MAPPING CLASS GROUP INVARIANTS FROM FACTORIZABLE HOPF ALGEBRAS

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Plan

Contents :

- A distinguished Hopf algebra object K in a class of monoidal categories
- A family of Frobenius algebra objects F^{ω} in a class of monoidal categories

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- Motivation :
  Invariants as correlation functions in not necessarily semisimple full CFT
- Tools : Factorizable ribbon Hopf algebras and coends

[JF, Carl Stigner, Christoph Schweigert]

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for a class of *finite braided* monoidal categories $\mathcal{C}$:

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  - a projective representation $\pi_{g,m}$ of Map_Σ on $\mathrm{Hom}_{\overline{\mathcal{C}} \boxtimes \mathcal{C}}(K^{\otimes g}, (F^\omega)^{\otimes m})$
  - an invariant $Cor_\Sigma \in \mathrm{Hom}_{\overline{\mathcal{C}} \boxtimes \mathcal{C}}(K^{\otimes g}, (F^\omega)^{\otimes m})$:

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- Problem still open for generic categories $\mathcal{C}$ in this class
- Understanding of factorization still widely open
- Results for subclass : $\boxed{\mathcal{C} = H\text{-mod}_{\mathrm{f.d.}}}$

  for H finite-dimensional factorizable ribbon Hopf $\Bbbk$-algebra

Plan:

Reminders :

- (I) Factorizable Hopf algebras
- (II) Coends

Combine these ingredients to identify

- (III) A coend K with a Hopf algebra structure in a class of categories
- (IV) Representations of mapping class groups

Reminder (I) :

- Hopf algebra $H \equiv (H, m, \eta, \Delta, \varepsilon, \mathrm{s})$ over a field $\mathbb{k}$
- Quasitriangular Hopf algebra (H, R) : R-matrix $R \in H \otimes H$

  $\Delta^{\mathrm{op}} = \mathrm{ad}_R \circ \Delta\,, \qquad (\Delta \otimes \mathrm{id}_H) \circ R = R_{13} \cdot R_{23}\,, \qquad (\mathrm{id}_H \otimes \Delta) \circ R = R_{13} \cdot R_{12}$
- Monodromy matrix :

  $$Q := R_{21} \cdot R \ \in H \otimes H$$
- Ribbon Hopf algebra (H, R, v) : ribbon element $v \in H$

  central , invertible , $\mathrm{s} \circ v = v\,, \qquad \varepsilon \circ v = 1\,, \qquad \Delta \circ v = (v \otimes v) \cdot Q^{-1}$
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- Left integral of H: $\Lambda \in H$ s.t. $m \circ (\mathrm{id}_H \otimes \Lambda) = \Lambda \circ \varepsilon$
- Right cointegral of H: $\lambda \in H^*$ s.t. $(\lambda \otimes \mathrm{id}_H) \circ \Delta = \eta \circ \lambda$

(I) Factorizable Hopf algebras

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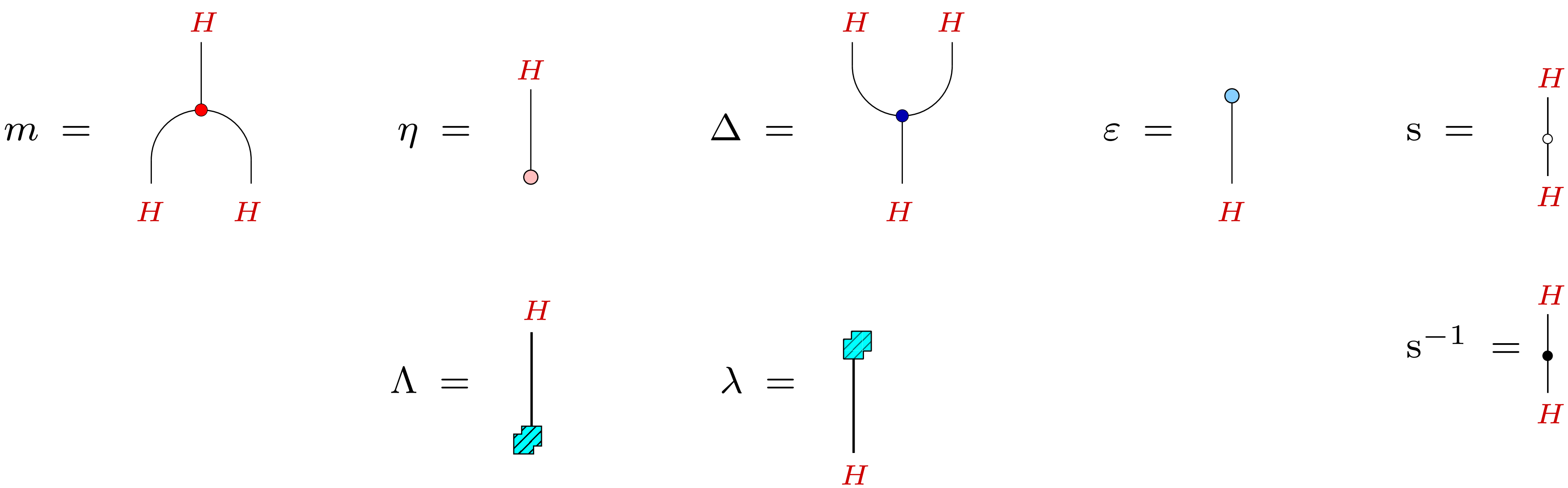
- *Graphical calculus* in the tensor category $\mathcal{V}ect_{\Bbbk}$ (strictified)

  — diagrams read from bottom to top —



Reminder (II) :

- Dinatural transformation

  Given a functor $G\colon\ \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{E}$ and an object $B \in \mathcal{E}$

  a dinatural transformation $G \Rightarrow B$

  is a family of morphisms $\varphi_X\colon\ G(X,X) \to B$ s.t.

$$\begin{array}{ccc} G(Y,X) & \xrightarrow{\ G(\mathrm{id}_Y,f)\ } & G(Y,Y) \\ {\scriptstyle G(f,\mathrm{id}_X)}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_Y} \\ G(X,X) & \xrightarrow{\quad \varphi_X \quad} & B \end{array}$$

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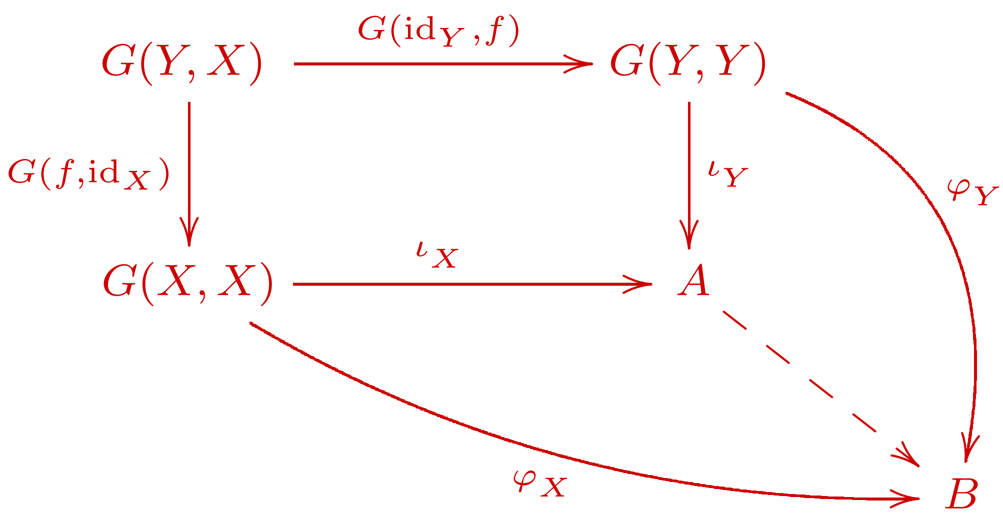
- Example : $G:\ \ \mathcal{V}ect_{\Bbbk}^{\mathrm{op}}\times\mathcal{V}ect_{\Bbbk}\to\mathcal{V}ect_{\Bbbk}$, $(V,W)\mapsto V^*\otimes_{\Bbbk}W$ — dinatural transformation $G\Rightarrow\Bbbk$: $\mathrm{tr}:\ G(V,V)\cong\mathrm{End}_{\Bbbk}(V)\to\Bbbk$

- Example : $G:\ \ \mathcal{D}^{\mathrm{op}}\times\mathcal{D}\to\mathcal{D}$, $(X,Y)\mapsto X^\vee\otimes Y$ — dinatural transformation $G\Rightarrow\mathbf{1}_{\mathcal{D}}$: $d:\ G(X,X)\to\mathbf{1}_{\mathcal{D}}$

- *Coend* (A, ι) for G:
  initial object in category
  of dinatural transformations
  $G \Rightarrow -$:



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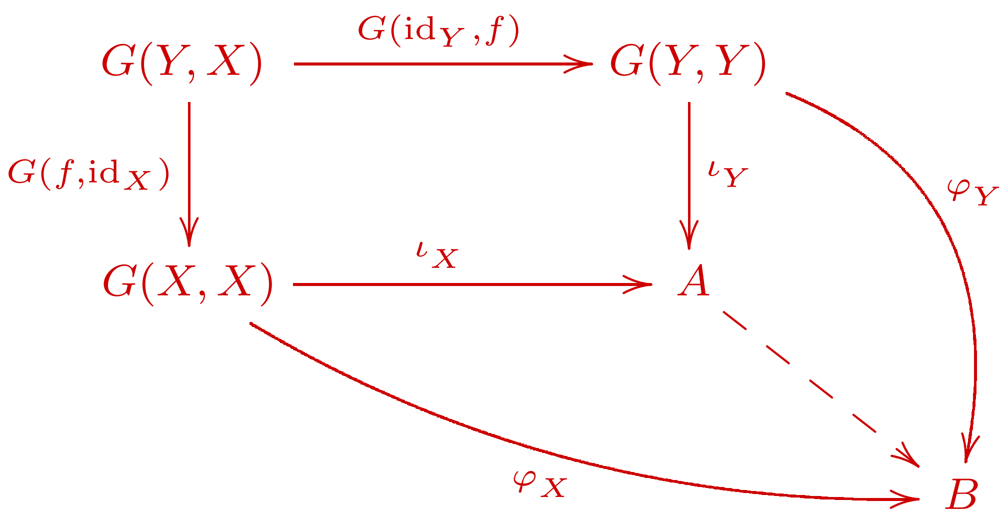

- Notation : $(A,\iota) = \int^X G(X,X)$
- unique up to unique isomorphism (if exists)

- Theorem : $\mathcal{D}$ finite abelian $\mathbb{k}$-linear ribbon category

  $\Longrightarrow$ the coend $L = \int^U U^\vee \otimes U$

  of $G\colon\ \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \ni (U,V) \longmapsto U^\vee \otimes V \in \mathcal{D}$

  exists and carries a natural structure of a Hopf algebra in $\mathcal{D}$

  [Majid, Lyubashenko, Kerler, ...]

- Special case : $\mathcal{D}$ modular $\Longrightarrow$ $L = \bigoplus_{i\in\mathcal{I}} S_i^\vee \otimes S_i$

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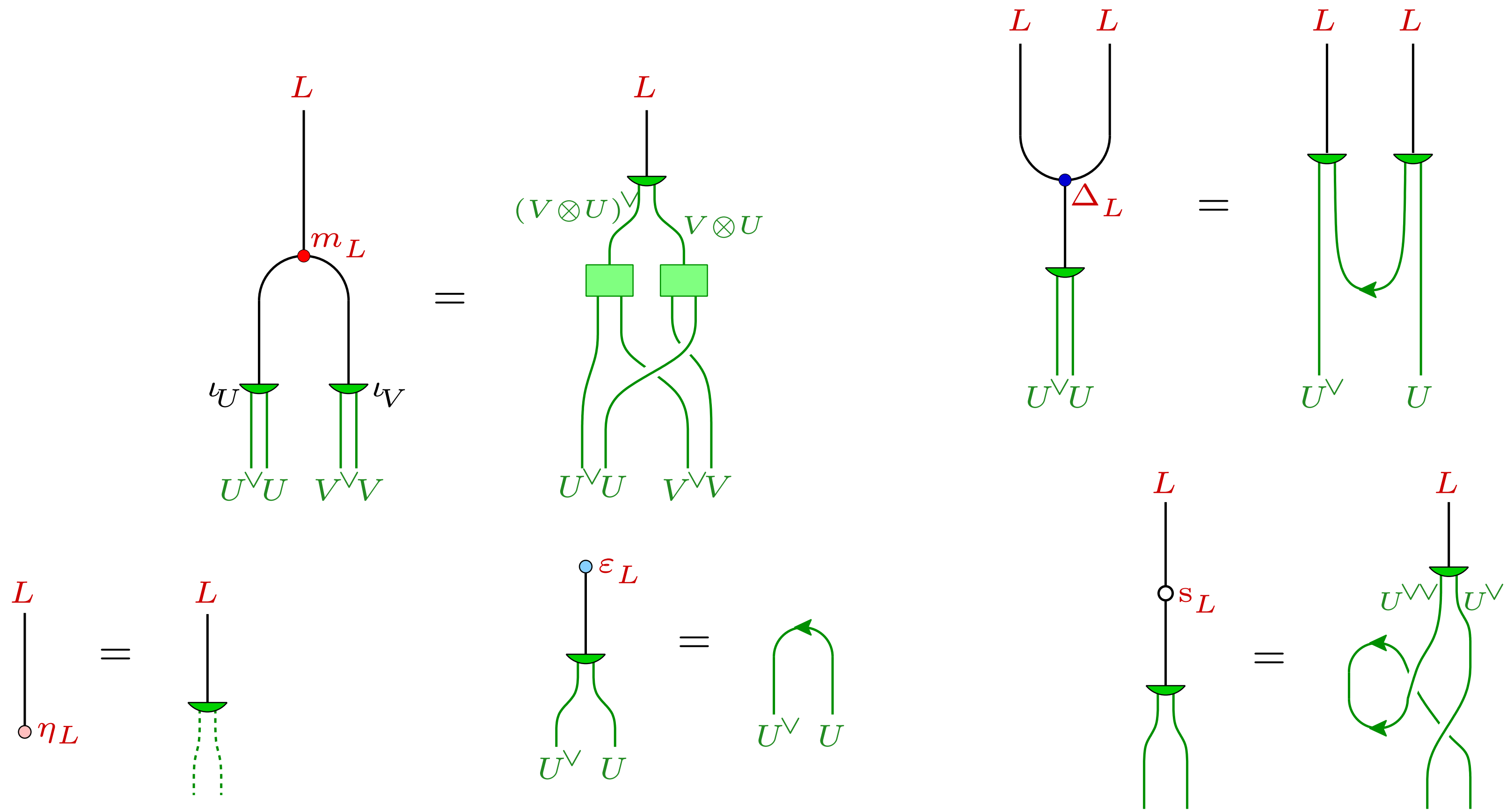

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  - projective rep $\pi_{g,m}$ of Map_Σ on $\mathrm{Hom}_\mathcal{C}(L^{\otimes g}, U_1 \otimes \cdots \otimes U_m)$
  - two-sided integral Λ_L
  - non-degenerate Hopf pairing [Lyubashenko 1995]

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- To give examples need to identify more structure on L

  First to be explained in which context the Hopf algebra L arises

The bulk handle Hopf algebra

- Category of our interest: $\mathcal{D} = \overline{\mathcal{C}} \boxtimes \mathcal{C}$ with $\mathcal{C}$ finite factorizable ribbon category
  - $\mathcal{C} = H$-mod for factorizable ribbon Hopf algebra H
    $\Longrightarrow$ $\overline{\mathcal{C}} \boxtimes \mathcal{C} \simeq H$-bimod (with particular braiding – see below)
  - Braided functor $\overline{\mathcal{C}} \boxtimes \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ (is equivalence for factorizable categories)
  - in CFT bulk fields are objects of $\overline{\mathcal{C}} \boxtimes \mathcal{C}$ ("left and right movers")

- Corollary: The coend $K = \int^X X^\vee \otimes X$

  of $F_\otimes :\ (\overline{\mathcal{C}} \boxtimes \mathcal{C})^{\mathrm{op}} \times (\overline{\mathcal{C}} \boxtimes \mathcal{C}) \ni (X,Y) \longmapsto X^\vee \otimes Y \in \overline{\mathcal{C}} \boxtimes \mathcal{C}$

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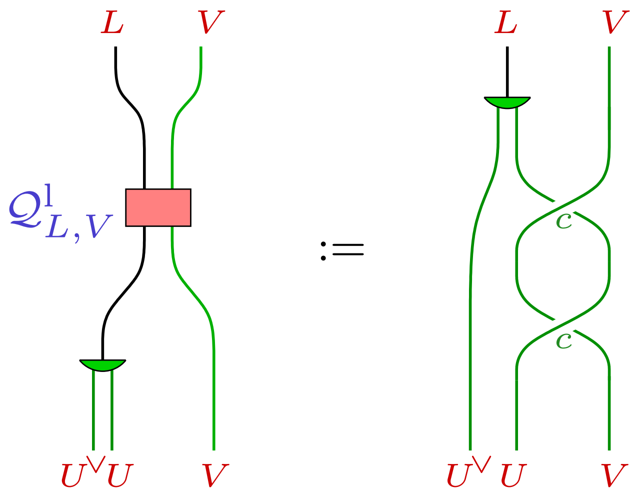
- Terminology : *Bulk handle Hopf algebra* K
  - Punctured torus is Hopf algebra in the bicategory of 3-cobordisms with corners

    [Yetter 1997]

- L Hopf algebra object in factorizable category $\mathcal{C}$
- For every object $V \in \mathcal{C}$: partial monodromy $\mathcal{Q}^{\mathrm{l}}_{L,V} \in \mathrm{End}_{\mathcal{C}}(L \otimes V)$



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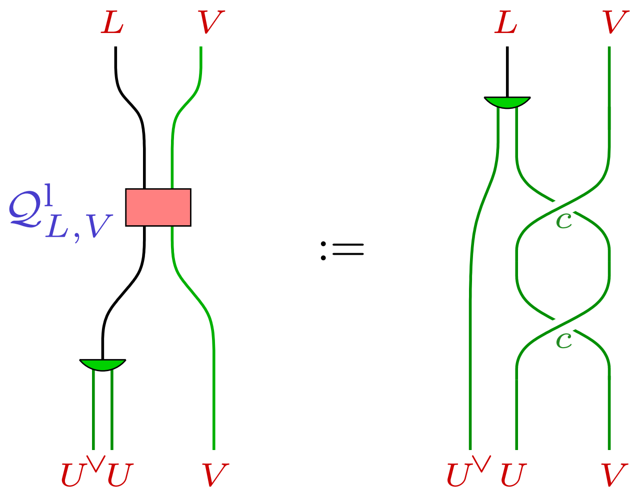

- Propos.: The partial monodromy $\mathcal{Q}^{\mathrm{l}}_{L,V}$
  endows $V \in \mathcal{C}$ with the structure of an L-module (V, ρ^L_V) :

$$\rho^L_V = (\varepsilon_L \otimes V) \circ \mathcal{Q}^{\mathrm{l}}_{L,V}$$

- Remark : extends to a fully faithful embedding $\mathcal{C} \to {}_L\mathcal{YD}^L$

- Theorem 1 : The coend

$$F = \int^U U^\vee \boxtimes U$$

of $\mathcal{C}^{\rm op} \times \mathcal{C} \ni (U,V) \longmapsto U^\vee \boxtimes V \quad \in \overline{\mathcal{C}} \boxtimes \mathcal{C}$

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Main results

- Theorem 1 : For any *ribbon automorphism* ω of $\mathcal{C}$ the coend

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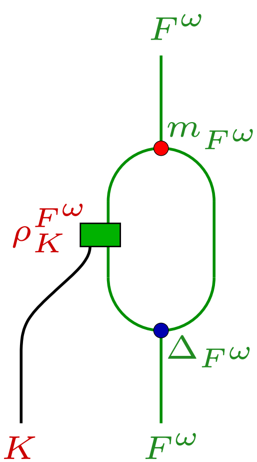
Cor$^\omega_{1;1,1}$ $:=$ 

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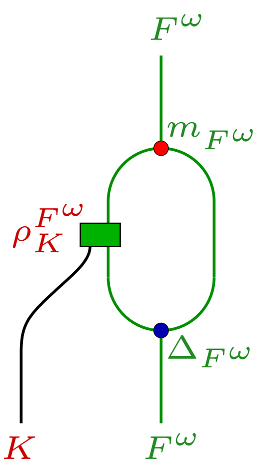

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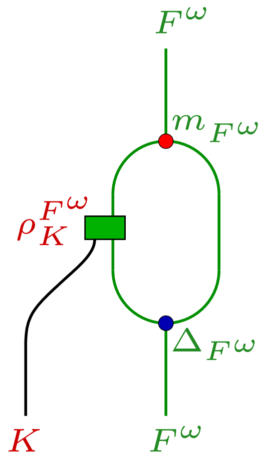

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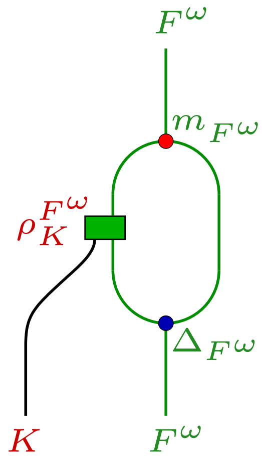

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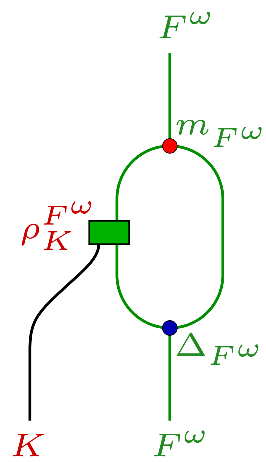
$\mathit{Cor}^{\omega}_{1;1,1} :=$ 

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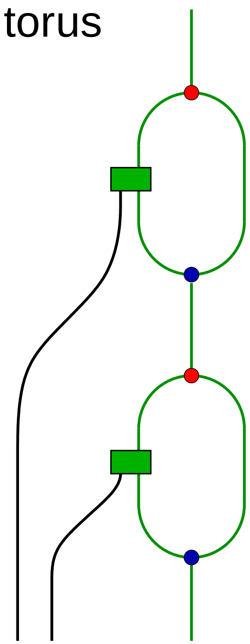
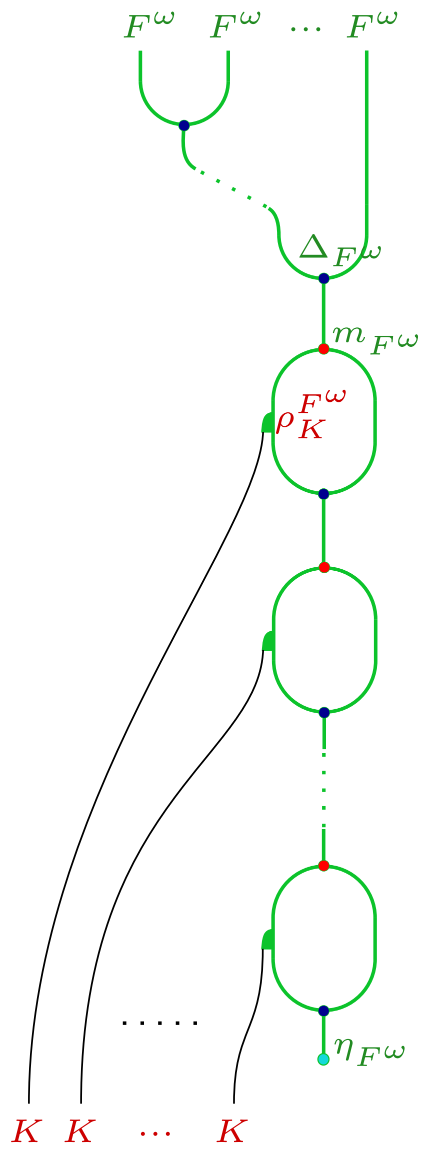
- Generalizes by "composition of handles" to higher genus :

$\mathit{Cor}^{\omega}_{2;1,1} =$ 

General form of the invariant

$\mathit{Cor}^{\omega}_{g;0,n} \;=$



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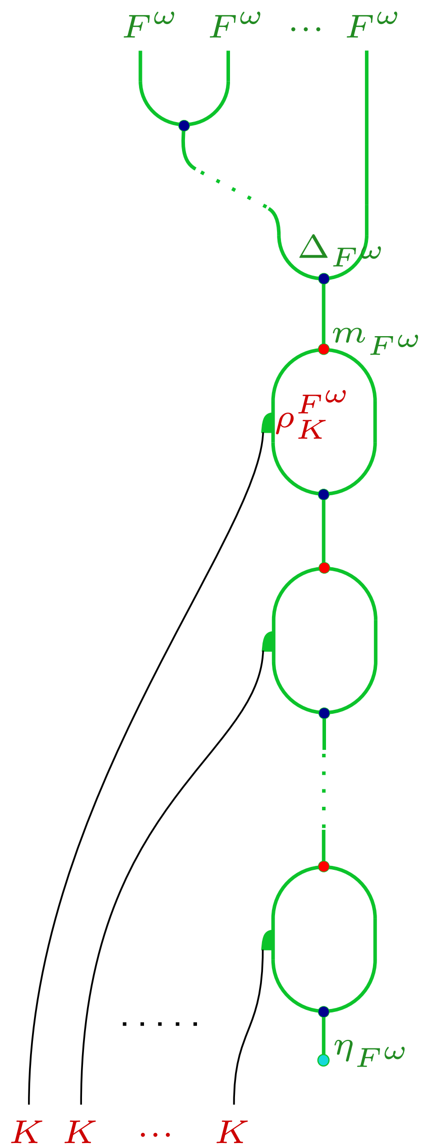

Dream : Generalizes to the relevant class of categories and obeys factorization

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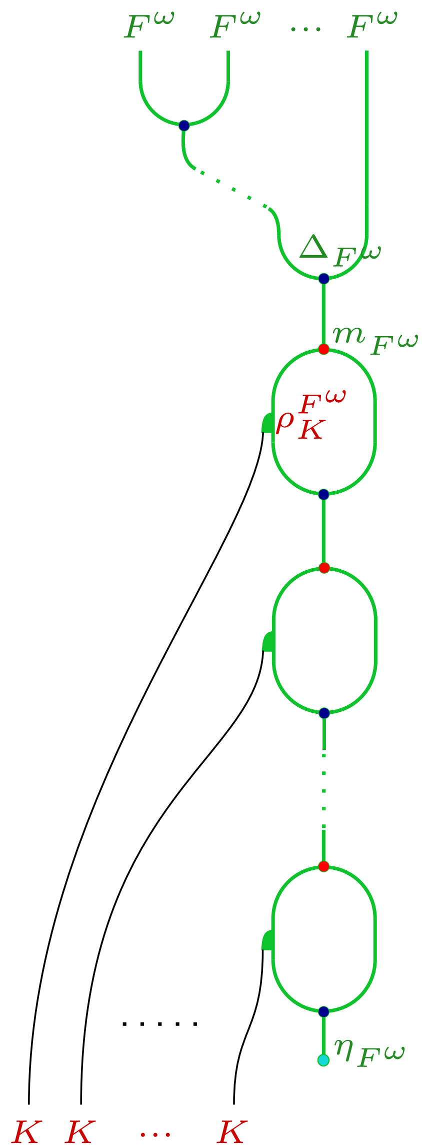
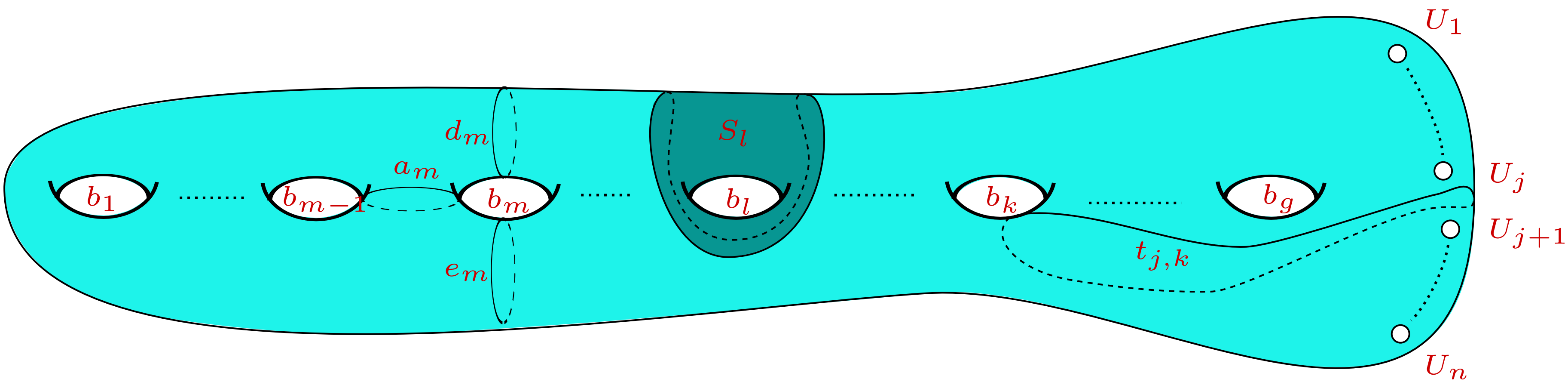


From now on : Details

Dream : Generalizes to the relevant class of categories and obeys factorization

Generators of Map_Σ

- Exact sequence $1 \longrightarrow B_{g;n} \longrightarrow \mathrm{Map}_{g;n} \longrightarrow \mathrm{Map}_{g;0} \longrightarrow 1$
- Thus generated by permutations of holes and Dehn twists



- Relations not needed here

The category H-bimod

- Equivalences of abelian categories :

$$H\text{-mod} \boxtimes H\text{-mod} \;\simeq\; (H {\otimes_{\Bbbk}} H)\text{-mod} \;\simeq\; (H {\otimes_{\Bbbk}} H^{\rm op})\text{-mod} \;\simeq\; H\text{-bimod}$$

- Extends to braided monoidal equivalence $\overline{H}\text{-mod} \boxtimes H\text{-mod} \;\simeq\; H\text{-bimod}$

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- Natural monoidal structure on the category H-bimod :

  pull back H-actions along coproduct

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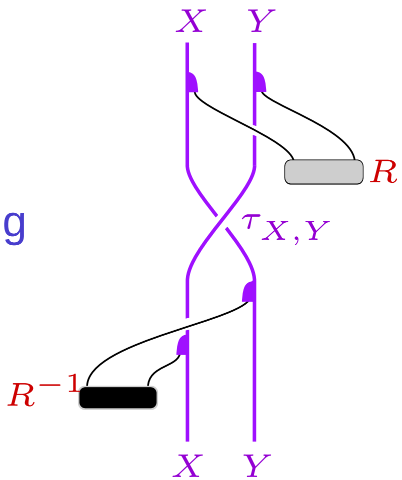
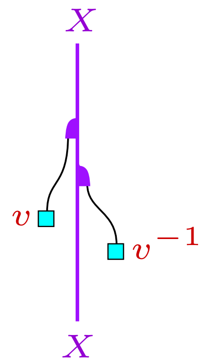
braiding



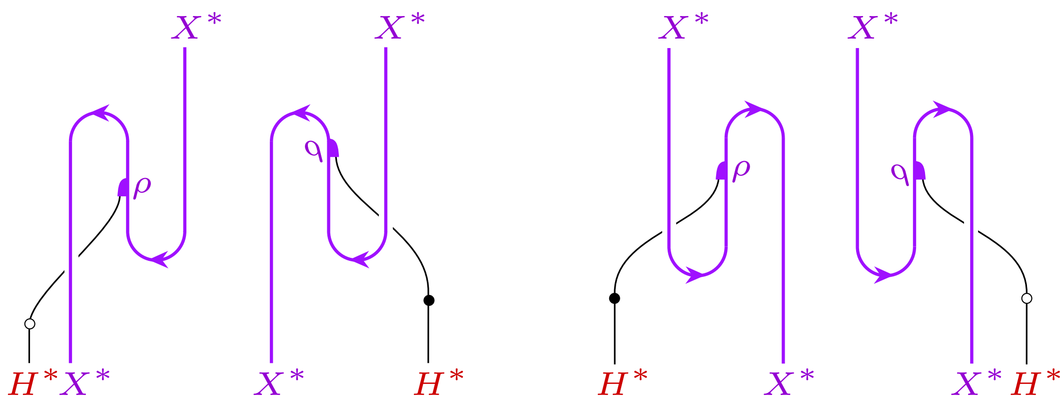
twist



WARNING : PICTURES DRAWN IN $\mathcal{V}ect_\Bbbk$

right and left dual

of H-bimodule X



The bulk handle Hopf algebra

- Bulk handle Hopf algebra K as an object of H-bimod $\simeq \overline{H}$-mod $\boxtimes H$-mod :
  - $K = H^* \otimes_{\Bbbk} H^*$ as vector space
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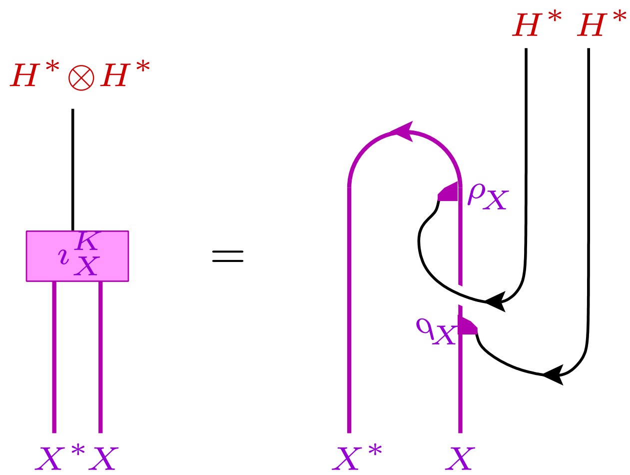
with dinatural family



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$$m_K = \ldots \qquad \mathrm{s}_K = \cdots$$

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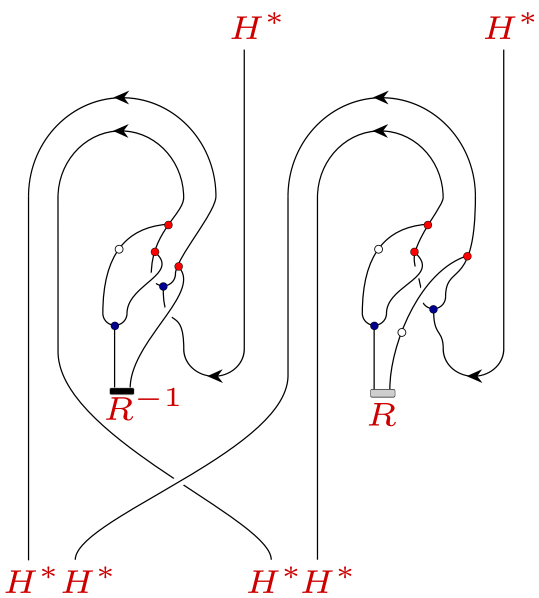
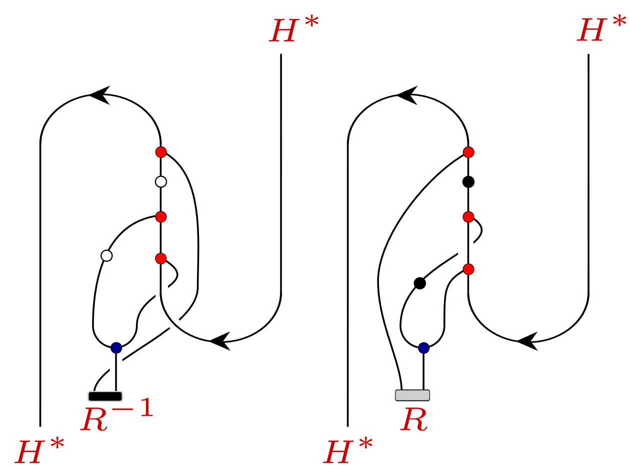
$m_K =$ 

$\mathrm{s}_K =$ 

The bulk Frobenius algebra

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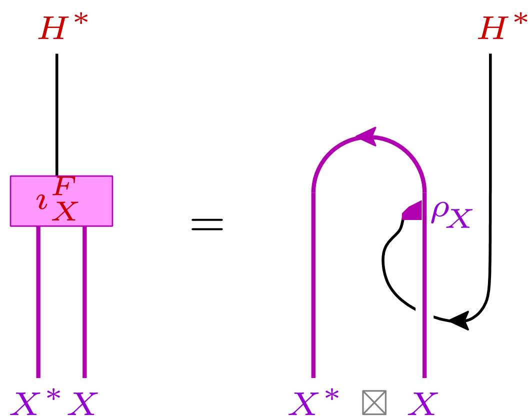

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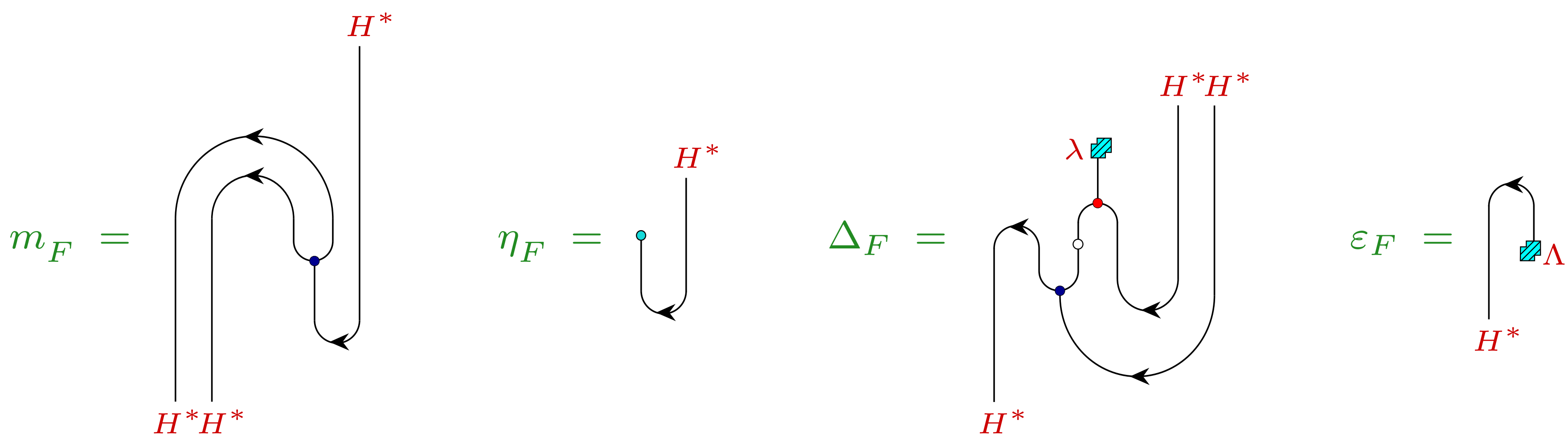
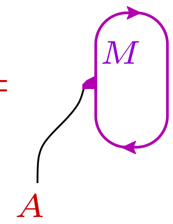


The Cardy-Cartan invariant

- Character : A associative algebra, M module

$\chi_M^A =$  $\in \mathrm{Hom}_{\mathcal{C}}(A, \mathbf{1})$

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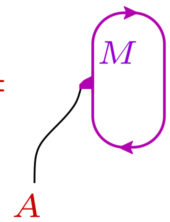
χ_M^A = 

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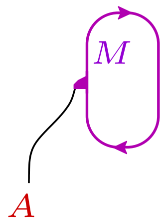
$$\chi_M^A \; = $$



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- Partition function $Z = \mathit{Cor}_{1;0} = \chi_F^K \in \mathrm{Hom}_{\overline{\mathcal{C}} \boxtimes \mathcal{C}}(K, \mathbf{1})$ obeys

$$\boxed{Z = \textstyle\sum_{i,j \in \mathcal{I}} c_{\bar{i},j} \, \chi_i^L \otimes \chi_j^L}$$

with c the Cartan matrix of the category H-mod :

$$c_{ij} = [P_j : S_i] = \dim_{\Bbbk} \mathrm{Hom}_H(P_i, P_j)$$

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  - Δ_F a bimodule morphism **?**

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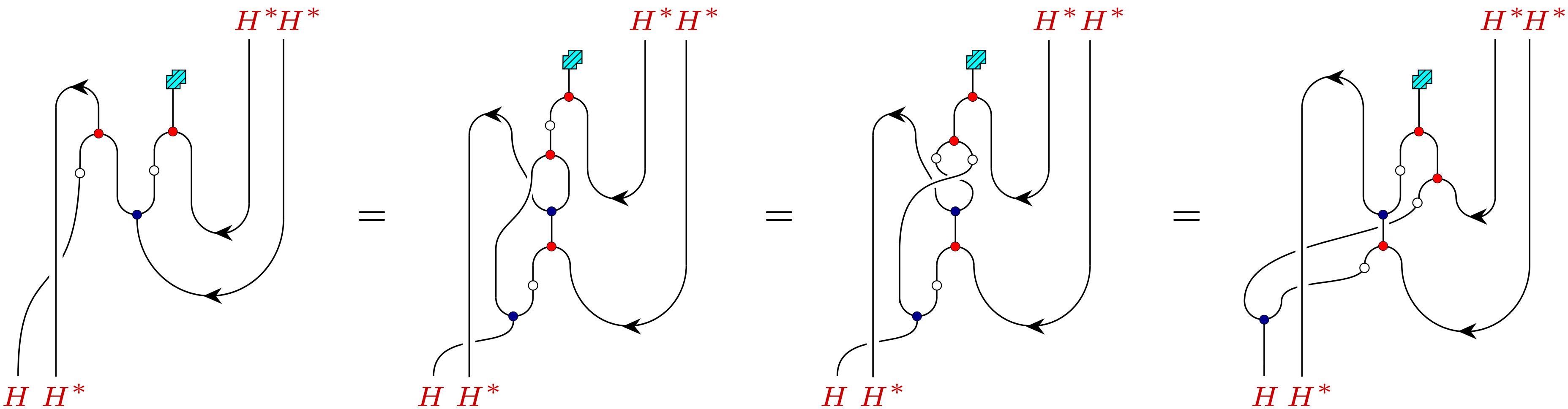
Left module morphism :



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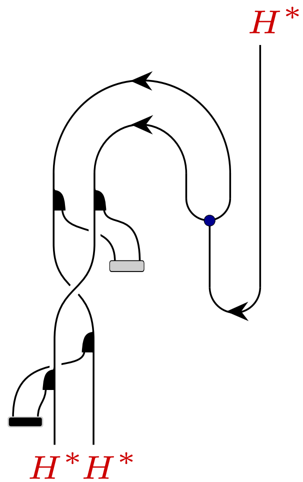
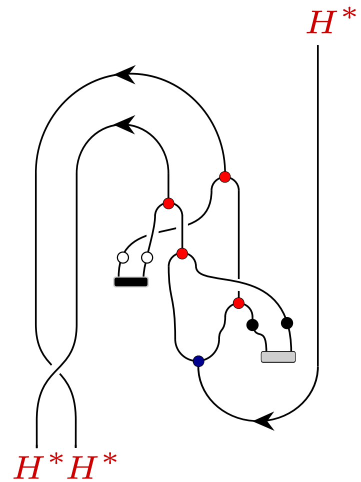
  - m_F commutative **?**

$m_F \circ c_{F,F} =$



$=$



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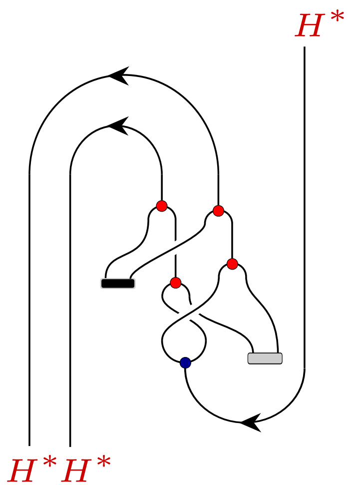

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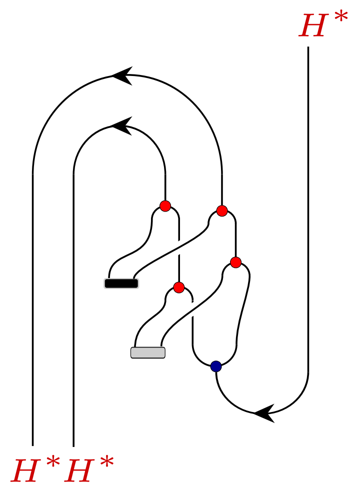

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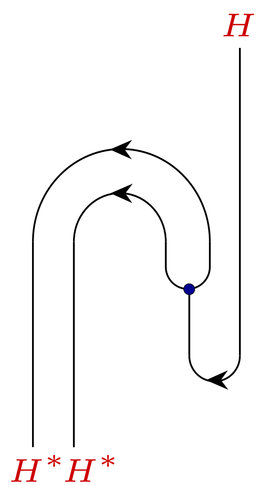

$= m_F$

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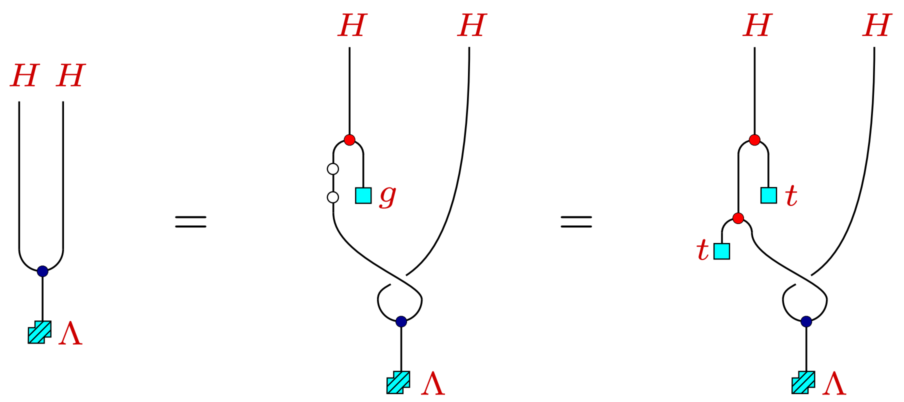
Follows from



[Radford 1992, 1994]

g = right modular element

$t = uv^{-1}$

The bulk Frobenius algebra

- Some sample checks :
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Twist by automorphism

- Generalization :
  - ω Ribbon Hopf algebra automorphism of H

$\Longrightarrow$ Twisted bimodule $F^\omega := {}^{\mathrm{id}_H}(F)^\omega$

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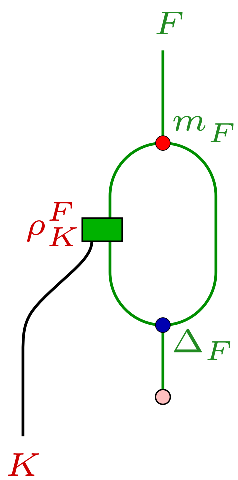
Rewriting the genus-1 invariant

- Invariant for $(0,1)$ -punctured torus :

$Cor_{1;0,1}$ $=$



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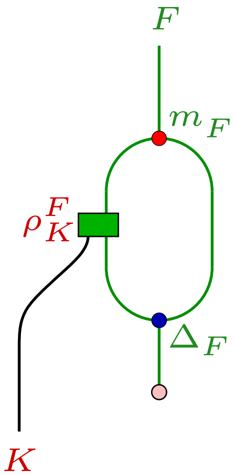
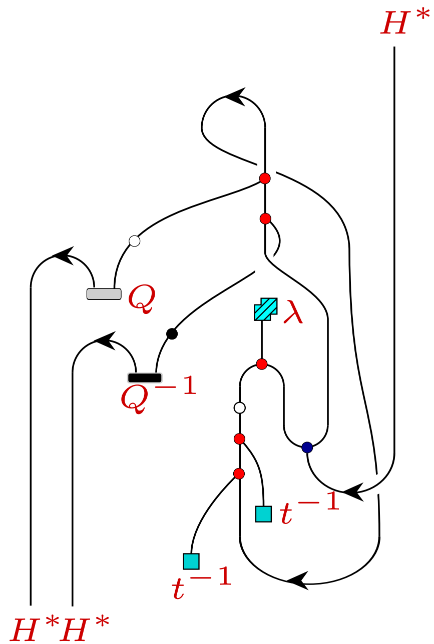
$Cor_{1;0,1}$ =  = 

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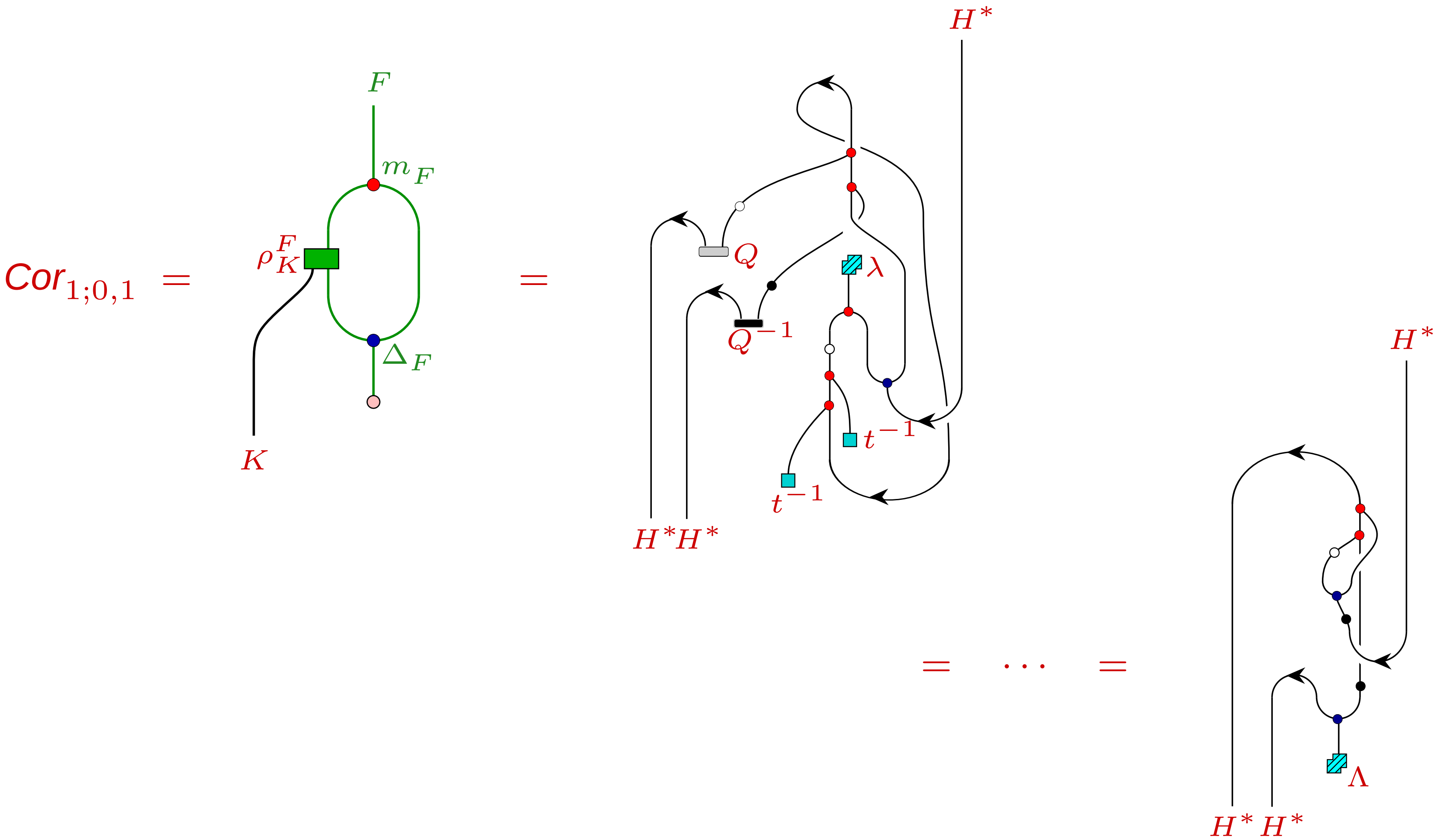
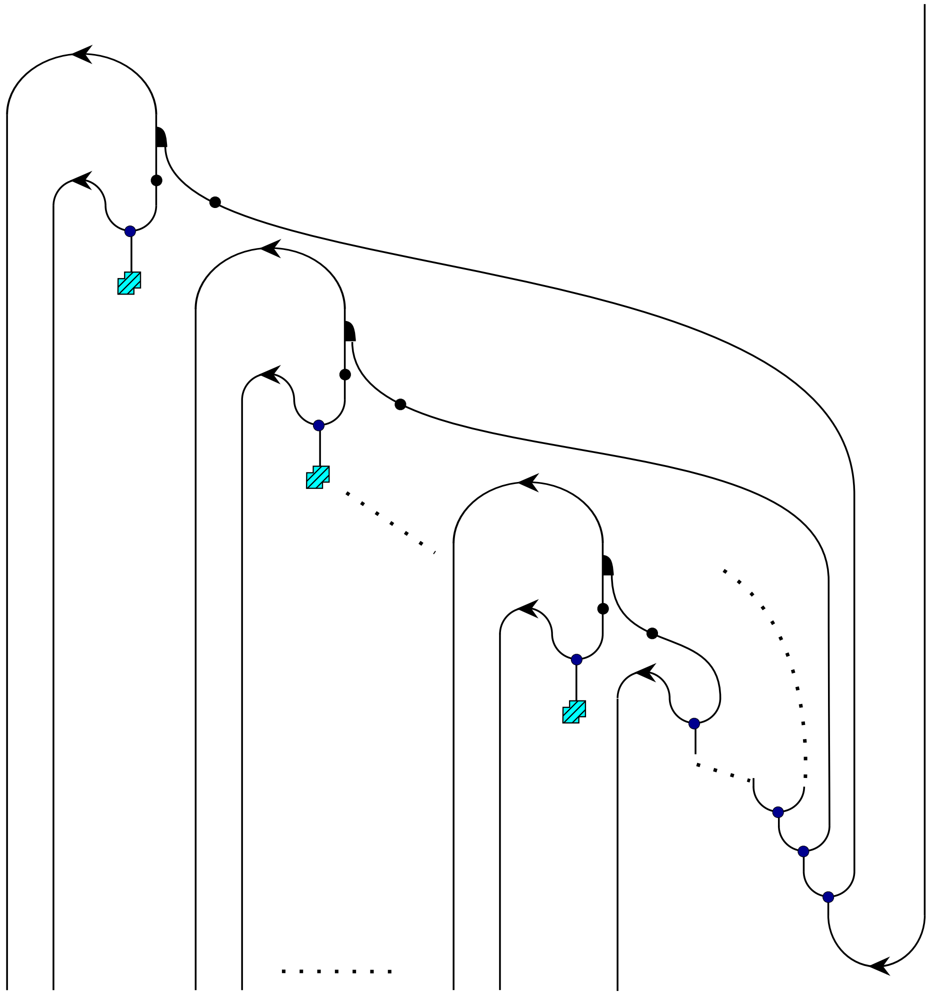


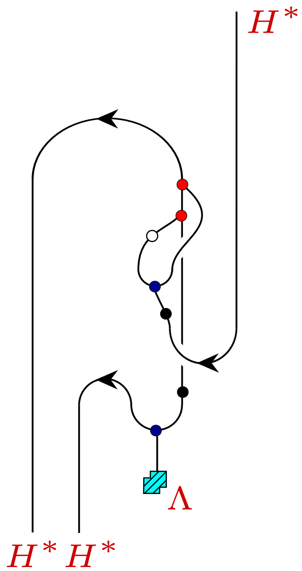
- Generalizes to higher genus :

$\mathit{Cor}_{g;1,1} =$

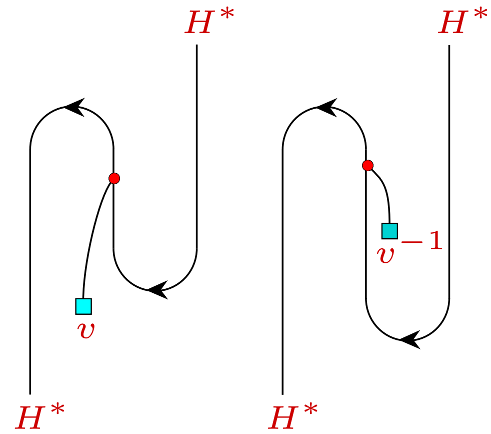


- Mapping class group invariance for $(0{,}1)$ -punctured torus :

$\mathit{Cor}_{1;0,1} \; =$

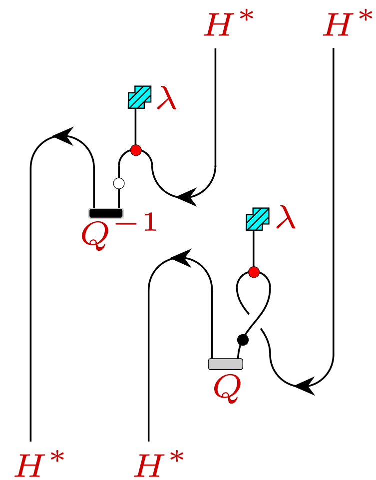


$\pi^{(K)}_{1,1}(T) \; =$



$$\pi^{(K)}_{1,1}(\Theta) = \theta_F = \mathrm{id}_F$$

$\pi^{(K)}_{1,1}(S) \; =$



Invariance of genus-1 correlators

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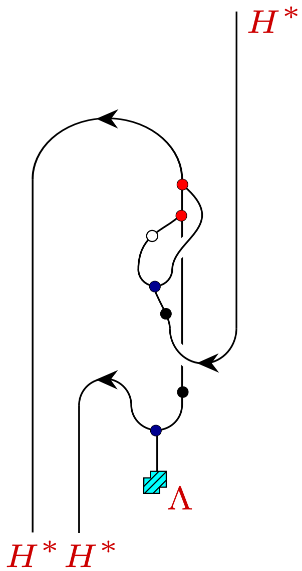
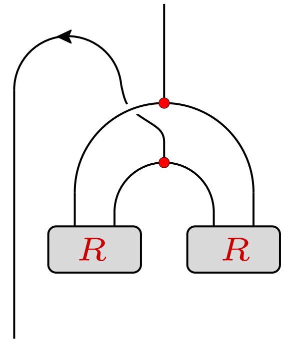
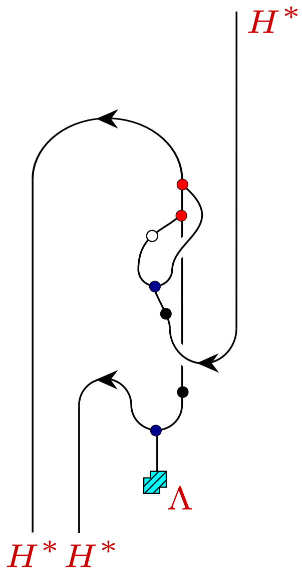


- Drinfeld map : $f_Q \;:=\; (d_H \otimes \mathrm{id}_H) \circ (\mathrm{id}_{H^*} \otimes Q) \;=$  $\in \mathrm{Hom}(H^*, H)$

  - intertwines left coadjoint action and left adjoint action
  - $f_Q(\lambda) = \Lambda = f_{Q^{-1}}(\lambda)$ (with choice of scalars up to ± 1)

- Mapping class group invariance for $(0{,}1)$ -punctured torus :

$\mathit{Cor}_{1;0,1} \;=$



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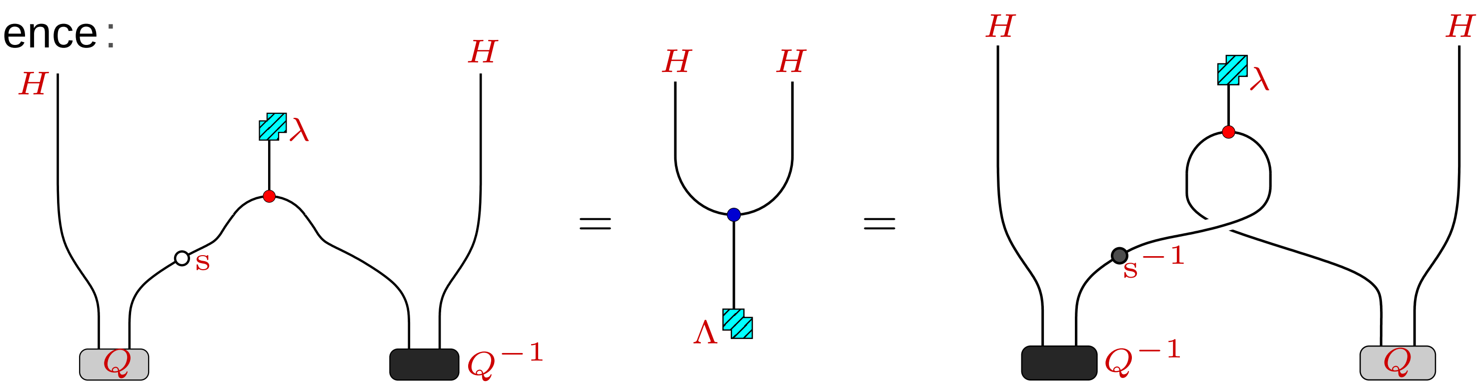

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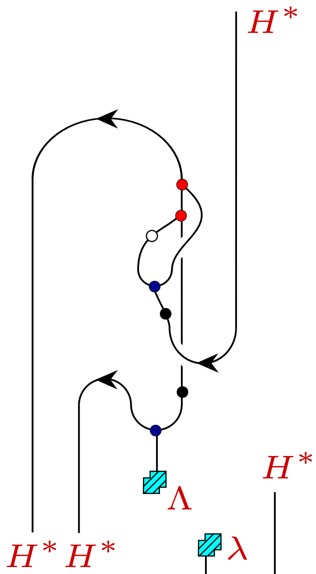
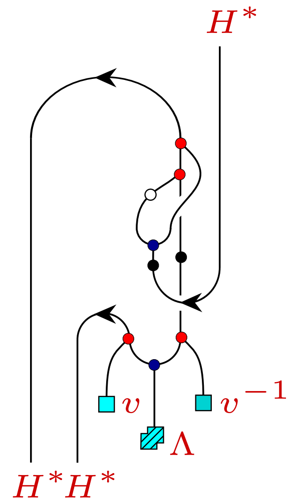


$\mathit{Cor}_{1;0,1} \circ \pi_{1,1}^{(K)}(T) \;=$



$= \;\mathit{Cor}_{1;0,1}$

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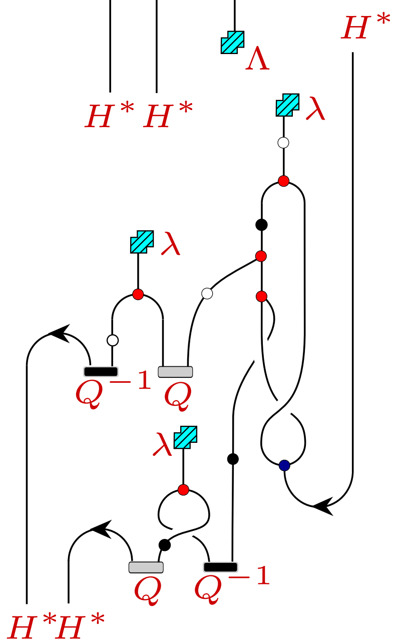

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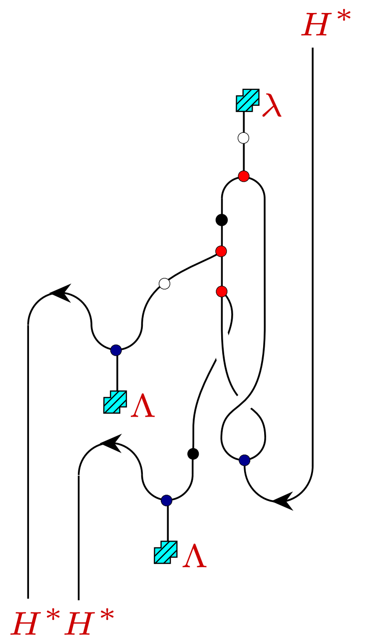

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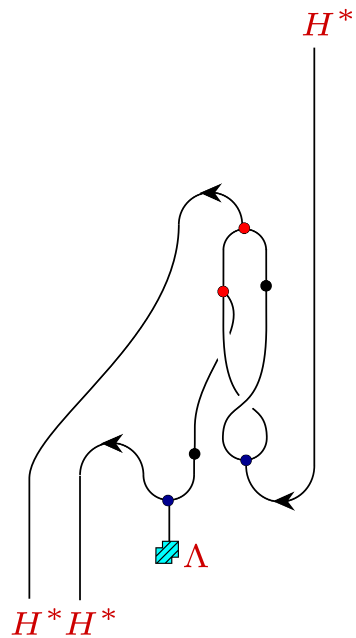

$= \;\mathit{Cor}_{1;0,1}$