Theoretical Introduction to Gromov-Witten invariants

Huailiang Chang

Department of Mathematics
The Hong Kong University of Science and Technology

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Abstract

The Material to be presented focus on theoretical/definition side.
The goal is the algebraic geometric part of the theory.

Outline

1. symplectic geometric origin
2. GW invariants: algebraic geometric approach
3. GW invariant of Calabi Yau threefold
4. GW invariant of surface with $p_g > 0$: Parker and Li :

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Symplectic geometric origin(Ruan-Tian...)

- Gromov Witten invariant is the Euler class of an infinite rank complex bundle over an infinite dim. complex manifold;
- As a path-integral by Ed. Witten, it is connected to many other math subjects, and conjectures.

Symplectic geometric origin(Ruan-Tian...)

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Symplectic origin

- Let M be a manifold and w is a closed two from.
- Condition: $\forall p \in M$ the pairing

$$w|_p : T_{p,M} \times T_{p,M} \longrightarrow \mathbb{R}$$

  is nondegenerate.
- We call w a symplectic form on M. The pair (M, w) is called a symplectic manifold.

For (M, w) we can find $J : T_M \to T_M$ such that

$$J^2 = -1 \qquad \text{(almost complex structure)}$$

and

$$w(v, Jv) \geq 0 \quad \text{(compatibility)}.$$

Gromov finds that the

$$\{J | J \text{ is compatible with } w\}$$

is modeled by R^∞.

For any genus g Riemann surface C and a smooth map $\phi : C \to M$. One has

$$d\phi : T_C \longrightarrow \phi^* T_M .$$

The object

$$\overline{\partial}\phi := d\phi \circ J_C - J_M \circ d\phi \in \Omega^{(0,1)}(\phi^* T_M)$$

measures how far two complex structures are compatible via ϕ.

- We call ϕ (pseudo)holomorphic if it satisfies

$$\overline{\partial}\phi = d\phi \circ J_C - J_M \circ d\phi = 0,$$

called Cauchy Riemann equation.

- Pick chart

$$p \in \mathbb{C} = \{z\} \subset C \quad \text{and} \quad \phi(p) \subset \mathbb{C}^n \subset M$$

then

$$\overline{\partial}\phi = (\frac{\partial \phi_1}{\partial \bar{z}}, \cdots, \frac{\partial \phi_n}{\partial \bar{z}}).$$

Hence

$$\overline{\partial}\phi = 0 <=> \phi = (\phi_1, \phi_2, \cdots, \phi_n) \text{ are holomorphic.}$$

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- Given a compact symplectic manifold (M, w) with a compatible almost complex structure J, fix $g \in \mathbb{Z}^{\geq 0}$ and $\beta \in H_2(M, \mathbb{Z})$.

- We want to collect the following object: a genus g Riemann surface C and a smooth map $\phi : C \to M$ such that $\phi_*[C] = \beta$.

- $\overline{M}_g^{sm}(M,\beta) := \{\phi : C \to M \mid \phi$ is smooth map,

$$g(C) = g, \phi_*[C] = \beta\}$$

  where C is a smooth or nodal Riemann surface of genus g.

- Brief $\mathcal{M}^\infty = \overline{M}_g^{sm}(M,\beta)$.

- For each $\phi \in \mathcal{M}^\infty$ we have a vector space

$$\mathcal{E}_\phi^\infty := \Omega_C^{(0,1)}(\phi^* T_M).$$

  It is infinite dimensional.

- Take union when ϕ runs over all point on $\mathcal{M}^\infty$;

$$\mathcal{E}^\infty = \cup_\phi \, \mathcal{E}^\infty_\phi$$

is a vector bundle over $\mathcal{M}^\infty$.

- The association

$$s_{GW}(\phi) := \overline{\partial}\phi$$

defines a section of the bundle $\mathcal{E}^\infty$ where

$$s_{GW}(\phi) = 0 <=> \phi \text{ is (pseudo) holomorphic map}$$

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Different choices of J_M does not change $\mathcal{E}^\infty, \mathcal{M}^\infty$ but changes

$$s_{GW} \in \Gamma(\mathcal{M}^\infty, \mathcal{E}^\infty)$$

and

$$\{s_{GW} = 0\} = \overline{M}_g(M, J_M, \beta).$$

[Gromov,Ruan-Tian]:
For generic choice of J_M

1. s_{GW} is a transverse section of $\mathcal{E}^\infty$
2. $\overline{M}_g(M, J_M, \beta)$ is smooth, and compact (some detail).

Symplectic GW

The fundamental class of $\overline{M}_g(M, J_M, \beta)$ is the euler class of the bundle $\mathcal{E}^\infty$ over $\mathcal{M}^\infty$

$$e(\mathcal{E}^\infty) \in H_k(\mathcal{M}^\infty)$$

where

$$k = 3g - 3 + \text{index of } \overline{\partial} = \int_\beta c_1(T_M) + (dim\, M - 3)(1 - g).$$

(vaguely $k = dim\, \mathcal{M}^\infty - rank\, \mathcal{E}^\infty$).

- Suppose Y is smooth projective variety and take $M = Y$, with w the canonical Kaehler form.

- [Li-Tian, Behrend-Fantechi]: There is an cycle class $[\overline{M}_g(Y,\beta)]^{vir} \in A_k(\overline{M}_g(Y,\beta))$ such that under the inclusion

$$\overline{M}_g(Y,\beta) \subset \overline{M}_g^{sm}(M,\beta) = \mathcal{M}^\infty,$$

as classes in $H_k(\mathcal{M}^\infty)$

$$e(\mathcal{E}^\infty) = [\overline{M}_g(Y,\beta)]^{vir}.$$

Idea from Kuranishi model (Jun Li)

From $s_{GW} \in \Gamma(\mathcal{M}^\infty, \mathcal{E}^\infty)$ and $\mathcal{X} := Z_{s_{GW}} = \overline{M}_g(Y, \beta)$ one has

$$0 \longrightarrow T_{\mathcal{X}} \longrightarrow T_{\mathcal{M}^\infty}|_{\mathcal{X}} \stackrel{ds_{GW}|_{\mathcal{X}}}{\longrightarrow} \mathcal{E}^\infty|_{\mathcal{X}} \stackrel{q}{\longrightarrow} \mathcal{O}b \longrightarrow 0.$$

Li-Tian and Behrend-Fantechi found, there is a subspace (cone)

$$\mathfrak{C} \subset \mathcal{O}b$$

which can be constructed in AG, such that

$$\mathfrak{C}^\infty := lim_{t\to\infty} t \cdot \Gamma_{s_{GW}} = q^*\mathfrak{C}$$

The way we learn Gromov Witten invariant in algebraic geometry.

- Y: smooth projective variety over $\mathbb{C}$, $\beta \in H_2(Y,\mathbb{Z})$.
- Denote
$$\overline{M}_g(Y,\beta) = \{u : C \to Y, u_*[C] = \beta\}$$
where C is a nodal curve of genus g and f is holomorphic.

This moduli space is a “Deligne-Mumford stack(singular orbifold)". It is compact (proper) and may be as singular as you wish.

Let

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle \pi} & & \\ \overline{M}_g(Y,\beta) & & \end{array}$$

denote the universal family of curves and the universal map.

- For every point $[u : C \to Y] \in \overline{M}_g(Y, \beta)$ we have two vector spaces

$$H^0(C, u^* T_Y) \quad \text{and} \quad H^1(C, u^* T_Y).$$

- Let u runs over every point, they glue to be

$$\pi_* g^* T_Y \quad \text{and} \quad R^1 \pi_* g^* T_Y$$

called sheaves over $\overline{M}_g(Y, \beta)$.

$H^i(C, u^*T_Y)$

- $H^0(C, u^*T_Y)$ is the space of infinitesimal deformations of u where C is fixed;
- $H^1(C, u^*T_Y)$ contains the obstructions of deforming u while C is fixed.

$H^1(C, u^*T_Y)$

For $T \subset T'$ a small extension of Artinian local scheme over $\mathbb{C}$ and

$$\begin{array}{ccc} T & \longrightarrow & \overline{M}_g(Y,\beta) \\ \downarrow & & \downarrow \\ T' & \longrightarrow & \overline{M}_g \end{array} \tag{2.1}$$

there exists an (obstruction)

$$o(T,T') \in H^1(C, u^*T_Y) \otimes_{\mathbb{C}} I_{T/T'},$$

$o(T,T') = 0$ if and only if (2.1) admits lift $T' \to \overline{M}_g(Y,\beta)$.

- If $T = \text{Spec } \mathbb{C}[\epsilon]/\epsilon^n \subset T' = \text{Spec } \mathbb{C}[\epsilon]/\epsilon^{n+1}$, $o(T, T') \in H^1(C, u^* T_Y)$ is called curvilinear obstructions.

- The set of curvilinear obstructions forms a cone $\mathfrak{C} \subset \mathcal{O}b$.

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- $\overline{M}_g =$ moduli space of nodal curves of genus g.
- Forgetful map

$$\overline{M}_g(Y, \beta) \to \overline{M}_g$$

- The deformation theory of $\overline{M}_g(Y, \beta)/\overline{M}_g$ admits its obstruction lies in $\mathcal{O}b := R^1\pi_* g^* T_Y$,
- B-F:The collection of "curvi-linear" obstructions form a cone

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- Define the virtual fundamental class of $\overline{M}_g(Y,\beta)$

$$[\overline{M}_g(Y,\beta)]^{vir} := 0^!([\mathfrak{C}]) \in A_*(\overline{M}_g(Y,\beta)),$$

  the Gysin map of $\mathfrak{C} \subset \mathcal{O}b$.

- The Gromov Witten invariant are defined to be intersections of special classes with $[\overline{M}_g(Y,\beta)]^{vir}$.

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Deformation invariance

The homology class of $\overline{M}_{g,n}(Y,\beta)^{vir}$ and hence the GW invariant is independent of holomorphic deformation of Y.

- The moduli of maps with n marked points is

$$\overline{M}_{g,n}(Y,\beta) = \{u : C \to Y | u \in \overline{M}_g(Y,\beta), \\ p_1, p_2, \cdots, p_n : \text{ smooth pts in } C\}$$

-

$$[\overline{M}_{g,n}(Y,\beta)]^{vir} \in A_k \overline{M}_{g,n}(Y,\beta).$$

- There is evaluation map

$$ev_i : \overline{M}_{g,n}(Y,\beta) \longrightarrow Y$$

sending $ev_i(u) = u(p_i)$.

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- Let $b_1, \cdots, b_n \in H^*(Y, \mathbb{Z})$ be Poincare dual of submanifolds $M_1, \cdots M_n \subset Y$.
- $$< b_1, .., b_n >_g^{(Y,\beta)} := \int_{\overline{M}_{g,n}(Y,\beta)^{vir}} ev_1^* b_1 \cdots ev_n^* b_n$$
  is the number of genus g curve "in" Y that represents β and touch $M_1, \cdots, M_n$.

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Questions about $\mathfrak{C} \subset \mathcal{O}b$

- Q1: What do we mean by a cone "lies in" a sheaf?
- Q2: How is the cone $\mathfrak{C} \subset \mathcal{O}b$ constructed?
- resolution of $\mathcal{O}b$ in [L-T]; cotangent complex used in [B-F] ; treat sheaf as a stack in [C-L] .

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If X is a Calabi-Yau threefold then the virtual dimension of $\overline{M}_g(X,\beta)$ is

$$\int_\beta c_1(X) + (n-3)(1-g) = 0.$$

Hence $[\overline{M}_g(X,\beta)]^{vir} \in H_0(\overline{M}_g(X,\beta),\mathbb{Q})$ and its degree is a rational number:

$$N_{g,\beta}(X) := \deg[\overline{M}_g(X,\beta)]^{vir} \in \mathbb{Q}.$$

Starting from Witten's path-integral treatment, the power series

$$F_g := \sum_\beta N_{g,\beta}(X) q^\beta$$

and

$$\sum_g F_g \lambda^{2g-2} = \sum_{\beta,g} N_{g,\beta}(X) q^\beta \lambda^{2g-2}$$

are related to many subjects: Variation of Hodge structure, Gauge theory, Analytic torsion,
via many conjectures: Gopakuma Vafa conjecture, Mirror Conjecture,... and hidden in many physics theories waiting for math version to be found: Matrix model, Gauged Linear Sigma Model, $g > 0$ B model, other string theories or M theory,....

CY three fold and hyperplane property:

Let $s = x_1^5 + \cdots + x_5^5 \in \Gamma(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(5))$ and

$$(s = 0) = Q \subset \mathbb{P}^4$$

is the Quintic threefold. One can show $K_Q \cong \mathscr{O}_Q$ and Q is a Calabi-Yau threefold. This is a typical example of CY 3fold.

Let

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathfrak{u}} & \mathbb{P}^4 \\ \downarrow{\scriptstyle \pi} & & \\ \overline{M}_g(\mathbb{P}^4, \beta) & & \end{array}$$

be the universal family of $\overline{M}_g(\mathbb{P}^4, \beta)$. There is

$$\overline{M}_g(Q, d) \subset \overline{M}_g(\mathbb{P}^4, d)$$

- For every point $[u : C \to \mathbb{P}^4] \in \overline{M}_g(\mathbb{P}^4, \beta)$ we have two vector spaces

$$H^0(C, u^*\mathscr{O}_{\mathbb{P}^4}(5)) \quad \text{and} \quad H^1(C, u^*\mathscr{O}_{\mathbb{P}^4}(5)).$$

- Let u runs over every point, they glue to be

$$V := \pi_* \mathfrak{u}^*\mathscr{O}_{\mathbb{P}^4}(5) \quad \text{and} \quad V' := R^1\pi_* \mathfrak{u}^*\mathscr{O}_{\mathbb{P}^4}(5)$$

called sheaves over $\overline{M}_g(Y, \beta)$.

By Riemann Roch theorem we have

$$\begin{aligned}
&\chi(C, u^*\mathscr{O}_{\mathbb{P}^4}(5)) \\
= \quad & \dim H^0(C, u^*\mathscr{O}_{\mathbb{P}^4}(5)) - \dim H^1(C, u^*\mathscr{O}_{\mathbb{P}^4}(5)) \\
= \quad & 5d + 1 - g.
\end{aligned}$$

For every point $[u : C \to \mathbb{P}^4] \in \overline{M}_g(\mathbb{P}^4, \beta)$ the section

$$s = x_1^5 + \cdots + x_5^5 \in \Gamma(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(5))$$

induces

$$u^* s \in H^0(C, u^* \mathscr{O}_{\mathbb{P}^4}(5))$$

such that

$$u^* s = 0 \text{ if and only if } u \in \overline{M}_g(Q, \beta).$$

We denote $\tilde{s}(u) = u^* s$ then

$$\tilde{s} \in \Gamma(\overline{M}_g(\mathbb{P}^4, \beta), \pi_* \mathfrak{u}^* \mathscr{O}_{\mathbb{P}^4}(5)) = \Gamma(\overline{M}_g(\mathbb{P}^4, \beta), V)$$

then $\tilde{s}$ is a section of V and its zero loci is $\overline{M}_g(Q, \beta)$.

- We should think $\tilde{s}$ measures how far a $[u : C \to \mathbb{P}^4]$ is being obstructed to become $u \in \overline{M}_g(Q, \beta)$.
- in this thinking, V is where the obstructions takes value in; we briefly say

$$V = \text{the obstructions of } \overline{M}_g(Q, \beta) \text{ relative to } \overline{M}_g(\mathbb{P}^4, \beta).$$

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When $g = 0$

- $\overline{M}_0(\mathbb{P}^4, \beta)$ is smooth (as an orbifold) of dimension $5d + 1$;
- If $g = 0$, $H^1(C, u^*\mathscr{O}_{\mathbb{P}^4}(5)) = 0, \forall u$. Hence V is a bundle over $\overline{M}_0(\mathbb{P}^4, \beta)$ of rank $5d + 1$.

Rough reason: $H^1(C, u^*\mathscr{O}_{\mathbb{P}^4}(5)) = H^0(C, \omega_C \otimes u^*\mathscr{O}_{\mathbb{P}^4}(-5))$ vanishes by tail reduction argument.

When $g = 0$

- We have smooth compact $\mathcal{M} := \overline{M}_0(\mathbb{P}^4, \beta)$ of dimension $5d + 1$, a $rk = 5d + 1$ bundle V over $\mathcal{M}$ with a section $\tilde{s}$ whose zero loci is $\overline{M}_0(Q, \beta)$.
- This is similar to Kuranishi model.
- Kontseviech says, as classes in $\overline{M}_0(\mathbb{P}^4, \beta)$

$$[\overline{M}_0(Q, \beta)]^{vir} = e(V).$$

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$g = 0$: Proof

From $\tilde{s} \in \Gamma(\mathcal{M}, V)$ and denote $X := (\tilde{s} = 0) = \overline{M}_0(Q, \beta)$ one has

$$0 \longrightarrow T_{X/\mathcal{M}_g} \longrightarrow T_{\mathcal{M}/\mathcal{M}_g}|_X \xrightarrow{d\tilde{s}|_X} V|_X \xrightarrow{q} \mathcal{O}b_{X/\mathcal{M}_g} \longrightarrow 0.$$

$$\tilde{\mathfrak{C}} := lim_{t\to\infty} t \cdot \Gamma_{\tilde{s}}$$

$$e(V) = 0^!(\mathfrak{C}).$$

Why is $0^!(\mathfrak{C}) = [X]^{vir}$?

Reason

$$0 \longrightarrow T_Q \longrightarrow T_{\mathbb{P}^4}|_Q \longrightarrow \mathscr{O}(5) \longrightarrow 0.$$

For any $[u : C \to Q] \in \overline{M}_0(Q, \beta)$, there are long exact sequences

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H^0(u^* T_Q) & \longrightarrow & H^0(u^* T_{\mathbb{P}^4}) & \longrightarrow & H^0(u^* \mathscr{O}(5)) & \longrightarrow & H^1(u^* T_Q) & \longrightarrow & 0 \\
 & & \downarrow \wr\wr & & \downarrow \wr\wr & & \downarrow \wr\wr & & \downarrow \wr\wr & & \\
0 & \longrightarrow & T_{X/\mathcal{M}_g}|_u & \longrightarrow & T_{\mathcal{M}/\mathcal{M}_g}|_u & \xrightarrow{d\tilde{s}|_X} & V|_u & \xrightarrow{q} & \mathcal{O}b_{X/\mathcal{M}_g}|_u & \longrightarrow & 0
\end{array}$$

Using deformation theory, one checks

$$\tilde{\mathfrak{C}} = q^* \mathfrak{C}.$$

Troubles when $g > 0$

- $\overline{M}_1(\mathbb{P}^4, \beta)$ is compact, but not smooth;
- $H^1(C, u^*\mathscr{O}(5))$ can be nonzero, hence $\tilde{V}$ is not a bundle; $e(\tilde{V})$ makes nosense.

Question

When $g > 0$, with such troubles, can still represent

$$[\overline{M}_g(Q,\beta)]^{vir}$$

in terms of

$$\overline{M}_g(\mathbb{P}^4,\beta)?$$

key difficulty in math

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(u^*T_Q) & \longrightarrow & H^0(u^*T_{\mathbb{P}^4}) & \longrightarrow & H^0(u^*\mathscr{O}(5)) & \longrightarrow & H^1(u^*T_Q) & \longrightarrow \\
 & & \downarrow || & & \downarrow || & & \downarrow || & & \downarrow || & \\
0 & \longrightarrow & T_{X/\mathcal{M}_g}|_u & \longrightarrow & T_{\mathcal{M}/\mathcal{M}_g}|_u & \xrightarrow{d\tilde{s}|_X} & V|_u = \mathcal{O}b_{X/\mathcal{M}}|_u & \xrightarrow{q} & \mathcal{O}b_{X/\mathcal{M}_g}|_u & \longrightarrow
\end{array}$$

is prolonged to

$$\begin{array}{ccccccc}
\longrightarrow & H^1(u^*T_{\mathbb{P}^4}) & \longrightarrow & H^1(u^*\mathscr{O}(5)) & \longrightarrow & 0 \\
 & \downarrow || & & \downarrow || & & \\
\longrightarrow & \mathcal{O}b_{\mathcal{M}/\mathcal{M}_g}|_u & \longrightarrow & \mathcal{O}b'_{\mathcal{M}/\mathcal{M}_g}|_u & \longrightarrow & 0.
\end{array}$$

What is, and how can we do with,

$$\mathcal{O}b_{X/\mathcal{M}_g}|_u \quad \text{and} \quad \mathcal{O}b'_{\mathcal{M}/\mathcal{M}_g}|_u ?$$

What is, and how can we do with, $\mathcal{O}b_{X/\mathcal{M}_g}|_u$ and $\mathcal{O}b'_{\mathcal{M}/\mathcal{M}_g}|_u$?

The first time I met this question is end of 2007, (4 yrs ago), when J. Li and I want to modify hyperplane property to compute $g = 1$ GW of quintic. We tried to cut $\overline{M}_1(\mathbb{P}^4, d)$ but failed.

I held the tiny and only hope, that physicists already know this. It turns out the hope is somehow true, incidentally, only after one can first build a bridge to drag a different subject, LG model, into algebraic geometry.

The new subject

Fan-Jarvis-Ruan-Witten theory: GW theory of singularities.

A singular space is

a Landau Ginzburg space (X, W) where X is an complex manifold and $W : X \to \mathbb{C}$ a holomorphic function with

$$(dW = 0) \text{ is compact } .$$

W is called “superpotential".

this section delay to 2nd talk