S^2 partition functions: Coulomb vs Higgs localization and vortices

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Introduction

Strongly coupled (gauge) quantum field theories ubiquitous in physics:
QCD, beyond the SM particle physics, condensed matter, inflation, . . .

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Other methods:

- Lattice (gauge) theory. Especially useful for numerical simulations.
- Large N expansion of gauge theories.
  Diagrammatic simplifies (only *planar* diagrams at leading order).
  't Hooft coupling $\lambda = Ng^2$. If $\lambda \gtrsim 1$ still a problem.
- AdS/CFT: use dual gravity description.
  Useful at large N and large λ.
- Integrability: exploit infinite number of conserved charges.
- SUSY: often full perturbative + non-perturbative computations exactly.
  Exploit dualities.

We will consider the last approach

Supersymmetry

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...however share many key features with more "conventional" theories

- 4d: confinement & chiral symmetry breaking (*cfr.* QCD)
- 3d: topological sectors (*cfr.* topological insulators)
  Chern-Simons = SUSY Chern-Simons
  Particle / vortex duality
- 2d: statistical models may show "accidental" SUSY (*cfr.* tricritical Ising)
  Particle / kink (soliton) duality

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- S^d partition functions:

Euclidean SUSY theory on S^d (not twisted as in [Witten 88; Vafa, Witten 94])

Compute path-integral: $$Z_{S^d}(t) = \int_{S^d} \mathcal{D}\Phi \; e^{-S[\Phi,t]}$$

Parameters t: from flat space Lagrangian & curved S^d

With enough SUSY, *exactly* computable with localization techniques.

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- Compute VEVs of SUSY operators (*e.g.* line operators) as well:

$$Z_{S^d}(t, \mathcal{O}) = \int_{S^d} \mathcal{D}\Phi\ \mathcal{O}\ e^{-S[\Phi,t]}$$

Examples

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S^4 with $\mathcal{N} = 2$ SUSY [Pestun 07]
S^3 with $\mathcal{N} = 2$ SUSY [Kupustin, Willett, Yaakov 09; Jafferis 10; Hama, Hosomichi, Lee 11]
S^5 with $\mathcal{N} = 1$ SUSY [Hosomici, Seong, Terashima 12; Kallen, Qiu, Zabzine 12; Kim, Kim 12]
S^2 with $\mathcal{N} = (2,2)$ SUSY [FB, Cremonesi 12; Doroud, Gomis, Le Floch, Lee 12]

- Generalizations: *e.g.* squashing of spheres

[Hama, Hosomichi, Lee 11; Imamura, Yokoyama 11;Hama, Hosomichi 12]

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S^5 with $\mathcal{N}=1$ SUSY [Hosomici, Seong, Terashima 12; Kallen, Qiu, Zabzine 12; Kim, Kim 12]
S^2 with $\mathcal{N}=(2,2)$ SUSY [FB, Cremonesi 12; Doroud, Gomis, Le Floch, Lee 12]

- Generalizations: *e.g.* squashing of spheres

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- Other manifolds: *e.g.* $S^{d-1} \times S^1$

Index: $$I(f) = \mathrm{Tr}\,(-1)^F\, e^{-\beta H}\, f_i^{\mathcal{O}_i}$$

4d with $\mathcal{N}=1$ SUSY [Gadde, Gaiotto, Pomoni, Rastelli, Razamat, Yan]
3d with $\mathcal{N}=2$ SUSY [Kim 09; Imamura, Yokoyama 11]
5d with $\mathcal{N}=1$ SUSY [Kim, Kim, Lee 12]

Physical information in $Z_{S^d}(t)$

$Z_{S^d}(t)$ is an interesting function:

- Information about the theory that can be computed exactly (and non-perturbatively) at strong coupling

  Very non-trivial new tests of conjectured dualities
  (4d S-duality, 4d Seiberg duality, Seiberg-like dualities, 3d & 2d mirror symmetry, . . .

  Conformal theories: exact VEVs of (local & non-local) operators $\langle \mathcal{O} \rangle$

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- Dualities $\rightarrow$ interesting (often not-yet-proven) mathematical identities

Physical information in $Z_{S^d}(t)$

- In 3d it provides a "central charge" [Jafferis 10; Jafferis, Klabanov, Pufu, Safdi 11] that decreases from fixed point to fixed point along RG flows

$$c_{\mathsf{3d}} = Z_{S^3}(t = t_{\mathsf{conf}})$$

No conformal anomaly in odd dimensions: $\langle T^\mu_\mu \rangle = 0$

Related to entanglement entropy

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- Very interesting mathematical structures

  AGT [Alday, Gaiotto, Tachikawa 09] 4d $\mathcal{N} = 2$ SUSY (S^4) $\leftrightarrow$ 2d Liouville

  $$Z^{\text{4d}}_{\text{inst}}(q, a, m) = \text{conformal blocks } (q, a, m)$$

  4d $\mathcal{N} = 2$ index $(S^3 \times S^1)$ $\leftrightarrow$ 2d topological theory (YM)
  [Gadde, Rastelli, Razamat, Yan 11; Gaiotto, Rastelli, Razamat 12]

  3d $\mathcal{N} = 2$ SUSY $\leftrightarrow$ 3d Chern-Simons
  [Dimofte, Gaiotto, Gukov 11; Cecotti, Cordova, Vafa 11]

2d theories

- We consider two-dimensional theories:

connection with strings and topological strings;

connection with geometry via non-linear sigma models [Witten 93] .

2d $\mathcal{N} = (2,2)$ SUSY
and a vector-like R-symmetry $U(1)_R$

$\rightarrow$ non-twisted SUSY preserved on S^2

Gauge theory of vector multiplets + chiral multiplets

Admit generic twisted superpotential $\widetilde{W}(\Sigma)$

Localization

Path-integral computed *exactly* with localization techniques.

Works even with certain (BPS) operator insertions (*e.g.* loop operators).

- Supersymmetric action S, and operators $\mathcal{O}$, w.r.t. supercharge $\mathcal{Q}$:

deform path-integral by $\mathcal{Q}$-exact action

$$Z = \int \mathcal{D}\Phi \; \mathcal{O} \; e^{-S-u\,S_{\text{loc}}}$$

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Path-integral does not depend on u.

- In the large u limit, semiclassical approximation becomes *exact*:

$$Z = \sum_{\text{BPS } \Phi_0} e^{-S[\Phi_0]} \; Z_{\text{1-loop}}[\Phi_0]$$

Summary of results

- Z_{S^2} computed with localization techniques

$\rightarrow$ integral over "Coulomb branch", sum over flux sectors:

$$Z_{S^2}(\text{masses}, \text{FI}, \text{R-charges}) = \sum_{\mathfrak{m}} \int d\sigma \; e^{-S_{\text{class}}} \; Z_{\text{vector}}^{\text{1-loop}} \; Z_{\text{chiral}}^{\text{1-loop}}$$

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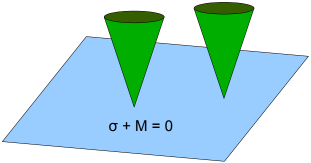
- Properties:
  - Expression very similar to 3d, 4d, 5d
  - Finite dimensional integral and series: easy to compute
  - One can check or conjecture 2d dualities (*e.g.* Hori-Tong)

Summary of results

Surprise: localization can be performed in a *different* way

→ sum over discrete "Higgs branch":

$$Z_{S^2} = \sum_{\text{Higgs vacua}} e^{-S_{\text{class}}} Z_{\text{1-loop}} Z_{\text{vortex}} Z_{\text{anti-vortex}}$$



Z_{vortex}: partition function of vortices on $\mathbb{R}^2_\varepsilon$ in Ω-background [Shadchin 06; Nekrasov 02]

Vortices at north pole, antivortices at south pole of S^2.

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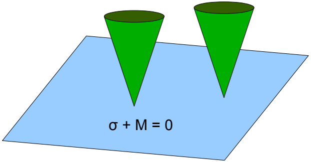

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- Higgs branch expression reminiscent of Pestun's S^4 result in terms of instanton partition function Z_{inst} [Nekrasov 02]
- Factorization as observed on S^3 [Pasquetti 11]
- $Z^{\text{Coulomb}}_{S^2} = Z^{\text{Higgs}}_{S^2}$ can be used to compute Z_{vortex}

Conclusions

- Z_{S^2} can be computed exactly in 2d with localization
- Conformal theories: VEVs computed exactly (*e.g.* Wilson lines)
- Dualities: new checks and new dualities
- The same Z_{S^2} can be written in two very different ways:

  Coulomb *vs* Higgs
- Provides a computationally powerful way to determine vortex partition functions
- Z_{vortex} is related to Gromov-Witten invariants of Kähler manifolds see [Jockers, Kumar, Lapan, Morrison, Romo 12]
- Open questions about similar phenomena in higher dimensions

Part II

Rigid supersymmetry on S^2

- Two-dimensional $\mathcal{N} = (2,2)$ theories with a vector-like $U(1)_R$ R-symmetry can be placed supersymmetrically on S^2 (2 complex supercharges):

$$\mathfrak{osp}^*(2|2) \cong \mathfrak{su}(2|1) \ \supset \ \mathfrak{su}(2) \times \mathfrak{u}(1)_R$$

No twisting!

Contained in global Euclidean superconformal algebra

$$\mathfrak{osp}(2|2, \mathbb{C}) \ \supset \ \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{u}(1)^2$$

Algebra:

$$[\delta_\epsilon, \delta_{\bar{\epsilon}}] = \mathcal{L}^A_\xi + \frac{i}{2r} \alpha R \qquad \xi^\mu = i \bar{\epsilon} \gamma^\mu \epsilon$$

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = 0 \qquad \alpha = i \bar{\epsilon} \epsilon$$

$$D_\mu \epsilon = \frac{i}{2r} \gamma_\mu \epsilon$$

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- Vector multiplet: $(A_\mu, \lambda, \bar\lambda, \sigma, \eta, D)$
  Chiral multiplet: $(\phi, \bar\phi, \psi, \bar\psi, F, \bar F)$

Euclidean signature: fields get complexified.

On S^2 freedom to choose R-charges q of chiral multiplets $\rightarrow$ couplings

Supersymmetric actions on S^2

Action constructed order by order in $\frac{1}{r}$ or by coupling to SUGRA [Festuccia, Seiberg 11]

- Yang-Mills action for vector multiplets:

$$\mathcal{L}_{YM} = \frac{1}{g^2}\,\mathrm{Tr}\left\{\frac{1}{2}\left(F_{12}-\frac{\eta}{r}\right)^2+\frac{1}{2}\left(D+\frac{\sigma}{r}\right)^2+\frac{1}{2}(D_\mu\sigma)^2+\frac{1}{2}(D_\mu\eta)^2-\frac{1}{2}[\sigma,\eta]^2 + \frac{i}{2}\bar{\lambda}\not{D}\lambda + \frac{i}{2}\bar{\lambda}[\sigma,\lambda] + \frac{1}{2}\bar{\lambda}\gamma_3[\eta,\lambda]\right\}$$

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- Twisted superpotential $\widetilde{W}(\Sigma)$

$$\mathcal{L}_{\widetilde{W}} = i\widetilde{W}'\left(D - iF_{12} + \frac{\sigma+i\eta}{r}\right) - \frac{i}{2}\widetilde{W}''\,\bar\lambda(1+\gamma_3)\lambda - \frac{i}{r}\widetilde{W}$$

and its anti-chiral counterpart $\widetilde{W}^*(\Sigma)$. We will take complex conjugate.

Twisted chiral superfield: $\Sigma = (\sigma + i\eta,\ \lambda,\ D - iF_{12})$

Special case: complexified Fayet-Iliopoulos term: $\widetilde{W}(z) = \frac{1}{2}\big(-\xi + \frac{i\theta}{2\pi}\big)z$

$$\mathcal{L}_{FI} = -i\xi\, D + i\frac{\theta}{2\pi}\, F_{12}$$

Supersymmetric actions on S^2

- Matter kinetic action for chiral multiplets (of R-charge q):

$$\mathcal{L}_{\mathsf{mat}} = |D_\mu \phi|^2 + \bar{\phi}\Big(\sigma^2 + \eta^2 + iD + \frac{iq}{r}\sigma + \frac{q(2-q)}{4r^2}\Big)\phi + \bar{F}F$$

$$+ \bar{\psi}\Big(-i\not{D} + i\sigma - \gamma_3\eta - \frac{q}{2r}\Big)\psi + i\bar{\psi}\lambda\phi - i\bar{\phi}\bar{\lambda}\psi$$

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- Superpotential ($R[W] = 2$):

$$\mathcal{L}_W = \sum_j \frac{\partial W}{\partial \phi_j} F_j - \frac{1}{2}\sum_{j,k} \frac{\partial^2 W}{\partial \phi_j \partial \phi_k}\psi_j\psi_k$$

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- Couple global flavor symmetries to external vector multiplets, give VEV to $\sigma^{\text{ext}} = -rD^{\text{ext}}$, $\eta^{\text{ext}} = rF_{12}^{\text{ext}}$.

σ^{ext} $\quad\rightarrow\quad$ real (or twisted) masses M

$\sigma^{\text{ext}} + \frac{iq}{2r}$ form a holomorphic pair.

Localization

- Supersymmetric action S and operators $\mathcal{O}$ w.r.t. supercharge $\mathcal{Q}$:

$$[\mathcal{Q}, S] = [\mathcal{Q}, \mathcal{O}] = 0$$

$\mathcal{Q}$-exact terms do not affect the path-integral:

$$\frac{\partial}{\partial u} \int \mathcal{D}\Phi \; \mathcal{O} \; e^{-S-u\,\{\mathcal{Q},\mathcal{P}\}} = 0$$

Z is sensitive only to $\mathcal{Q}$-cohomology (in space of functionals).

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- Choose exact deformation action with positive definite bosonic part:

$$S_{\text{loc}} = u \sum_{\text{fermions } \chi} \mathcal{Q}\left((\overline{\mathcal{Q}\chi})\, \chi \right) \qquad\qquad S_{\text{loc}}\big|_{\text{bos}} = u \sum_{\chi} \left|\mathcal{Q}\chi\right|^2$$

In $u \to \infty$ limit, only BPS configurations $\mathcal{Q}\chi = 0$ contribute:

$$Z = \sum_{\Phi_0 \,|\, \mathcal{Q}\chi = 0} e^{-S[\Phi_0]} \; Z_{\text{1-loop}}[\Phi_0]$$

Localization on S^2

- Choose "equivariant" supercharge:

$$\mathcal{Q}^2 = J + \frac{R}{2} + i\Lambda(\sigma, \eta)$$

Form a superalgebra $\mathfrak{su}(1|1)$.

North and south pole: fixed points of J.

At north (south) pole looks like topological (anti-topological) A-twist

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- All actions constructed before are $\mathcal{Q}$-exact, except the *twisted superpotential*

Z_{S^2} depends on $\widetilde{W}$, (complexified) real masses M and external fluxes $\mathfrak{n}$

Coulomb branch localization

Euclidean path integral: complexified fields $\Rightarrow$ choose a contour.

- Regard A_μ, σ, η, D real, and $(\lambda, \bar{\lambda})$, $(\psi, \bar{\psi})$, $(\phi, \bar{\phi})$, $(F, \bar{F})$ complex conjugates

$$\mathcal{L}_{YM} = \operatorname{Tr} \mathcal{Q}\big[(\overline{\mathcal{Q}\lambda})\lambda + \lambda^\dagger(\overline{Q\lambda^\dagger})\big] \qquad \mathcal{L}_\psi = \operatorname{Tr} \mathcal{Q}\big[(\overline{\mathcal{Q}\psi})\psi + \psi^\dagger(\overline{Q\psi^\dagger})\big]$$

Solve BPS equations:

$$0 = \mathcal{Q}\lambda = \mathcal{Q}\lambda^\dagger \qquad 0 = \mathcal{Q}\psi = \mathcal{Q}\psi^\dagger$$

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Simple BPS configurations:

$$\sigma = -r\, D = \text{constant} \qquad F_{12} = \frac{\eta}{r} \equiv \frac{\mathfrak{m}}{2r^2} \qquad [\sigma, \mathfrak{m}] = 0$$

$$\phi = F = 0$$

This is a “Coulomb branch” (very similar to S^3 case)

Coulomb branch localization

The S^2 partition function is:

$$\boxed{Z_{S^2} = \frac{1}{|\mathcal{W}|} \sum_{\mathfrak{m}} \int \Big(\prod_j \frac{d\sigma_j}{2\pi} \Big) \; Z_{\text{class}}(\sigma, \mathfrak{m}) \; Z_{\text{gauge}}(\sigma, \mathfrak{m}) \; \prod_{\Phi} Z_{\Phi}(\sigma, \mathfrak{m}; M, \mathfrak{n})}$$

The one-loop determinants are:

$$Z_{\text{gauge}} = \prod_{\alpha \in G\,,\, \alpha > 0} \Big(\frac{\alpha(\mathfrak{m})^2}{4} + \alpha(\sigma)^2 \Big)$$

$$Z_{\Phi} = \prod_{\rho \in R_\Phi} \frac{\Gamma\Big(\frac{R[\Phi]}{2} - i\rho(\sigma) - i f^a[\Phi] M_a - \frac{\rho(\mathfrak{m}) + f^a[\Phi] n_a}{2} \Big)}{\Gamma\Big(1 - \frac{R[\Phi]}{2} + i\rho(\sigma) + i f^a[\Phi] M_a - \frac{\rho(\mathfrak{m}) + f^a[\Phi] n_a}{2} \Big)}$$

The classical action is:

$$Z_{\text{class}} = e^{-4\pi i \xi \,\mathrm{Tr}\, \sigma - i\theta \,\mathrm{Tr}\, \mathfrak{m}} \exp\Big\{ 8\pi i r \,\mathbb{R}\mathrm{e}\, \widetilde{W}\big(\tfrac{\sigma}{r} + i \tfrac{\mathfrak{m}}{2r} \big) \Big\}$$

We isolated the linear piece in $\widetilde{W}$ (Fayet-Iliopoulos term)

Some simple checks

- Give large twisted mass to a chiral multiplet: $w = \rho(\sigma) + f^a M_a \to \pm\infty$

$$Z_\Phi \;\to\; e^{8\pi i r \, \mathbb{R}\mathrm{e}\, \widetilde{W}_{\mathsf{eff}}}$$

$$\widetilde{W}_{\mathsf{eff}}(\Sigma) = -\frac{1}{4\pi}\Sigma_s \big[\log(-ir\Sigma_s) - 1 \big] \qquad\qquad \Sigma_s = \rho(\Sigma) + f^a M_a$$

reproduces the correct one-loop running of FI term

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- $U(1)$ with 1 fundamental X of charge Q:

$$Z_{S^2} = \frac{1}{Q^2}\sum_{n=0}^{|Q|-1} \exp\left[2ie^{-2\pi\xi/Q}\sin\left(\frac{\theta - 2\pi n}{Q}\right)\right]$$

Mirror symmetry [Hori, Vafa 00] : twisted chiral Σ, Y with

$$\widetilde{W} = \frac{1}{4\pi}\big[\Sigma\big(QY - \tau(\mu)\big) + i\mu e^{-Y}\big]$$

The on-shell action evaluated at critical points precisely reproduces Z_{S^2}.

Higgs branch localization

- In the Euclidean theory fields are complexified and we can choose a contour.

Allow σ, D to be complex in BPS eqns $\quad \rightarrow \quad$ Higgs branches and vortex solutions

Motivated by [Pasquetti 11] we might hope to be able to perform localization in such a way that vortices (and *not* Coulomb branch) contribute.

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- In the Euclidean theory fields are complexified and we can choose a contour.

Allow σ, D to be complex in BPS eqns $\quad \rightarrow \quad$ Higgs branches and vortex solutions

Motivated by [Pasquetti 11] we might hope to be able to perform localization in such a way that vortices (and *not* Coulomb branch) contribute.

- Trick: introduce *exact* FI term ζ and impose D-term equation:

$$\mathcal{L}_H = \mathcal{Q}\,\mathrm{Tr}\left[\frac{\epsilon^\dagger\lambda - \lambda^\dagger\epsilon}{2i}(\phi\phi^\dagger - \zeta\,\mathbb{1})\right] \;=\; i\Big(D + \frac{\sigma}{r}\Big)(\phi\phi^\dagger - \zeta\,\mathbb{1}) + \dots$$

D appears quadratically in localizing action $\mathcal{L}_{\text{loc}} = u\big(\mathcal{L}_{YM} + \mathcal{L}_H + \mathcal{L}_\psi\big)$

$$\text{Gaussian path-integral} \qquad \rightarrow \qquad D + \frac{\sigma}{r} + i(\phi\phi^\dagger - \zeta\mathbb{1}) = 0$$

A posteriori: $D \notin \mathbb{R}$.

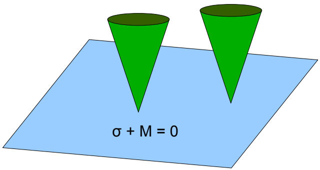
Higgs BPS configurations

When gauge group gets completely broken, and with generic real masses M:

- Higgs branches: $\phi\phi^\dagger = \zeta\, \mathbb{1}_N \qquad \big(\sigma + M\big)\phi = 0 \qquad F_{12} = \eta = 0$

$\rightarrow$ vacua where N chirals get VEV, at fixed positions on Coulomb branch

$$\sigma_a = -M_{l_a} \qquad\qquad a = 1, \ldots, N$$



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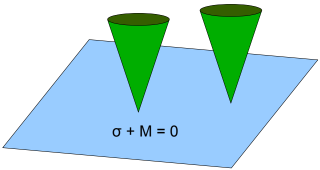

E.g.: $U(N)$ with (N_f, N_a) flavors: $\vec{l} \in C(N, N_f)$

color-flavor locking phases

$$U(N)\times S\big[U(N_f)\times U(N_a)\big] \overset{\text{c-f locking}}{\rightarrow} S\big[U(N)\times U(N_f-N)\big]\times U(1)\times SU(N_a)$$

Higgs BPS configurations

- Vortices at north pole, antivortices at south pole

  size of vortices $\sim\ 1/\sqrt{\zeta}$

  Limit $\zeta \to \infty$:

$$\text{NP:} \qquad D_{\bar{z}}\phi = 0 \qquad F_{12} = -\big(|\phi|^2 - \zeta\mathbb{1}\big)$$

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Close to poles: same action as 2d Ω-background $\mathbb{R}^2_\epsilon$ [Shadchin 06]

$$\mathcal{Q}^2 = J + \frac{R}{2} + i\sigma \qquad \to \qquad \varepsilon = \frac{1}{r},\ a = -iM_{\text{eff}}$$

identifying equivariant parameters with S^2 parameters.

Sum over BPS vortices $\quad \to \quad$ vortex partition function.

Vortex partition function is equivariant volume of the vortex moduli space:

$$Z_{\text{vortex}}(z, \varepsilon, a) = \sum_{k=0}^{\infty} z^k Z_k(\varepsilon, a) \qquad z = e^{-2\pi\xi - i\theta}$$

Higgs branch localization

Result:

$$\boxed{Z_{S^2} = \sum_{\text{vacua}} e^{-4\pi i\xi \sum_{j=1}^{N} \sigma_j} \, Z'_{\text{1-loop}} \, Z_{\text{v}} \, Z_{\text{av}}}$$

with

$$Z_{\text{v}} = Z_{\text{vortex}}\big((-1)^N z \, , \, \tfrac{1}{r} \, , \, -iM_{\text{eff}}\big)$$
$$Z_{\text{av}} = Z_{\text{vortex}}\big((-1)^N \bar{z} \, , \, -\tfrac{1}{r} \, , \, iM_{\text{eff}}\big)$$

and

$$z = e^{-2\pi xi - i\theta} \; .$$

$Z'_{\text{1-loop}}$ does not include the N non-vanishing chiral multiplets.

Vortex partition function of SQCD

$U(N)$ with (N_f, N_a) flavors (assume $N_f \geq N_a$):

k-vortex moduli space in a given vacuum $\vec{l}$ is a symplectic quotient

ADHM-like: Higgs branch of an $\mathcal{N} = 2$ quantum mechanics [Hanany, Tong 03; Eto, Isozumi, Nitta, Ohashi, Sakai 05] , dimensional reduction of a 2d $\mathcal{N} = (0, 2)$ $U(k)$ gauge theory

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Z_k is the equivariant volume, getting contribution from fixed points of unbroken symmetry (color-flavor locked phase):

$$U(1)_\varepsilon \times S\big[U(N) \times U(N_f - N)\big] \times U(1) \times SU(N_a)$$

Equivariant parameters: ε , a_i , $\tilde{a}_j$

Vortex partition function of SQCD

The result can be written as a contour integral

[Nekrasov, Shadchin 04; Dimofte, Gukov, Hollands 10]

$$Z_k = \oint \Big[\prod_{j=1}^{k} \frac{d\varphi_j}{2\pi i} \Big] \; \mathcal{Z}_{\text{vec}}(\varphi, \varepsilon) \; \mathcal{Z}_{\text{fund}}(\varphi, \varepsilon, a) \; \mathcal{Z}_{\text{antifund}}(\varphi, \varepsilon, \tilde{a})$$

where

$$\mathcal{Z}_{\text{vec}} = \frac{1}{\varepsilon^k k!} \prod_{i<j}^{k} \frac{\varphi_i - \varphi_j)^2}{(\varphi_i - \varphi_j)^2 - \varepsilon^2}$$

$$\mathcal{Z}_{\text{fund}} = \prod_{j=1}^{k} \prod_{r \in \vec{l}} \frac{1}{\varphi_j - a_r} \prod_{s \notin \vec{l}} \frac{1}{a_s - \varphi_j - \varepsilon}$$

$$\mathcal{Z}_{\text{antifund}} = \prod_{j=1}^{k} \prod_{f=1}^{N_a} (\tilde{a}_f + \varphi_j)$$

Contour encircles multi-poles parametrized by $\vec{k} \in \mathbb{Z}^N_{\geq 0}$ with $\sum k_i = k$:

$$\{\varphi_j\} = \{a_r + (l_r - 1)\varepsilon \mid r \in \vec{l}, \; l_r = 1, \ldots, k_r\}$$

One-to-one correspondence between multi-poles and equivariant fixed points.

Vortex partition function of SQCD

- Sum over residues at the poles:

$$Z_k = \varepsilon^{(N_a - N_f)k} \sum_{\substack{\vec{k} \in \mathbb{Z}^N_{\geq 0} \\ |\vec{k}|=k}} \prod_{r \in \vec{l}} \frac{\prod_{f=1}^{N_a} \left(\frac{\tilde{a}_f + a_r}{\varepsilon}\right)_{k_r}}{k_r! \prod_{\substack{s \in \vec{l} \\ s \neq r}} \left(\frac{a_s - a_r}{\varepsilon} - k_r\right)_{k_s} \prod_{j \notin \vec{l}} \left(\frac{a_j - a_r}{\varepsilon} - k_r\right)_{k_r}}$$

For a $U(1)$ gauge theory, Z_{vortex} reduces to hypergeometric function ${}_{N_a}F_{N_f-1}$

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- Explicitly verify that this Z_k plugged into the Higgs branch localization formula agrees with the Coulomb branch expression

To evaluate Coulomb branch integral, close the contour of integration and sum over residues

Dualities

Equality of Z_{S^2} for pair of theories ($\rightarrow$ conjecture duality):

$$U(N) \text{ with } (N_f, 0) \quad \leftrightarrow \quad U(N_f - N) \text{ with } (N_f, 0) \qquad N_f > 1$$

$$SU(N) \text{ with } (N_f, 0) \quad \leftrightarrow \quad SU(N_f - N) \text{ with } (N_f, 0)$$

[Hori,Tong 06]

$$U(N) \text{ with } (N_f, N_a) \quad \leftrightarrow \quad U(N_f - N) \text{ with } (N_f, N_a) \qquad N_f > N_a + 1$$

$$N_f N_a \text{ singlets} + W = \tilde{q} M q$$

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$$U(N) \text{ with } (N_f,N_a) \quad \leftrightarrow \quad U(N_f-N) \text{ with } (N_f,N_a) \qquad N_f>N_a+1$$
$$N_f N_a \text{ singlets} + W=\tilde{q}Mq$$

- Unitary: use Higgs branch expression
  - 1-1 correspondence of vacua $\vec{l}\in C(N,N_f)$
  - Classical action + 1-loop determinants easily coincide
  - To prove coincidence of Z_k $\forall k$, use contour integral expression

- Special unitary: perform Fourier transform

$$Z^{(N_F,0)}_{SU(N)}(b;a_j)=\int_0^{2\pi}\frac{d\theta}{2\pi}\int_{-\infty}^{+\infty}4\pi\,d\xi\,e^{4\pi i\xi}\;Z^{(N_f,0)}_{U(N)}(\xi,\theta;a_j)$$

S^2 partition function and Gromow-Witten invariants

[Jockers, Kumar, Lapan, Morrison, Romo 12] have recently observed that when the 2d GLSM theory flows to a conformal non-linear σ-model on a compact CY,
Z_{S^2} computes the full quantum genus-zero Kḧaler potential on the Kähler moduli space of the CY:

$$Z_{S^2} = e^K$$

Does not need to know what the mirror is (and do the computation in the mirror)

K computes Gromow-Witten invariants.

Conclusions

We have computed the p.f. of a 2d $\mathcal{N} = (2,2)$ theory on S^2. Generalizations:

- include twisted chiral superfields (mirror symmetry)
- squash S^2
- higher genus Riemann surfaces?
- $\mathcal{N} = (0,2)$ supersymmetry?

Explore the connection with non-linear σ-models and Gromov-Witten invariants

Alternative localization allowing (some) complex fields

- compute Z_{vortex} in absence of ADHM-like construction
- does it work in higher dimensions?