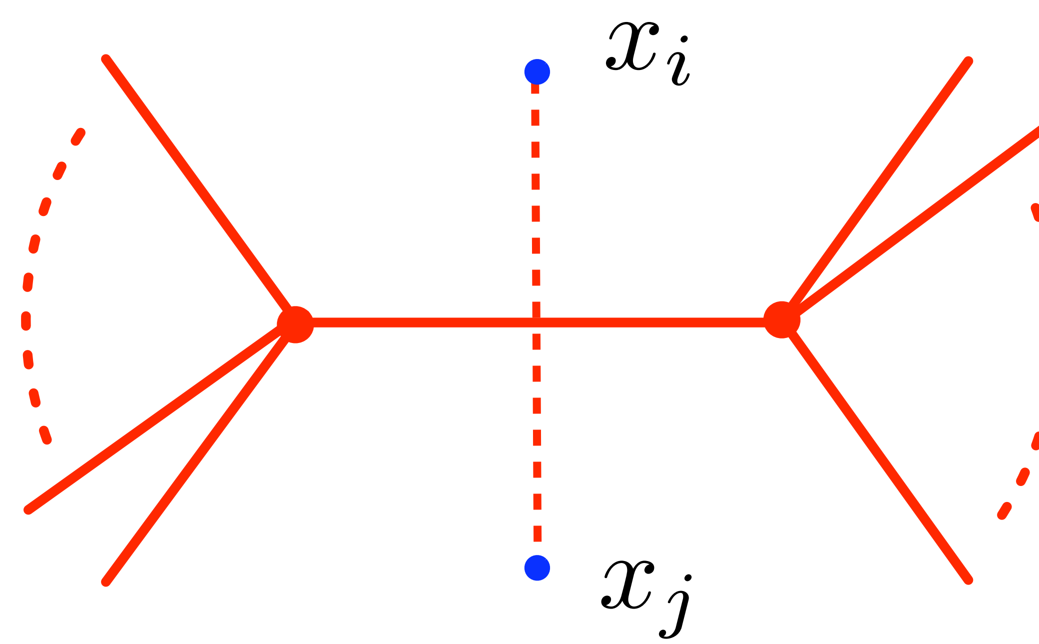
$$\langle \mathrm{W}[C_n]\rangle = \langle \mathrm{W}[C_{n-1}]\rangle + \sum_{i=3}^{n-1} [n-1, n, 1, i-1, i]\, \langle \mathrm{W}[C_i]\rangle\, \langle \mathrm{W}[C_i']\rangle$$
$$+ \int \mathrm{D}A\, \mathrm{D}B\, [n-1, n, 1, A, B]\, \langle \mathrm{W}[C_{n+2}^{AB}]\rangle$$

[Arkani-Hamed,Bourjaily,Cachazo, Caron-Huot,Trnka; Bullimore, DS]

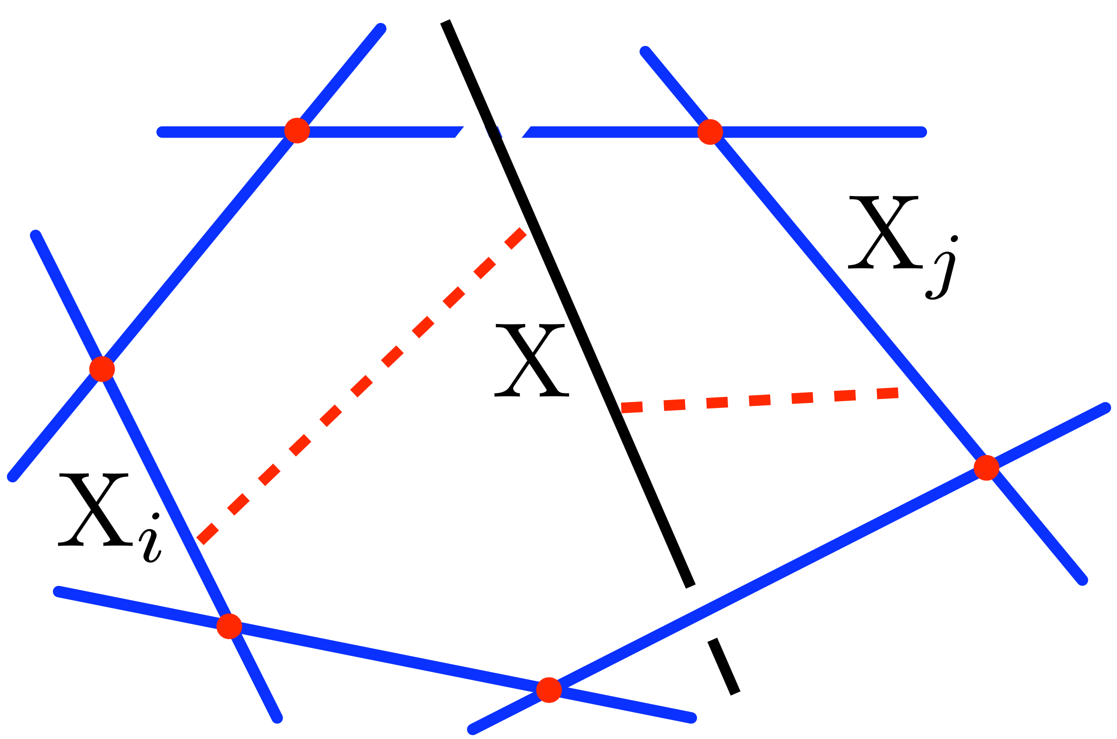
Resulting expressions contain spurious, non-local poles

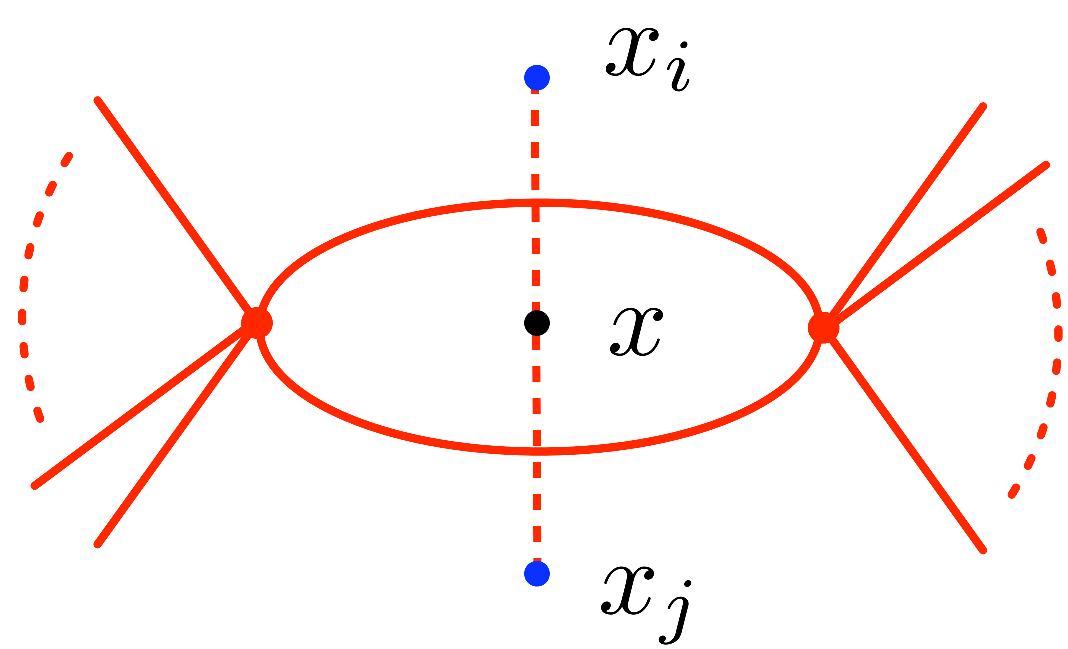
Superloop in sd $\mathcal{N} = 4$ $\longleftrightarrow$ Tree superamplitude





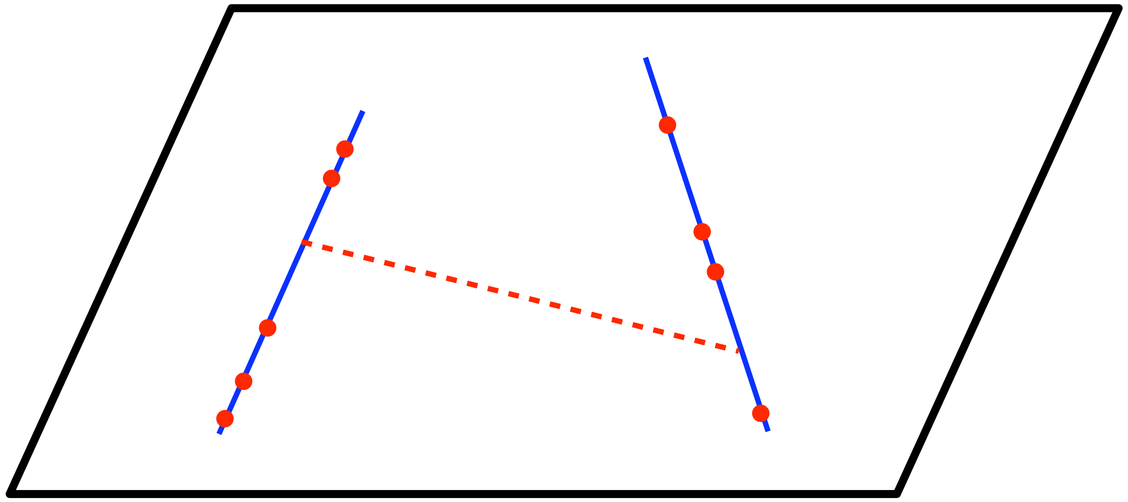
Superloop in full $\mathcal{N} = 4$ $\longleftrightarrow$ Planar superamplitude



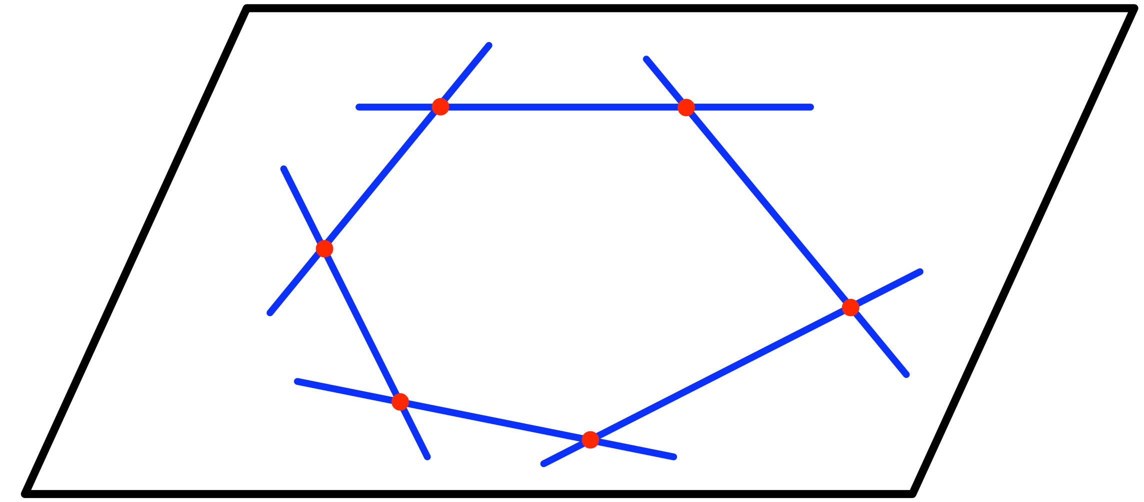


Grassmannians

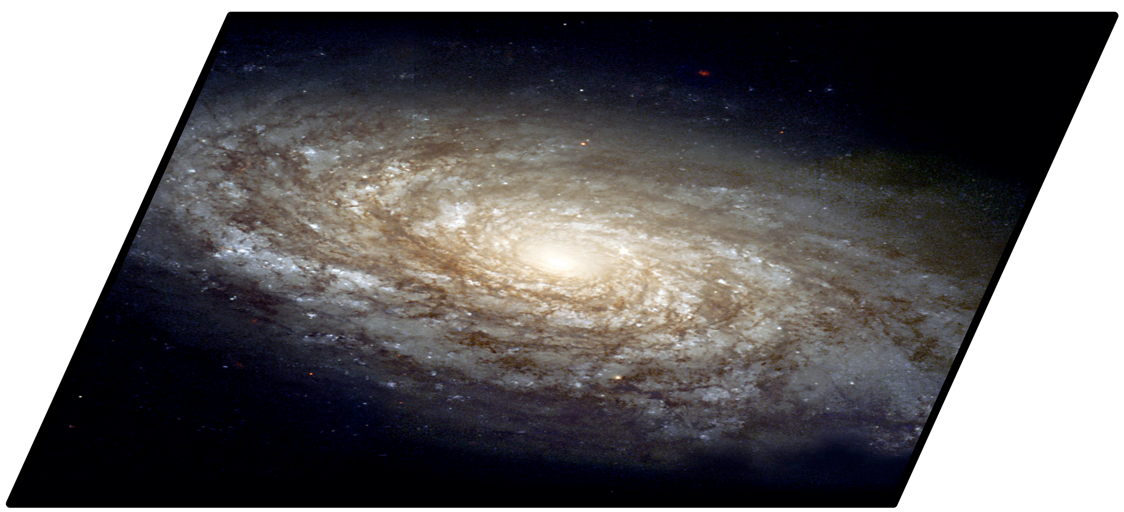
twistor space



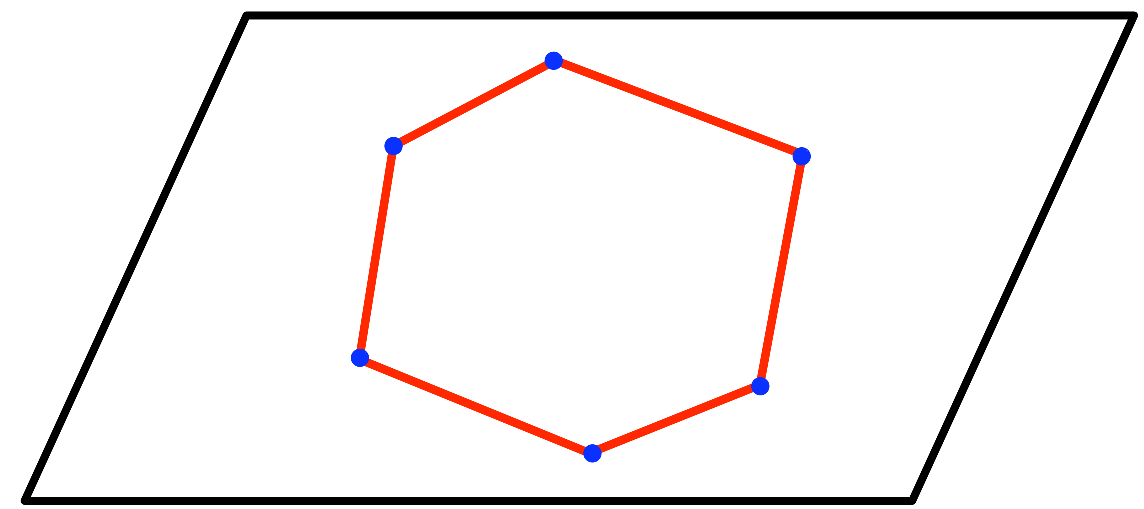
momentum twistor space



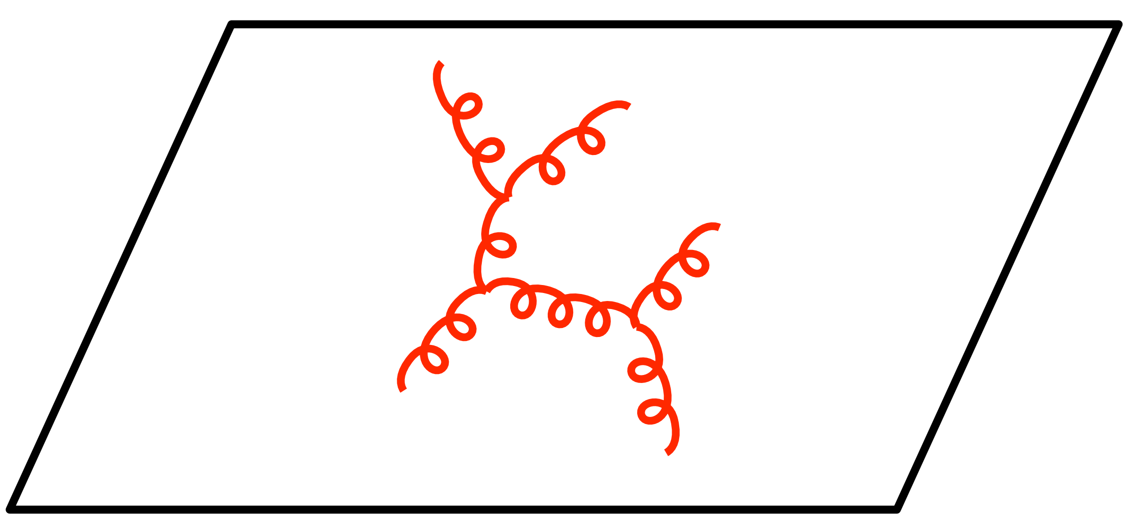
conformal symmetry



dual conformal symmetry



momentum space



Dual descriptions are each representations of a more invariant underlying picture

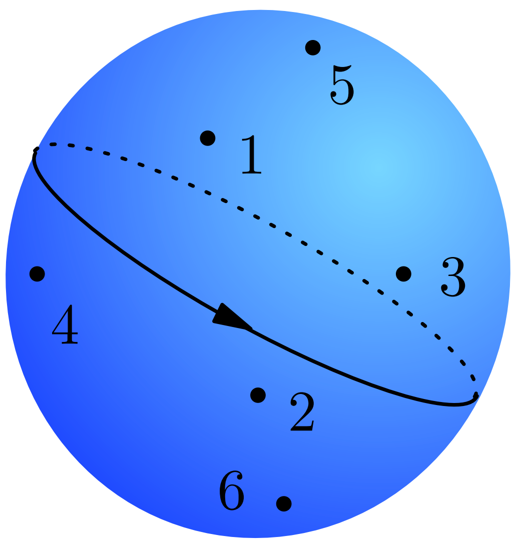
Grassmannian contour integral

[Arkani-Hamed,Cachazo *et al.*; Mason,DS]

$$\oint_\Gamma \frac{\mathrm{D}^{k(n-k)}C}{(12\cdots k)(23\cdots k{+}1)\ldots(n1\cdots k{-}1)} \prod_{r=1}^{k} \bar\delta^{4|4}(C_{ri}\mathcal{Z}_i)$$

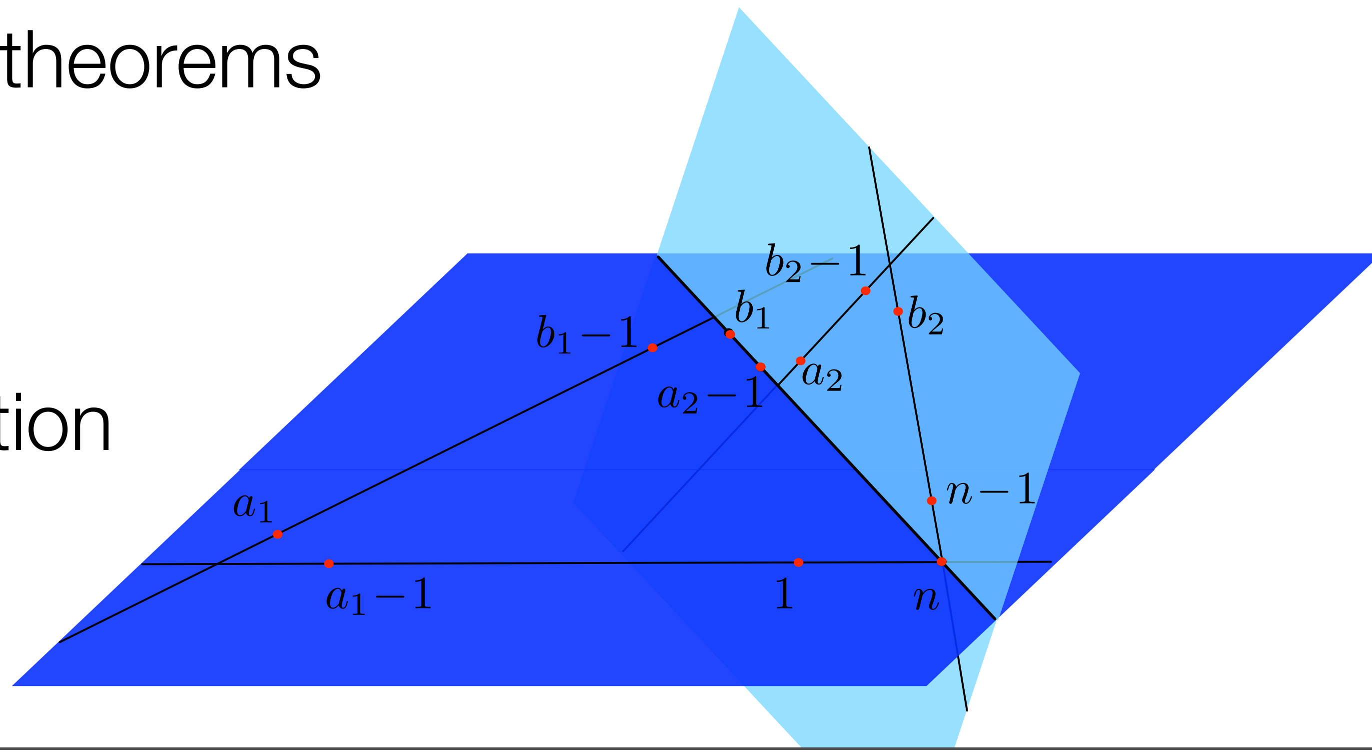
takes identical form on both sides; neither is preferred

- Describes scattering without relying on a space-time interpretation!



Different amplitude decompositions = residue theorems

Contains ***all-loop*** information



Original Superconformal Symmetry	$+$	Dual Superconformal Symmetry	$=$	Infinite Dimensional Yangian

[Beisert,Ricci,Tseytlin,Wolf; Berkovits,Maldacena; Drummond,Henn,Plefka]

- Reflection of integrability of planar $\mathcal{N}=4$ SYM

The Yangian is represented on (either) twistor space by

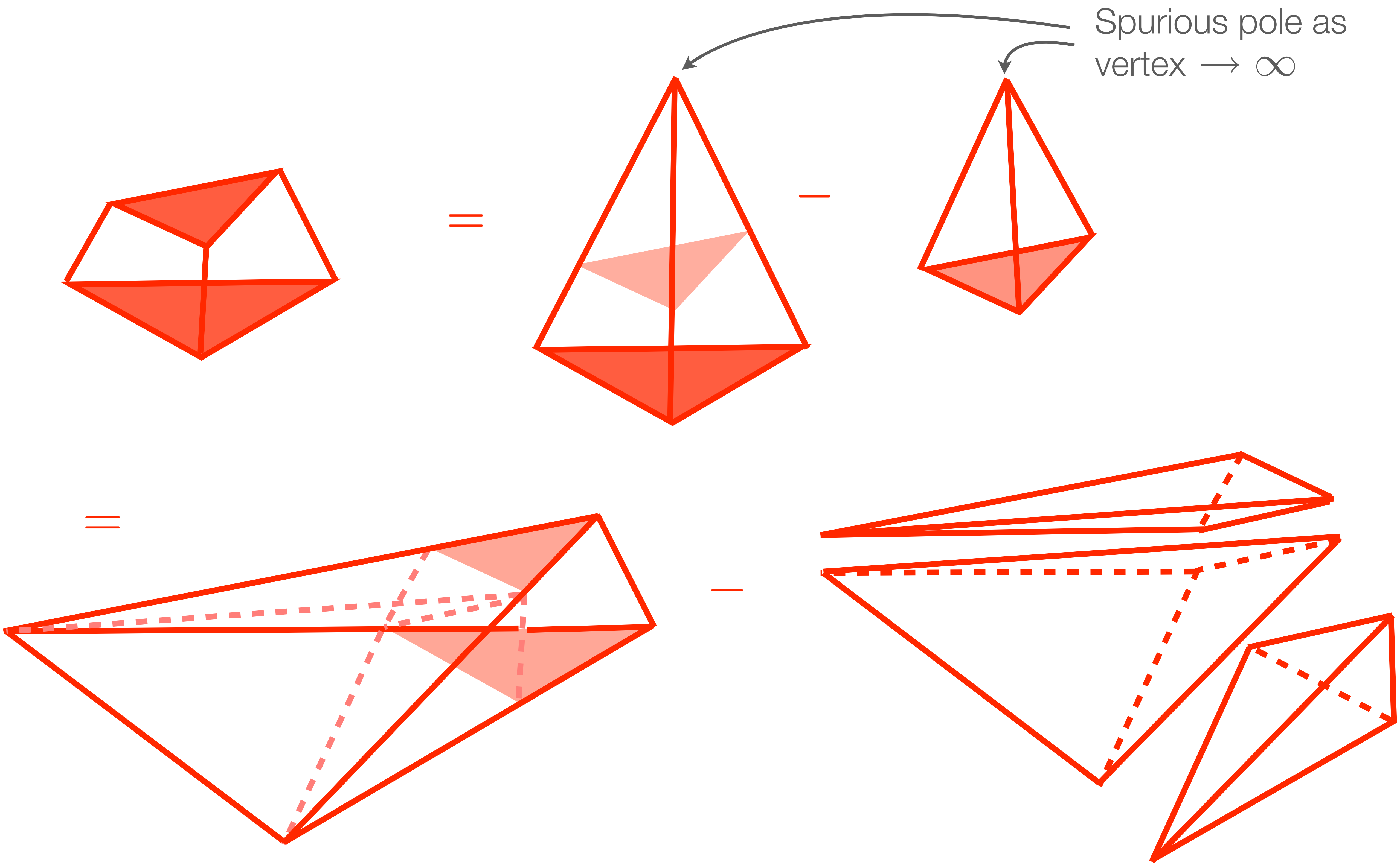
$$J^{(0)} = \sum_i \mathcal{Z}_i \frac{\partial}{\partial \mathcal{Z}_i} \qquad J^{(1)} = \sum_{i>j} \mathcal{Z}_i \left(\mathcal{Z}_j \cdot \frac{\partial}{\partial \mathcal{Z}_i} \right) \frac{\partial}{\partial \mathcal{Z}_j} - (i \leftrightarrow j)$$

plus infinitely many higher generators

- Grassmannian formula is uniquely fixed by Yangian

[Drummond,Ferro]

The amplitudes are defined by a polytope [Hodges]

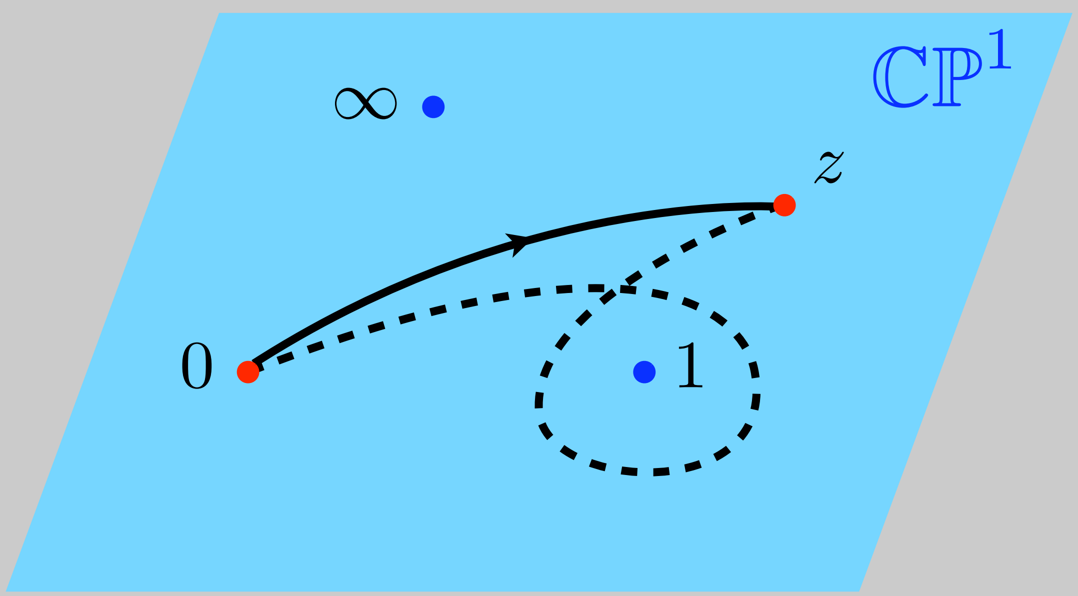


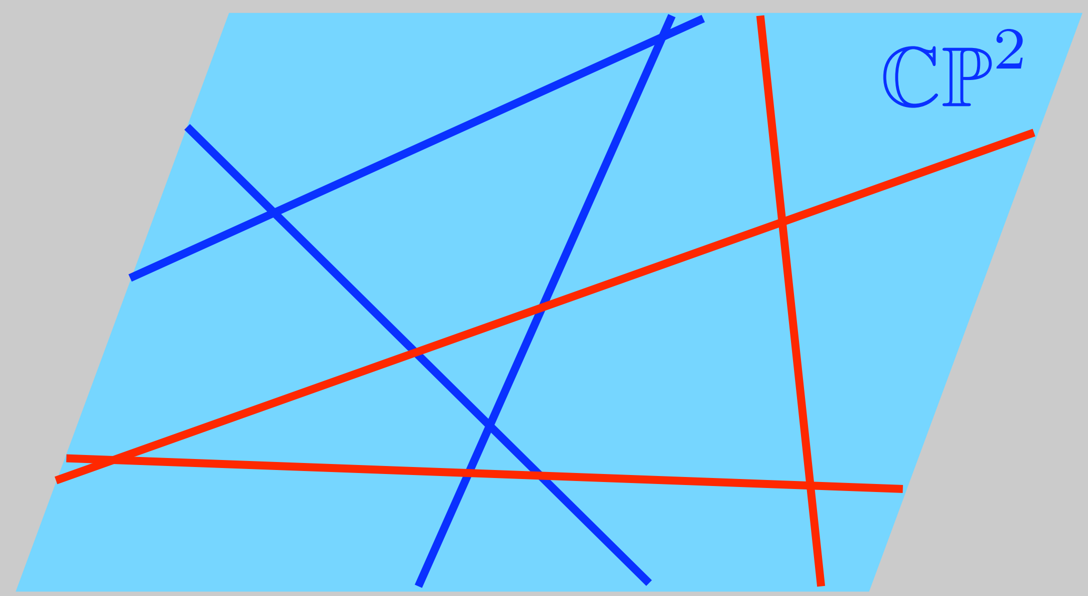
- Different triangulations give different representations

At loop level, ℓ-loop amplitudes are given in terms of polylogarithms of *transcendentality* 2ℓ

- Polylogarithms represent relative homology classes on Grassmannians[Goncharov; Hain,MacPherson]

$$\mathrm{Li}_1(z) = \int_0^z \frac{\mathrm{d}t}{1-t} \qquad \mathrm{Li}_2(z) = \int_0^z \frac{\mathrm{d}t}{1-t} \circ \frac{\mathrm{d}s}{s}$$

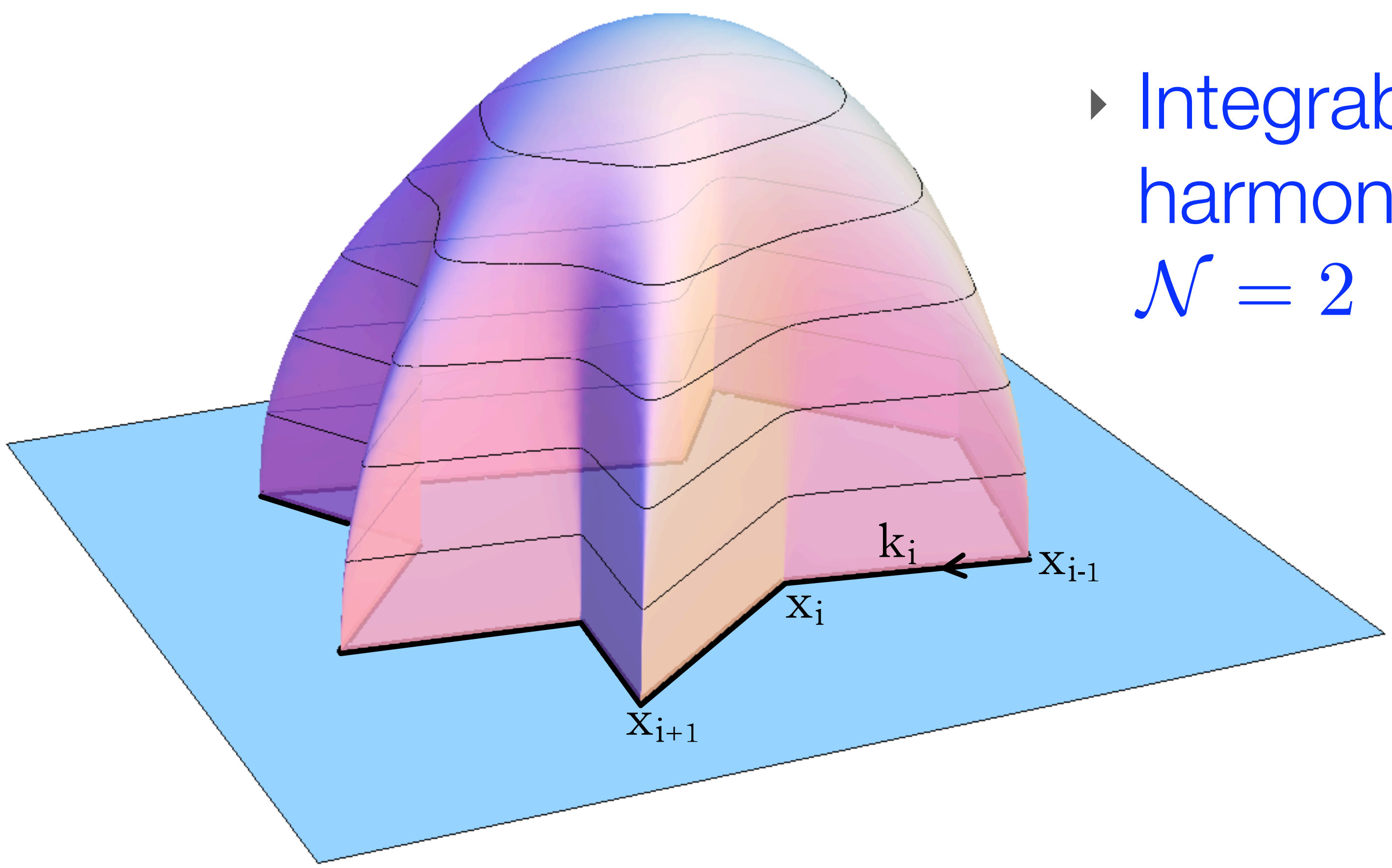




Suggests loops proper may be obtained from the Grassmannian integral on contours with boundary

Towards Strong Coupling

At strong coupling the MHV amplitude is given by a minimal surface in AdS_5 [Alday,Gaiotto,Maldacena,Sever,Vieira]



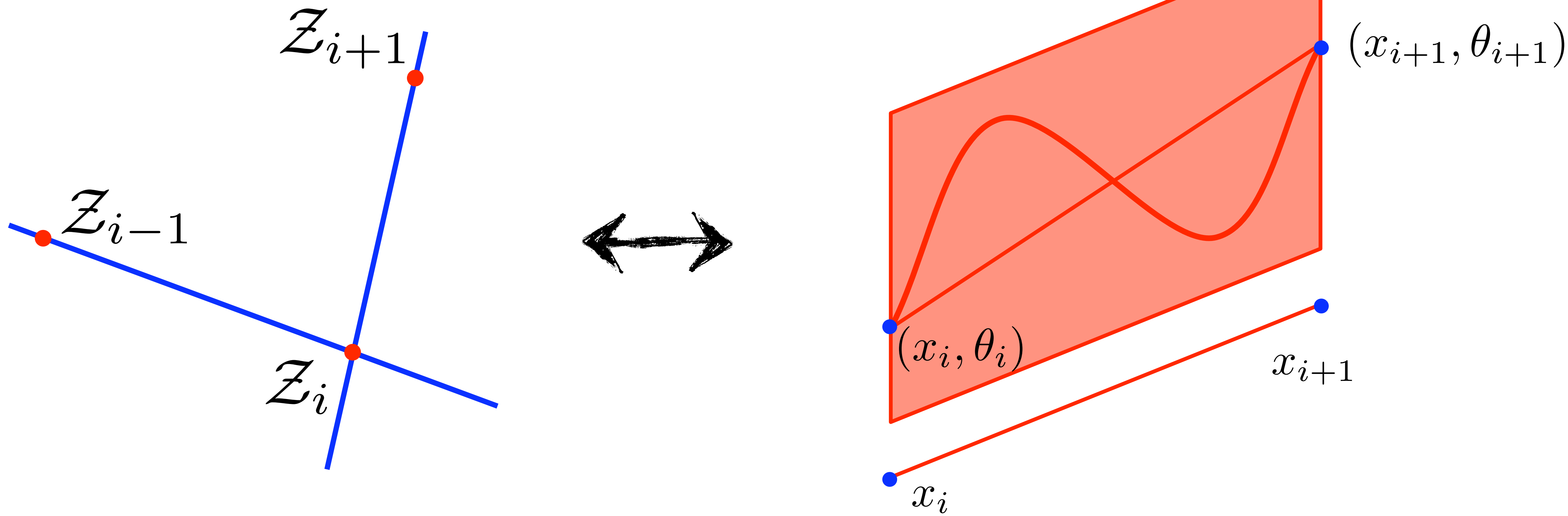
- Integrable system related to both harmonic maps and wall-crossing in $\mathcal{N} = 2$ gauge theories[Gaiotto,Moore,Neitzke]

The strongly-coupled superamplitude should be given by IIB action for a minimal surface on supercoset

- Much simpler than including vertex operators & perturbing around bosonic background

work in progress...

$\mathcal{Z} \in \mathbb{CP}^{3|4}$ defines a totally null *super*-ray



$$\mu^{\dot\alpha} = \mathrm{i} x^{\alpha\dot\alpha}\lambda_\alpha \qquad \chi^a = \theta^{\alpha a}\lambda_\alpha$$

The corresponding space-time superloop is

$$\frac{1}{N}\mathrm{Tr}\,\mathrm{P}\exp\left(\mathrm{i}\oint \mathbb{A}\right) \qquad \mathbb{A} = A_{\alpha\dot\alpha}\mathrm{d}x^{\alpha\dot\alpha} + \Gamma_{\alpha a}\mathrm{d}\theta^{\alpha a}$$

where superconnection obeys integrability conditions

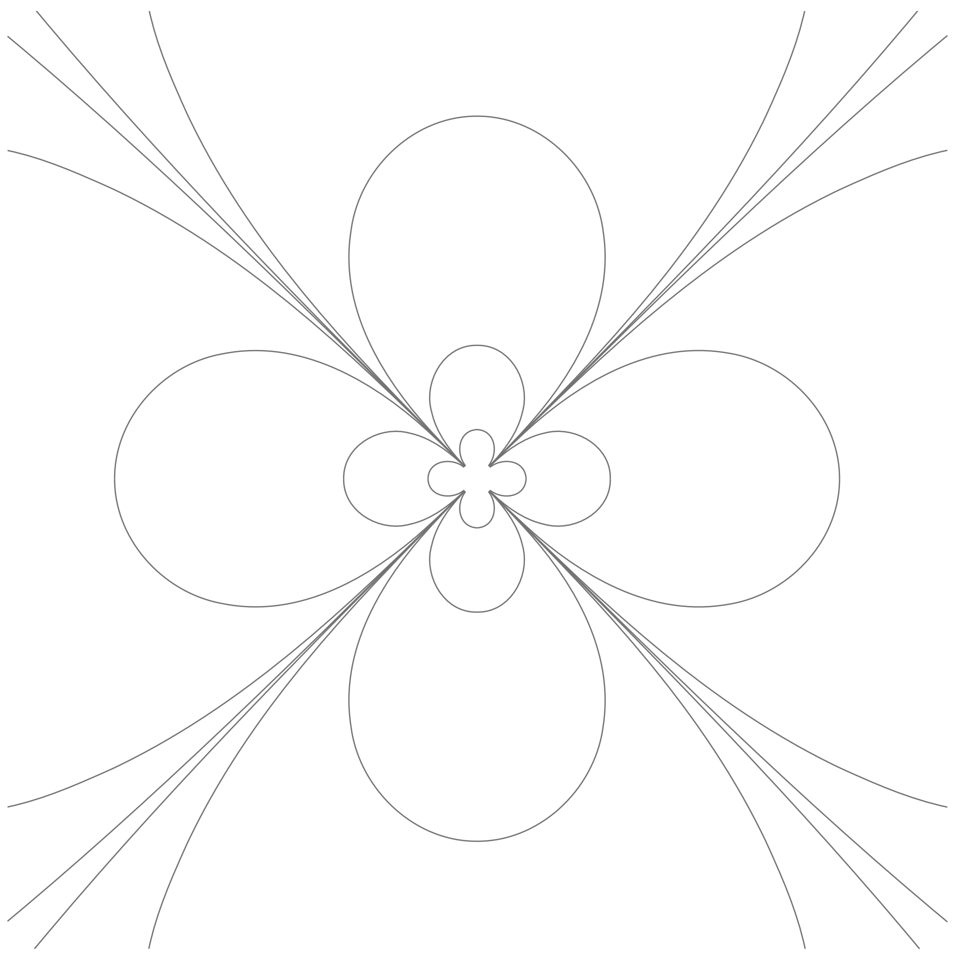
$$\lambda^\alpha\lambda^\beta\left[\nabla^{\mathrm{bos}}_{\alpha\dot\alpha}, \nabla^{\mathrm{ferm}}_{\beta b}\right] = 0 \qquad \lambda^\alpha\lambda^\beta\left\{\nabla^{\mathrm{ferm}}_{\alpha a}, \nabla^{\mathrm{ferm}}_{\beta b}\right\} = 0$$

Integrability along super null rays is worldline κ-symmetry [Witten]

- Agrees with restriction of Type IIB worldsheet κ-symmetry to boundary [Ooguri,Rahmfeld,Robins,Tannenhauser]

Sigma model into coset described by graded Lax connection. Pohlmeyer reduce to account for Virasoro constraints of string [Grigoriev,Tseytlin; Mikhailov,Schafer-Nameki]

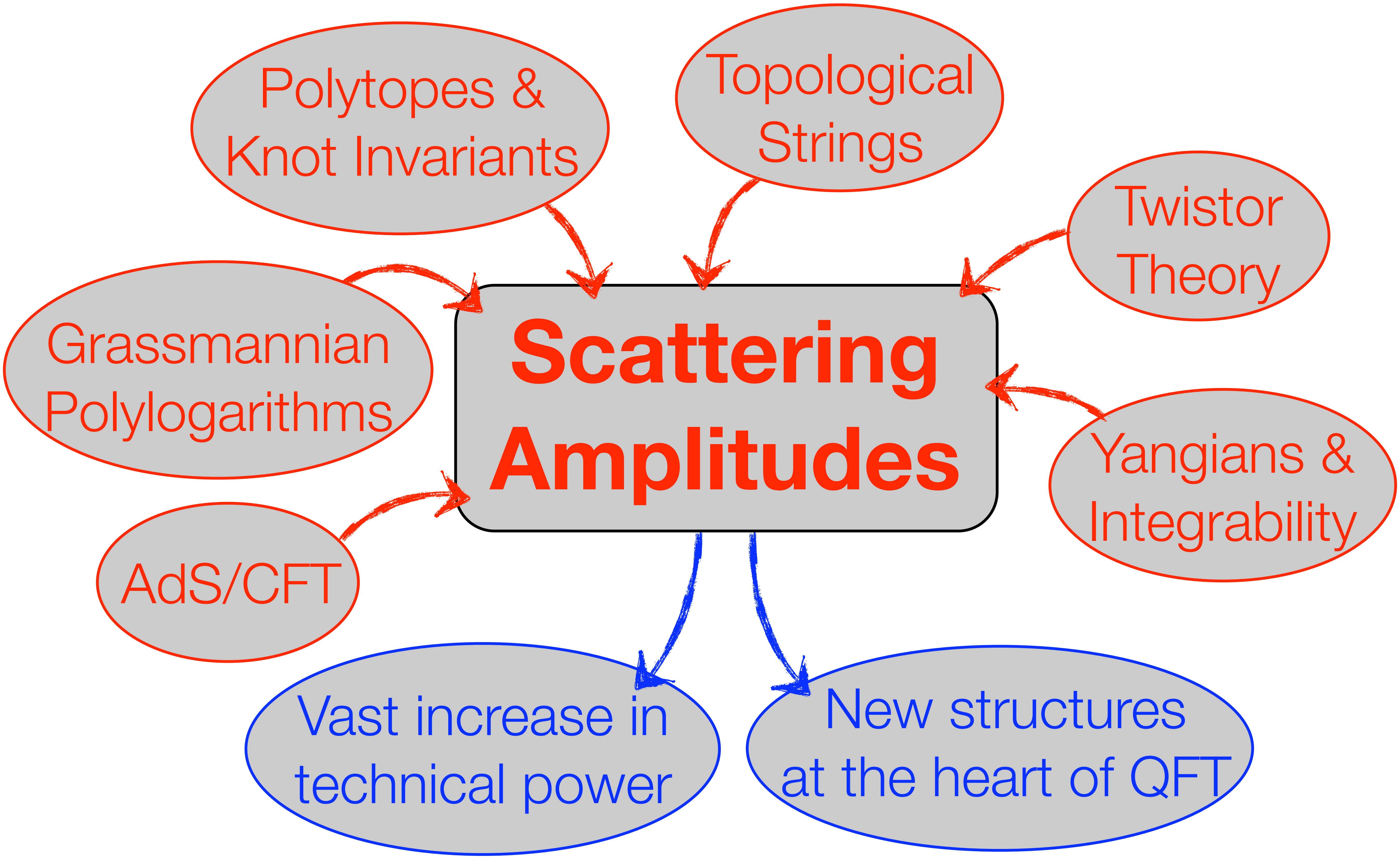
[Gaiotto,Moore,Neitzke]



- Expect a flat $\mathrm{PSL}(4|4;\mathbb{C})$ connection on $\mathbb{CP}^1$, wildly ramified at a single point
- Different *super*twistors associated to each Stokes sector

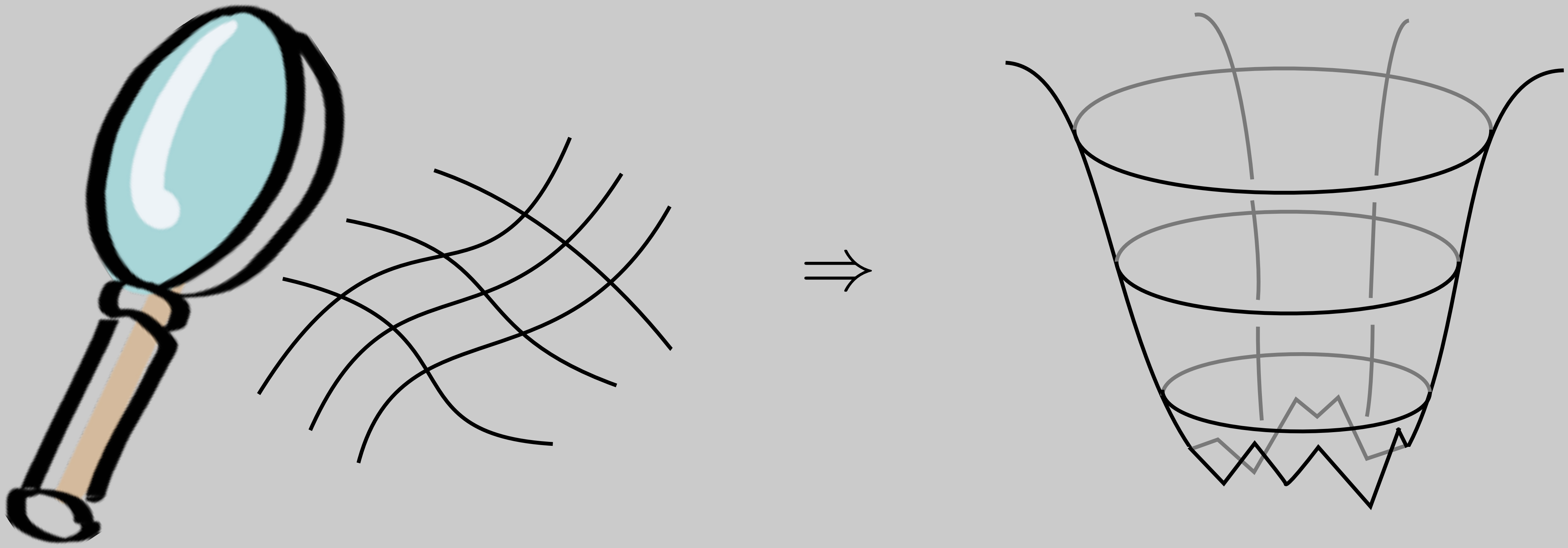
work in progress...

Conclusions


Polytopes &
Knot Invariants
Topological
Strings
Twistor
Theory
Grassmannian
Polylogarithms
Scattering
Amplitudes
Yangians &
Integrability
AdS/CFT
Vast increase in
technical power
New structures
at the heart of QFT

Scattering amplitudes have many remarkable properties that are completely invisible from the perspective of Feynman diagrams

They are also among precious few observables that can still exist in a diffeomorphism invariant quantum theory



Reformulating amplitudes in ways that do not rely on space-time is important preparation for the case that there *is* no space-time

Hopefully, the great technical progress is a good sign that we're on the right track...