Exact results in super-conformal field theories

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Oct 23, 2012

Part I

- Motivation: quantifying the number of degrees of freedom in odd dimensions
- Relation to entanglement entropy
- Brief review of 3d susy QFT
- Exactly calculating F

Part II

- Motivation: non-perturbative exact results from non-renormalizable field theories

  Also AdS_6/CFT_5 duality, $N^{5/2}$ d.o.f.s

- Review of 5d SCFTs and their AdS_6 duals

- Localization on S^5

- Free energy at large N

Part I

c-theorems

c-theorems in various dimensions

- A measure of the number of degrees of freedom per volume per energy scale in interacting field theories. It should decrease along rg flow.

- Generalizes the number of fields in weakly interacting systems. Massive objects decouple in the IR, so it decreases.

- Monotonicity requires unitarity, locality, and Lorentz invariance.

c-theorems

- Most obvious conjecture is the high temperature thermal free energy. Works in 1+1 dimensions.
- In higher dimensions, not constant along conformal manifolds. Also, in 2+1, there are explicit counter-examples.
- Such a quantity is useful for constraining rg flows non-perturbatively.

Even dimensions: local anomalies

- In 2d, the coefficient of the trace anomaly famously has this property. RG flow is the gradient flow for this quantity.

  [Zamolodchikov]

- In 4d, $16\pi^2\langle T^\mu{}_\mu\rangle = c(\mathrm{Weyl})^2 - 2a(\mathrm{Euler})$, and a plays this role.

  [Cardy; Komargodski Schwimmer]

- In odd dimensions, there are no anomalies, so this has long been an open problem.

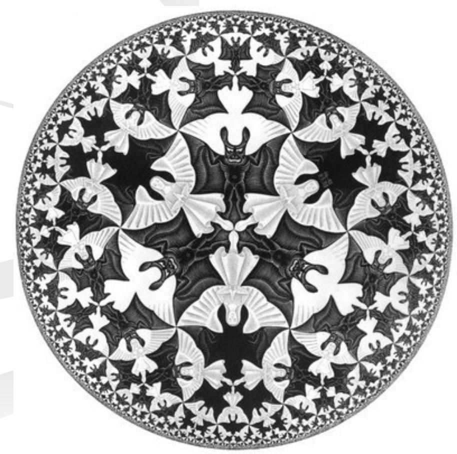
The rough idea

- "Measure" the amount of stuff by observing the response to a curved background.
- Most of the response is via complicated propagation of the CFT fields. However there are special *local* interactions induced for the background fields.
- These have a definite sign due to causality.

Natural holographic version

- The scale parameter is geometrized. The new coordinate and dynamical gravity is emergent. The bulk physics becomes approximately local in some limit (roughly large number of d.o.f. and strong coupling).

- Now just the “size” of the space in Planck units.



Sphere partition function

- Any conformal field theory can be put on the sphere – it is conformal to flat space.
- IR finite observable in odd dim, but non-local.
- Weyl invariance is maintained, hence 1-point functions vanish, and Z is constant along conformal manifolds.

Is the S³ partition function well-defined?

- In general, a calculation in an effective theory with a lower cutoff Λ' < Λ differs by a local effective action for the background fields.

$$\int \sqrt{g}, \qquad \int \sqrt{g}\, \mathcal{R}$$

- In even dimensions, have the Euler density, which integrates to a number.

$$\mathcal{E}_4 = \tfrac{1}{4} R_{ijkl} R_{abcd} \epsilon^{ijab} \epsilon^{klcd}$$

$$Z_\Lambda = Z_{\Lambda'}\ e^{const\ E} = \left(\frac{\Lambda}{\Lambda'}\right)^a$$

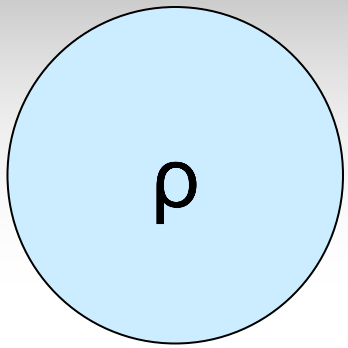
- In odd dimensions, all such terms depend on the radius of the sphere – they correspond to power law divergences.

- Therefore, the odd-dimensional sphere partition function is a well-defined number for conformal field theories.

- Gravitational Chern-Simons term integrates to a number, but it only affects the phase.

$$\frac{i}{4\pi}\int \mathrm{Tr}\left(\omega \wedge d\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega\right)$$

Entanglement entropy



$$S_{ent} = \mathrm{Tr}(\rho \log \rho)$$

- Recently shown that $-S + r\partial_r S$ is smaller for the IR fixed point than in the UV, but it is not obviously intrinsically defined.

  [Casini Huerta]

- Equal to the sphere partition function up to possible subtleties involving divergences.

  [Casini Huerta Myers]

From entanglement to spheres

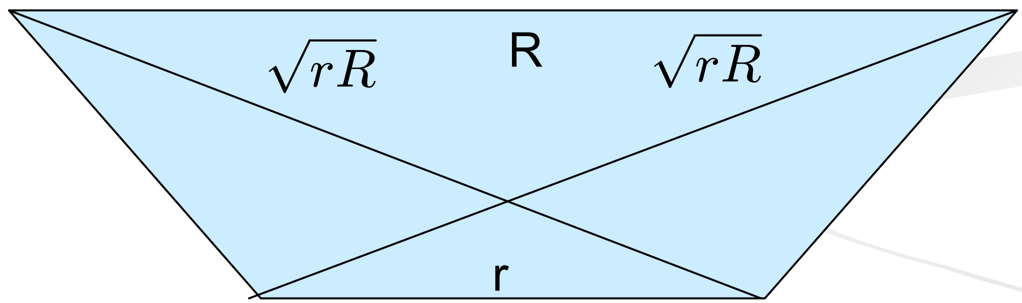
- The ball in flat space is conformal to the hemisphere in $S^{d-1} \times R$. Its casual diamond can be mapped by a time dependent Weyl rescaling to the static patch in de Sitter. The CFT vacuum maps to the euclidean vacuum.
- The reduced density matrix of any QFT in the static patch is thermal at the dS temperature.
- Analytic continuation of the static patch is S^d.

$$\rho = \frac{e^{-\beta H}}{\mathrm{tr}(e^{-\beta H})} \qquad S = \mathrm{tr}(\rho \log \rho) = \mathrm{tr}(\rho(-\beta H - \log Z)) = -\log Z$$

Sketch of the proof of the monotonity of entenglement entropy

- Strong sub-additivity

$$S(A) + S(B) \geq S(A \cup B) + S(A \cap B)$$



$$2S(\sqrt{rR} \geq S(R) + S(r)$$

[Casini Huerta]

$$rS(r) + S(r) \leq 0$$

Chern-Simons-matter theories

- The Yang-Mills coupling, $\frac{1}{g_{YM}^2}\int d^3x\ \mathrm{Tr}(F^{\mu\nu}F_{\mu\nu})$, is dimensionful in 3d – it is an irrelevant operator that renders the theory free in the UV.

- The Chern-Simons coupling, $\frac{k}{4\pi}\int \mathrm{Tr}(A\wedge F)$, is dimensionless – in fact a topological field theory

- When coupled to matter, gives rise to non-trivial IR fixed points with a tunable coupling.

Can supersymmetrize. Here with 4 supercharges:

$$S^{\mathcal{N}=2} = \int \frac{k}{4\pi}(A \wedge dA + \frac{2}{3}A^3) + D_\mu \bar{\phi}_i D^\mu \phi_i + i\bar{\psi}_i \gamma^\mu D_\mu \psi_i$$

$$-\frac{16\pi^2}{k^2}(\bar{\phi}_i T^a_{R_i}\phi_i)(\bar{\phi}_j T^b_{R_j}\phi_j)(\bar{\phi}_k T^a_{R_k} T^b_{R_k}\phi_k) - \frac{4\pi}{k}(\bar{\phi}_i T^a_{R_i}\phi_i)(\bar{\psi}_j T^a_{R_j}\psi_j)$$

$$-\frac{8\pi}{k}(\bar{\psi}_i T^a_{R_i}\phi_i)(\bar{\phi}_j T^a_{R_j}\psi_j).$$

Note that this action has classically marginal couplings. It is has been argued that it does not renormalize, up to shift of k, and so is a CFT.

Exact result for SUSY theories

- One may preserve supersymmetry on the sphere, even without conformal invariance.

$$SU(2|1) \times SU(2) \subset OSp(2|4)$$

- Requires a choice of R-symmetry, since the R-charge appears in the algebra – one turns on a background value of the R-gauge field.

- Localization (non-renormalization theorem) implies that 1-loop is exact.

Superconformal symmetries on S^3

- The conformal group in 3d is USp(4) = SO(3,2).
- In Euclidean signature, one has the real form USp(2,2) = SO(4,1).
- On S^3, the USp(2) × USp(2) = SO(4) subgroup acts as rotations of the sphere.
- The $N = 2$ superconformal group is OSp(2|4).
- The R-symmetry is SO(2) = U(1).

Localizing the path integral

- In Euclidean path integrals, the meaning of supersymmetry is that the expectation values of Q(..) vanish.

[Witten]

- This can be used to show that the full partition function localizes to an integral over Q-fixed configurations. There is a 1-loop determinant from integrating out the other modes.

$$S_{loc} = \{Q, V\}, \quad [Q^2, V] = 0 \qquad Z(t) = \int \prod d\Phi \; e^{-S - tS_{loc}}$$

$$\frac{d}{dt} Z = -\int \prod d\Phi e^{-S - tS_{loc}} \{Q, V\} = 0$$

The recipe

$$Z = \int \prod_{\text{Cartan}} \frac{d\sigma}{2\pi} \; \exp\left[\frac{i}{4\pi} k \; \mathrm{tr}\sigma^2\right] \; \mathrm{Det}_{\mathrm{Ad}} \left(\sinh \frac{\sigma}{2}\right)$$

$$\times \prod_{\substack{\text{chirals} \\ \text{in rep } R_i}} \mathrm{Det}_{R_i} \left(e^{\ell\left(1-\Delta_i + i\frac{\sigma}{2\pi}\right)}\right)$$

$$\ell(z) = -z \log\left(1 - e^{2\pi i z}\right) + \frac{i}{2}\left(\pi z^2 + \frac{1}{\pi}\mathrm{Li}_2(e^{2\pi i z})\right) - \frac{i\pi}{12}$$

$$\partial_z \ell(z) = -\pi z \cot(\pi z)$$

[Kapustin Willett Yaakov; DJ; Hama Hosomichi S. Lee]

|Z| extremization

- Using the vanishing of 1-point functions in a CFT, one can show that

$$\partial_\Delta |Z| = 0$$

at the superconformal value of Δ.

- In fact, |Z| is always minimized.

[Closset Dumitrescu Festuccia Komargodski Seiberg]

Punchlines

- The S^3 partition function measures the number of degrees of freedom.

- With N=2 susy, the IR partition function can be computed exactly from a UV Lagrangian as a function of R-charge parameterized curvature couplings.

- $|Z|$ is minimized by the IR superconformal R, determining the superconformal R-charge exactly.

Part II

5d SCFTs

Exact results from non-renormalizable field theories?

- Usually a situation in which one is simply ignorant of the UV $\Rightarrow$ No.

- But sometimes there is a non-trivial UV fixed point, or gravity. In the former case, one may try to demand background field Weyl covariance.

- The idea: Higher order operators won't affect certain supersymmetric quantities.

$\mathcal{N}$=1 supersymmetric 5d Yang-Mills

- Describes a relevant deformation from the UV point of view. Renders the theory weakly coupled in the IR.

$$\frac{1}{g_{YM}^2}\int F^{\mu\nu}F_{\mu\nu}$$

- Various 5d SCFTs, the (2,0) and (1,0) theory in 6d, and no known non-trivial CFTs in higher dimensions. Very hard to write Lagrangians.

An amusing pattern

2d	3d	4d	5d	6d
N	$N^{3/2}$	N^{2}	$N^{5/2}$	N^{3}

For supersymmetric CFTs with the "simplest" gravity duals. Other scalings are possible, so not to be taken too seriously.

Partition function on S^5

- Well-defined because of the lack of anomalies in odd dimensions.
- A measure of the number of d.o.f.s per unit volume per unit energy scale. Perhaps it's monotonic under rg flow.
- Constant under exactly marginal deformations.

Localization in 3d

[Kapustin Willett Yaakov; DJ; Hama Hosomichi S. Lee]

- Here one adds Q-exact irrelevant super-renormalizable operators to an IR CFT, which render it free in the UV.

$$\frac{1}{g_{YM}^2} \int F^{\mu\nu} F_{\mu\nu}$$

- The partition function is independent of the radius of the sphere, so can compute the IR CFT result from the simplified UV, up to local terms in the background fields.

5d theories

- There is no conformal extension of maximal susy in 5d. Maximal susy YM lifts to 6d.

- There is the F(4) exceptional superconformal algebra, containing the non-conformal N=1 susy

- Many theories with vector and hypermultiplets. Some have non-trivial UV fixed points.

[Seiberg]

Topological flavor symmetry

$$j = *\,\mathrm{tr}(F \wedge F)$$

- This current can be coupled to a background vector multiplet. The real scalar in that multiplet couples to the Yang-Mills kinetic term.

- There are charged instanton-solitons. Satisfy a BPS bound, with central charge $Z = \frac{1}{g_{eff}^2} I$

- These become light at strong coupling.

USp(2N) theories

- The simplest class of 5d CFTs with gravity duals are the UV fixed points associated to USp(2N) gauge theories with a hypermultiplet in the anti-symmetric representation, and $N_f < 8$ hypers in the fundamental.

  [Seiberg; Intriligator Morrison Seiberg]

- Arises in string theory from N D4 branes in massive IIA with D8 branes and O8 planes.

See Bergman Rodriguez-Gomez for an orbifold generalization.

Coulomb branch and effective coupling

- There is a real scalar in 5d vector multiplet, whose vevs describe the Coulomb branch.

- On the moduli space, the Yang-Mills coupling receives 1-loop corrections from the hypers and massive vectors. CFT at the origin, when YM deformation is turned off.

$$\frac{1}{g^2_{\mathrm{eff},i}(\sigma)} = \frac{1}{g^2_{\mathrm{YM}}} + \frac{1}{12\pi^2}(8 - N_f)\sigma_i$$

The UV CFT

- The UV theory has an enhanced E_{N_f} flavor symmetry, in which the instanton-solitons play a crucial role. It contains both the flavor symmetry of the fundamentals, and the topological symmetry. Confirmed recently in the superconformal index. [H.-C. Kim, S.-S. Kim, K. Lee]

- The YM deformation is like a real mass from the UV point of view.

Gravity dual

[Brandhuber Oz]

- Massive IIA AdS_6 solution. The internal space is half of an S^4, the coupling blows up at the equator. The exceptional flavor symmetry is associated to that singular boundary. This is 4-form flux wrapping the S^4.

$$ds^2 = \frac{1}{(\sin\alpha)^{1/3}}\left[L^2\frac{-dt^2+d\vec{x}^2+dz^2}{z^2} + \frac{4L^2}{9}\left(d\alpha^2+\cos^2\alpha\, ds^2_{S^3}\right)\right]$$

$$\frac{L^4}{\ell_s^4} = \frac{18\pi^2 N}{8-N_f} \qquad e^{-2\phi} = \frac{3(8-N_f)^{3/2}\sqrt{N}}{2\sqrt{2}\pi}(\sin\alpha)^{5/3}$$

Supersymmetry on the five sphere

- It is possible to preserve supersymmetry on S^5, without conformal invariance. One has SU(1|4), which contains the SO(6) rotations, and a U(1) subgroup of the flat space SU(2) R-symmetry.

- The Killing spinors square to Hopf isometries, with no fixed points. The base space is CP^2.

$$v^\mu = \xi^I \gamma^\mu \xi_I \qquad \{Q, \tilde{Q}\} = J + R$$

Background fields

- The possible supersymmetry preserving curvature couplings of a 5d N=1 theories can be determined by taking the $M_{pl} \rightarrow \infty$ limit of off-shell 5d supergravity, and finding configurations of the background fields that are invariant under some rigid supersymmetry. They need not satisfy any equation of motion or reality conditions.

- $SU(2)_R$ gauge fields and scalars, graviphoton and dilaton, anti-symmetric 2-form and scalar.

[Kugo Ohashi]

- On the round S^5, only the scalars in the above list will preserve isometries. The R-scalar breaks the $SU(2)_R$ to $U(1)$.

- The spinor equation is $\delta\Psi_\mu \sim \nabla_\mu \eta_i + \gamma_\mu t_i^j \eta_j = 0$

- The resulting susy Yang-Mills action

$$S_{\rm YM} = \frac{1}{g_{\rm YM}^2} \int {\rm tr}\Big(\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2} D_\mu\sigma D^\mu\sigma - Y_{ij}Y^{ij} + 4\sigma t^{ij} Y_{ij} - 8t^2\sigma^2 + 2i\lambda^i(\gamma^\mu D_\mu + t_i^j)\lambda_j - 2[\lambda^i, \lambda_i]\sigma \Big)$$

[Hosomichi Seong Terashima]

Localizing the path integral

- Adding Q-exact terms, the path integral reduces to an integral over a finite dimensional space.

$$v^\mu F_{\mu\nu} = 0, \quad \epsilon^{\mu\nu\rho\sigma\tau} v_\mu F_{\nu\rho} = F_{\sigma\tau}, \quad D_\mu \sigma = 0$$

- The localizing term for the vector multiplet is not rotationally invariant.

- All fields in the hypermulitplets are localized to 0 by a rotationally invariant Q-exact action.

[Kallen Zabzine; Hosomichi Seong Terashima; Kallen Qiu Zabzine; Kim Kim]

Classical action

- The only part of the original action is that is not Q-exact is the Yang-Mills term.

- It differs from a (non-rotationally invariant) Q-exact action by the supersymmetric completion of the instanton action.

$$S_{\rm inst} = \frac{1}{g_{\rm YM}^2} \int v \wedge {\rm Tr}(F \wedge F) + {\rm Tr}(\sigma^2) + \text{fermion terms}$$

Matrix integral

- The instantons are not well-understood, but in the 0-instanton sector, the 1-loop determinants are known.

$$Z = \frac{1}{\mathcal{W}} \int_{\text{Cartan}} d\sigma \, e^{-\frac{4\pi^3 r}{g_{\text{YM}}^2} \text{tr}\sigma^2 + \frac{\pi k}{3} \text{tr}\sigma^3} \det_{\text{Ad}} \left(\sin(i\pi\sigma) e^{\frac{1}{2} f(i\sigma)} \right)$$

$$\times \prod_I \det_{R_I} \left(\cos^{\frac{1}{4}}(i\pi\sigma) e^{-\frac{1}{4} f\left(\frac{1}{2} - i\sigma\right) - \frac{1}{4} f\left(\frac{1}{2} + i\sigma\right)} \right) + \text{instanton contributions}$$

$$f(y) = \frac{i\pi y^3}{3} + y^2 \log\left(1 - e^{-2\pi i y}\right) + \frac{iy}{\pi} \text{Li}_2\left(e^{-2\pi i y}\right) + \frac{1}{2\pi^2} \text{Li}_3\left(e^{-2\pi i y}\right) - \frac{\zeta(3)}{2\pi^2}$$

[Kallen Qiu Zabzine; Kim Kim; Kallen Minahan Nedelin Zabzine]

Leading behavior

- At large values of σ, this simplifies.

$$Z = \frac{1}{\mathcal{W}} \int_{\rm Cartan} d\sigma \, e^{-F(\sigma)}$$

$$F(\sigma) = \frac{4\pi^3 r}{g_{\rm YM}^2} {\rm tr}\sigma^2 + {\rm tr}_{\rm Ad} F_V(\sigma) + \sum_I {\rm tr}_{R_I} F_H(\sigma)$$

$$F_V(y) \approx \frac{\pi}{6} y^3 - \pi y, \qquad F_H(y) \approx -\frac{\pi}{6} y^3 - \frac{\pi}{8} y$$

Effective gauge coupling

- Recall that on the Coulomb branch, the gauge coupling is corrected. This is captured by the cubic terms in the 1-loop determinants.

- But it is the same effective coupling that weights the instantons $Z \sim \left(2(8 - N_f)\sigma + \frac{r}{g^2_{\rm YM}}\right) I$

- In the instanton background, the cubic terms in the 1-loop determinant must be unchanged, and the linear term must shift in this way.

[Seiberg]

Cancelling forces

- In the USp(2N) case, one has

$$F(\sigma_i) = \sum_{i \neq j} [F_V(\sigma_i - \sigma_j) + F_V(\sigma_i + \sigma_j) + F_H(\sigma_i - \sigma_j) + F_H(\sigma_i + \sigma_j)]$$

$$+ \sum_i [F_V(2\sigma_i) + F_V(-2\sigma_i) + N_f F_H(\sigma_i) + N_f F_H(-\sigma_i)]$$

- Note that the cubic long-range interactions cancel between vectors and hypers. This causes the clump of eigenvalues to spread to size $\sqrt{N}$

Large N saddle

- The eigenvalues form a density, and one can easily minimize the action.

$$F \approx -\tfrac{9\pi}{8} N^{2+\alpha} \int dx\, dy\, \rho(x)\rho(y)\,(x-y+x+y) + \tfrac{\pi(8-N_f)}{3} N^{1+3\alpha} \int dx\, \rho(x) x^3$$

$$\rho(x) = \tfrac{2x}{x_*^2}\,, \qquad x_*^2 = \tfrac{9}{2(8-N_f)}$$

$$F = -\frac{9\sqrt{2}\pi N^{5/2}}{5\sqrt{8-N_f}}$$

Exponential suppression of instantons

- Since the eigenvalues spread out to large values on the Coulomb branch, the effective gauge coupling is small. Therefore one is justified in ignoring the instanton contributions, even in this calculation for the CFT. This is a special feature of large N.

- Interestingly, the YM quadratic term wouldn't change the saddle as long as $\frac{r}{g_{YM}^2} \ll \sqrt{N}$

The sphere free energy

- To compare to gravity, we should compute the on-shell action. However this receives contributions from the singularity, so instead…

Entanglement entropy

- For conformal field theories in any dimension, the sphere partition function equals the entanglement entropy in the conformally invariant vacuum.

[Casini Huerta Myers]

- Consider a solid ball in R^{d-1}. Trace over the exterior, and compute the entropy of the resulting interior density matrix.

$$S = \mathrm{Tr}(\rho \log \rho)$$

From entanglement to spheres

- The ball in flat space is conformal to the hemisphere in $S^{d-1} \times R$. Its casual diamond can be mapped by a time dependent Weyl rescaling to the static patch in de Sitter. The CFT vacuum maps to the euclidean vacuum.
- The reduced density matrix of any QFT in the static patch is thermal at the dS temperature.
- Analytic continuation of the static patch is S^d.

$$\rho = \frac{e^{-\beta H}}{\mathrm{tr}(e^{-\beta H})} \qquad S = \mathrm{tr}(\rho \log \rho) = \mathrm{tr}(\rho(-\beta H - \log Z)) = -\log Z$$

Holographic prescription

- One finds a minimal surface in AdS, completely wrapping the internal manifold, that asymptotes to the edge of the entangling region (a ball of radius R) in the CFT on the boundary of AdS.

$$S = \frac{2}{(2\pi)^6 \ell_s^8} \int d^8x \, e^{-2\phi} \sqrt{g}$$

- The entanglement entropy is the minimized generalized area.

[Ryu Takayanagi; Klebanov Kutasov Murugan]

Perfect agreement!

$$S = \frac{20(N)^{5/2}}{27\pi^3\sqrt{2(8-N_f)}}\int \frac{\rho(z)^3\sqrt{1+\rho'(z)^2}}{z^4}(\sin\alpha)^{\frac{1}{3}}(\cos\alpha)^3 dz\wedge d\alpha\wedge \mathrm{vol}_{S^3}\wedge \mathrm{vol}_{S^3}$$

- Minimized by $\rho(z) = \sqrt{R^2 - z^2}$
- Non-universal divergences in R and R^3 are removed.
- Exactly matches the field theory result

$$F = -\frac{9\sqrt{2}\pi N^{5/2}}{5\sqrt{8-N_f}}$$

Punchlines

- Exact results may sometimes be extracted from non-renormalizable field theories.

- Instantons are exponentially suppressed at large N in the five sphere partition function.

- Perfect match between AdS_6 and CFT5.