Two-Sphere Partition Functions and Gromov-Witten Invariants

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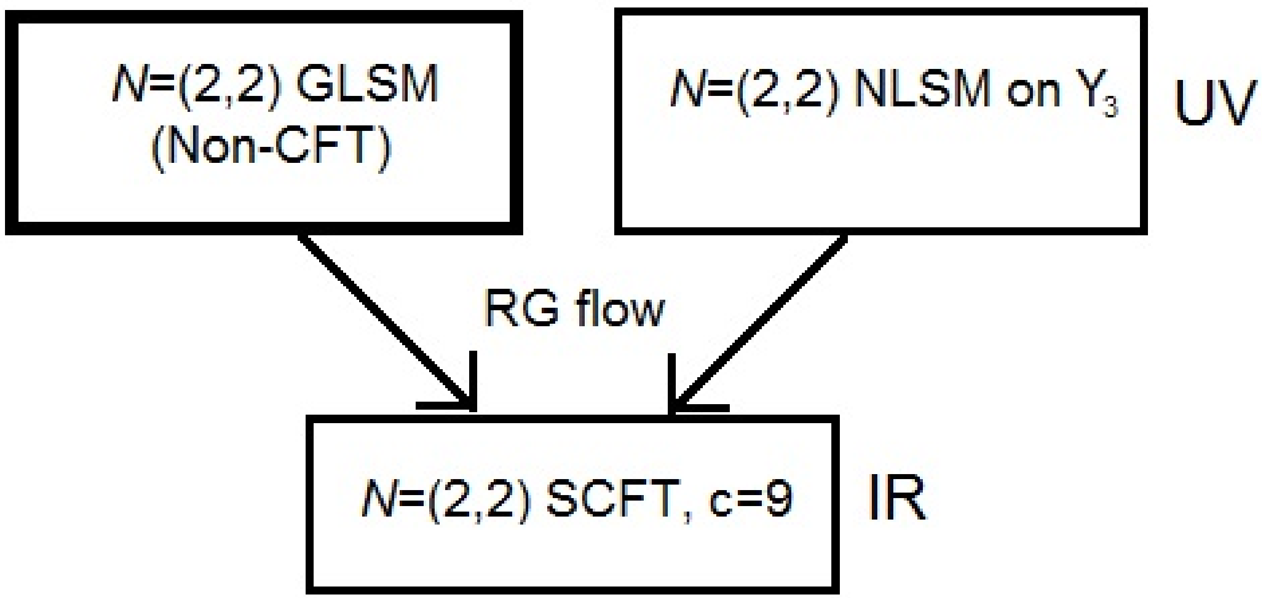
H. Jockers, V. Kumar, J. Lapan, D. Morrison and M. R., arXiv:1208.6244 [hep-th]
H. Jockers, V. Kumar, J. Lapan, D. Morrison and M. R., arXiv:1205.3192 [hep-th]

Part I: Introduction

Big Picture

String compactification on $\mathbb{R}^{1,3} \times Y_3$



Gauged Linear Sigma Models (GLSM)

Data:

- G Compact Lie group. $G = (\prod_j G_j) \times U(1)^s$, $\mathfrak{g} = \mathrm{Lie}(G)$
- Matter: $\rho : G \to GL(V)$, V: $\mathbb{Q}$-graded vector space.
- Superpotential: G-invariant holomorphic polynomial $W \in \mathrm{Sym}(V^*)$ $\mathrm{gr}(W) = 2$
- $r_l + i\theta_l \in \mathbb{C}$, $\theta_l \sim \theta_l + 2\pi$, $l = 1, \ldots, s$

Gauged Linear Sigma Models (GLSM)

$$L_{\mathsf{GLSM}} = L_{\mathsf{YM}} + L_{\mathsf{kin}} + L_{\mathsf{FI}} + L_{\mathsf{W}}$$

- Dynamical fields:

(A, λ, σ) Vector multiplet

(ϕ, ψ) Chiral multiplet

- $\mathcal{M}_{CS}$: coefficients of W. Complex structure moduli.

- $\mathcal{M}_K$: $q_l = e^{r_l + i\theta_l} \in \mathbb{C}^*$ Kähler moduli.

GLSM low energy dynamics

Look for critical points of the effective potential.
For example, simplest abelian case $G = U(1)$, $V = \mathbb{C}^n$, $(\phi_1, \ldots, \phi_n) \in V$.
$t \in G \ \rho(t) \cdot (\phi_1, \ldots, \phi_n) = (t^{Q_1}\phi_1, \ldots, t^{Q_n}\phi_n) \ Q_i \in \mathbb{Z}$

$$U(\phi, \sigma) = \sum_{a=1}^{n} \left| \frac{\partial W}{\partial \phi_a} \right|^2 + (\mu(\phi) - r)^2 + |\sigma|^2 \left(\sum_a Q_a^2 |\phi_a|^2 \right)$$

$$\mu(\phi) = \sum_a Q_a |\phi_a|^2$$

Therefore, we want to solve for

$$\frac{\mu^{-1}(r)}{U(1)} \qquad \frac{\partial W}{\partial \phi_a} = 0$$

GLSM low energy dynamics

Then, we expand all the fields

$$\phi = \phi_{\text{classical}} + \tilde{\phi}$$

integrate out massive fields

$$|\tilde{\phi}|^2 m_\phi^2 \subset L_{\text{GLSM}}(\phi_{\text{classical}} + \tilde{\phi})$$

GLSM Low energy dynamics

Example: smooth quintic $\{F_5(x) = 0\} \subset \mathbb{P}^4[x_0, \ldots, x_4]$

$G = U(1)$, $V = \mathbb{C}^6$, $(p, x_0, \ldots, x_4) \in V$.

$t \in G \ \rho(t) \cdot (p, x_0, \ldots, x_4) = (t^{-5}p, tx_0, \ldots, tx_4)$

$$W = pF_5$$

$$\mu(x, p) = \sum_a |x_a|^2 - 5|p|^2$$

GLSM Low energy dynamics

- $r > 0$: $x \neq 0$ $p = 0$ $\sigma = 0$

$$\left(\frac{\mu^{-1}(r)}{U(1)}\right)_{\{p=0\}} \cong \mathbb{P}^4[x_0, \ldots, x_4]$$

Massive fluctuations: $\tilde{p}$, $\tilde{x}$ normal to $\{F_5(x) = 0\} \subset \mathbb{P}^4$. NLSM on the quintic.

- $r < 0$: $p \neq 0$ $x = 0$ $\sigma = 0$

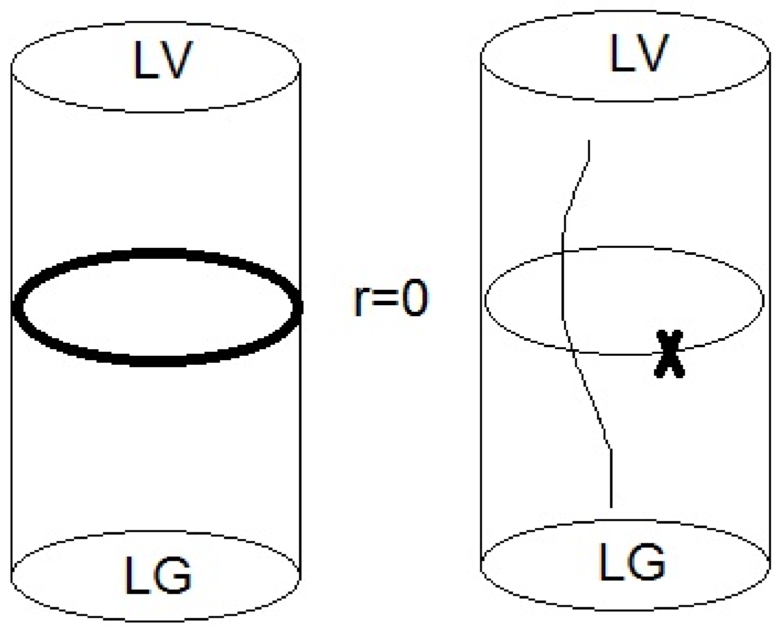
$$|p| = -\sqrt{\frac{-r}{5}}$$

Massive fluctuations: $\tilde{p}$. LG orbifold by $\mathbb{Z}_5 \subset G$ with potential $W = -\sqrt{\frac{-r}{5}} F_5(x)$

GLSM Low energy dynamics

$r = 0 \Rightarrow p = x = 0$ $\sigma \in \mathbb{C}$ Massive fluctuations: $\tilde{p}$, $\tilde{x}$

(r, θ) space:



Part II: Two-Sphere Partition Functions and Gromov-Witten Invariants

Outline

- Kähler and CS moduli spaces
- Exact partition function on S^2
- Conjecture and computing Gromov-Witten invariants
- Examples
- Future directions

Quantum Kähler moduli

We have to be careful when integrating massive $\tilde{p}$, $\tilde{x}$ fluctuations.

An effective potential for σ is generated

$$\widetilde{W}_{\text{eff}}(\sigma) = -\sigma\,[\tau + 5\log(-5)] \qquad \tau = ir + \frac{\theta}{2\pi}$$

Only at critical points $\frac{\partial \widetilde{W}_{\text{eff}}}{\partial \sigma} = 0$ a Coulomb branch emerges and the theory is singular.

The quantum Kähler moduli space is given by $\mathcal{M}_K = (\mathbb{C}^*)^s \setminus \Delta$

Abelian vs. Nonabelian GLSMs

The case $G = U(1)^s$ is naturally suited to CI in toric varieties

$$\{J_1(x) = \cdots = J_r(x) = 0\} \subset X$$

just take the superpotential

$$W = \sum_{a=1}^{r} p_a J_a$$

The nonabelian case is still under much less control

- $G = U(N) \Rightarrow$ CI in Grassmanians.

- $G = SU(k), Usp(k), O(k) \times U(1)^s \Rightarrow$ non-CI more general determinantal varieties.

$\mathcal{M}_{CS}$ vs. $\mathcal{M}_K$

- $\mathcal{M}_{CS}$ and $\mathcal{M}_K$ are Kähler manifolds (indeed special Kähler).

- $Y^\vee$ mirror of Y

$$\mathcal{M}_{CS}(Y^\vee) \cong \mathcal{M}_K(Y)$$

$$\mathcal{M}_{CS}(Y) \cong \mathcal{M}_K(Y^\vee)$$

$\mathcal{M}_{CS}$ vs. $\mathcal{M}_K$

Consider $\xi \in U \subset \mathcal{M}_{CS}$, describing Y_ξ.
$\Omega \in H^{3,0}(Y_\xi)$ non-vanishing holomorphic 3-form and $\mathcal{K}_{CS}(\xi, \bar{\xi})$ the Kähler potential of $\mathcal{M}_{CS}$.

$$e^{-\mathcal{K}_{CS}} = i \int_{Y_\xi} \Omega(\xi) \wedge \overline{\Omega(\xi)}$$

If $\{\gamma_i\}$ forms a symplectic basis of $H_3(Y_\xi)$, then

$$e^{-\mathcal{K}_{CS}} = i \Pi_i \omega^{ij} \overline{\Pi_j} \qquad \Pi_i = \int_{\gamma_i} \Omega(\xi)$$

$\mathcal{M}_{CS}$ **vs.** $\mathcal{M}_K$

Consider $t \in U \subset \mathcal{M}_K$ close to a LV point.

$J = \sum_{l=1}^{h^{1,1}(Y)} t^l \omega_l$ the Kähler form of Y. $\omega_l \in H^2(Y)$ forms an integral basis.

$$\begin{aligned} e^{-\mathcal{K}_K(t,\bar{t})} = & -\frac{i}{6}\sum_{\ell,m,n}\kappa_{\ell mn}(t^\ell-\bar{t}^\ell)(t^m-\bar{t}^m)(t^n-\bar{t}^n)+\frac{\zeta(3)}{4\pi^3}\chi(Y) \\ & +\frac{2i}{(2\pi i)^3}\sum_{\eta\in H^2(Y,\mathbb{Z})}N_\eta\Big(\mathrm{Li}_3(q^\eta)+\mathrm{Li}_3(\bar{q}^\eta)\Big) \\ & -\frac{i}{(2\pi i)^2}\sum_{\eta,\ell}N_\eta\Big(\mathrm{Li}_2(q^\eta)+\mathrm{Li}_2(\bar{q}^\eta)\Big)\eta_\ell(t^\ell-\bar{t}^\ell)\ , \end{aligned}$$

where

$$\kappa_{\ell mn} = \int_Y \omega_\ell\wedge\omega_m\wedge\omega_n$$

$$\mathrm{Li}_k(q) = \sum_{n=1}^{+\infty}\frac{q^n}{n^k}\ , \text{ and } q^\eta = \exp\left(2\pi i\int_\eta J\right) = e^{2\pi i\sum_\ell \eta_\ell t^\ell}\ .$$

S^2 partition function of GLSM

Recently, the exact partition function for a GLSM on S^2 have been computed.

$$Z_{S^2} = \frac{1}{|\mathcal{W}|} \sum_{\mathfrak{m} \in \mathbb{Z}^{\mathrm{rank}(G)}} \int \left(\prod_{\mu=1}^{\mathrm{rank}(G)} \frac{d\sigma_\mu}{2\pi} \right) Z_{\mathrm{class}} Z_{\mathrm{gauge}} \prod_A Z_{\Phi_A},$$

where

$$Z_{\mathrm{gauge}} = \prod_{\alpha>0} \left(\frac{\alpha(\mathfrak{m})^2}{4} + \alpha(\sigma)^2 \right)$$

$$Z_{\Phi_A} = \prod_{\rho \in R_A} \frac{\Gamma\left(\frac{\mathrm{gr}[\phi_A]}{2} - i\rho(\sigma) - \frac{\rho(\mathfrak{m})}{2}\right)}{\Gamma\left(1 - \frac{\mathrm{gr}[\phi_A]}{2} + i\rho(\sigma) - \frac{\rho(\mathfrak{m})}{2}\right)}$$

$$Z_{\mathrm{class}} = \exp\left(-4\pi i r\,\mathrm{Tr}(\sigma) - i\theta\,\mathrm{Tr}(\mathfrak{m})\right)$$

S^2 partition function of GLSM

So, write

$$z_\ell = e^{-2\pi r_\ell + i\theta_\ell}$$

Then, we conjecture

$$Z_{S^2}(z_\ell, \bar{z}_\ell) = e^{-\mathcal{K}_K(z_\ell, \bar{z}_\ell)}$$

Computing genus zero G-W invariants

- Evaluate $Z_{S^2}(z_\ell, \bar{z}_\ell) = e^{-\mathcal{K}_K}$

- Read from the $\zeta(3)$ term the transformation $\mathcal{K}_K \to \mathcal{K}_K + f(z) + \overline{f(z)}$ to normalize it to $\frac{\zeta(3)\chi(Y)}{4\pi^3}$.

- Read the holomorphic part of the coefficient of $\log \bar{z}_m \log \bar{z}_n$ to get

$$-\frac{i}{2(2\pi i)^2}\kappa_{\ell mn} t^\ell \ ,$$

  which allows us to extract the flat coordinates t^ℓ of the form

$$t^\ell = \frac{\log z_\ell}{2\pi i} + t^\ell_{(0)} + f^\ell(z) \ .$$

- Invert this relation to obtain the z_ℓ as a function of t^ℓ,

$$z_\ell = e^{2\pi i t^\ell_{(0)}}\big(q_\ell + O(q^2)\big) \,,$$

- Read off the rational Gromov-Witten invariants from the coefficients in the q-expansion

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Example: Quintic threefold

Field	$U(1)$	gr
Φ_a	$+1$	$2\mathsf{v}$
P	-5	$2-2n\mathsf{v}$

The partition function gives (set $\mathsf{v}=0^+$)

$$Z_{\text{quintic}} = \sum_{m\in\mathbb{Z}} e^{-i\theta m} \int_{-\infty}^{\infty} \frac{d\sigma}{2\pi}\, e^{-4\pi i r\sigma} \, (Z_\Phi)^5 \, Z_P \ ,$$

where

$$Z_\Phi := \frac{\Gamma\big(0^+ - i\sigma - \frac{m}{2}\big)}{\Gamma\big(1-0^+ + i\sigma - \frac{m}{2}\big)} \ , \qquad Z_P := \frac{\Gamma\big(1-0^+ + 5i\sigma + \frac{5m}{2}\big)}{\Gamma\big(0^+ - 5i\sigma + \frac{5m}{2}\big)} \ .$$

then, consider for example the LV point $r \gg 1$

$$Z_{\text{quintic}} = \text{Res}_{\epsilon=0} (z\bar{z})^{-\epsilon} \ \frac{\pi^4 \sin(5\pi\epsilon)}{\sin^5(\pi\epsilon)} \left| \sum_{k=0}^{\infty} (-z)^k \, \frac{\Gamma(1+5k-5\epsilon)}{\Gamma(1+k-\epsilon)^5} \right|^2 \ .$$

Example: Quintic threefold

Interestingly the normalized partition function is

$$e^{-K'} = -\frac{1}{8\pi^3}\frac{Z_{\text{quintic}}}{X^0(z)\overline{X^0(z)}} \ ,$$

where

$$X^0(z) = \sum_{k=0}^{\infty} \frac{\Gamma(1+5k)}{\Gamma(1+k)^5} \ (-z)^k \ .$$

Next, we determine the mirror map through the coefficient of the $\log^2 \bar{z}$ term, which yields

$$t = t_{(0)} + \frac{1}{2\pi i}\Big(-770\, z + 717\,825\, z^2 + \ldots\Big) \ .$$

Inverting the mirror map, we find can read the integral genus zero Gromov-Witten invariants

$$2\,875\,, \quad 609\,250\,, \quad 317\,206\,375\,, \quad 242\,467\,530\,000\,, \ldots \ ,$$

Determinantal Varieties

Set a compact complex algebraic variety V, $\dim_{\mathbb{C}} V = D$. Also two vector bundles $\mathcal{E}$ and $\mathcal{F}$ over V

$$\mathcal{E} \to V \qquad \mathcal{F} \to V \qquad \text{rk}(\mathcal{F}) = m, \text{rk}(\mathcal{E}) = n$$

A: generic global holomorphic section of $\text{Hom}(\mathcal{E}, \mathcal{F}) = \mathcal{E}^* \otimes \mathcal{F}$

$$Z(A,k) = \{\phi \in V | \text{rk}(A(\phi)) \leq k\}$$

In particular consider $V = \mathbb{P}^D$ and $m = n$ (also $\mathcal{E} = \mathcal{O}_{\mathbb{P}^D}^{\oplus n}$, $\mathcal{F} = \mathcal{O}_{\mathbb{P}^D}^{\oplus n}(1)$)

$$X_A = \tilde{Z}(A,k) = \left\{(\phi, x) \in \mathbb{P}^D \times G(n-k,n) | A(\phi)|_x = 0\right\}$$

Example: Determinantal Calabi-Yau

Let's look at a nonabelian example, $G = U(2) \times U(1)$

Field	$U(1)$	$U(2)$	$U(1)_{\mathrm{V}}$
$\Phi_{a=1,\ldots,8}$	$+1$	1	$2\mathrm{v}_\phi$
$P_{i=1,\ldots,4}$	-1	$\overline{\Box}$	$2 - 2\mathrm{v}_x - 2\mathrm{v}_\phi$
$X_{i=1,\ldots,4}$	0	$\Box$	$2\mathrm{v}_x$

superpotential ($A(\Phi) = \sum_a A^a \Phi_a$, A^a: 4×4 matrices)

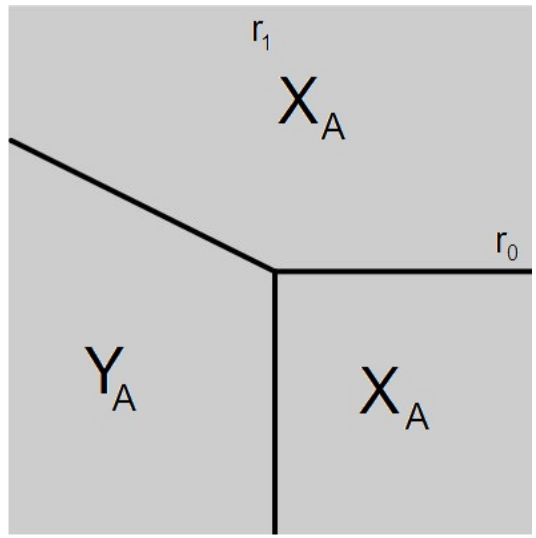
$$W = \sum_{i,j} \mathrm{Tr}(P_i A(\Phi)_{ij} X_j)$$

D-terms

$$U(1) : \sum_a |\phi_a|^2 - \mathrm{Tr}(p^\dagger p) = r_0$$

$$U(2) : pp^\dagger - x^\dagger x = r_1 \mathbf{1}_{2\times 2}$$

Example: Determinantal Calabi-Yau



$$X_A = \{(\phi, x) \in \mathbb{P}^7 \times G(2,4) | A(\phi)x = 0\}$$

$$Y_A = \{(x, p) \in \mathbb{P}(\mathcal{U}^{\oplus 4}) | \sum_{i,j} p_i A^a_{ij} x_j = 0\}$$

N_{m_0,m_1}	$m_0=0$	$1/2$	1	$3/2$	2	$5/2$	3
$m_1=0$	–		56		0		0
$1/2$		192		896		192	
1	56		2 544		23 016		41 056
$3/2$		896		52 928		813 568	
2	0		23 016		1 680 576		35 857 016
$5/2$		192		813 568		66 781 440	
3	0		41 056		35 857 016		3 074 369 392
$7/2$		0		3 814 144		1 784 024 064	
4	0		23 016		284 749 056		96 591 652 016
$9/2$		0		6 292 096		20 090 433 088	
5	0		2 544		933 789 504		1 403 214 088 320
$11/2$		0		3 814 144		105 588 804 096	
6	0		56		1 371 704 192		10 388 138 826 968
$13/2$		0		813 568		277 465 693 248	
7	0		0		933 789 504		41 598 991 761 344
$15/2$		0		52 928		380 930 182 784	
8	0		0		284 749 056		93 976 769 192 864
$17/2$		0		896		277 465 693 248	
9	0		0		35 857 016		122 940 973 764 384
$19/2$		0		0		105 588 804 096	
10	0		0		1 680 576		93 976 769 192 864
$21/2$		0		0		20 090 433 088	
11	0		0		23 016		41 598 991 761 344
$23/2$		0		0		1 784 024 064	
12	0		0		0		10 388 138 826 968
$25/2$		0		0		66 781 440	
13	0		0		0		1 403 214 088 320
$27/2$		0		0		813 568	
14	0		0		0		96 591 652 016
$29/2$		0		0		192	
15	0		0		0		3 074 369 392
$31/2$		0		0		0	

Future directions

- We presented a method to compute the exact Kähler potential of the quantum Kähler moduli $\mathcal{M}_K$ of $\mathcal{N} = (2, 2)$ SCFTs.

- The conjecture is supported against many nontrivial checks: intersection numbers, known GW invariants, etc.

- Study the effect of boundaries: put the theory on a hemisphere.

- Can we compute higher genus GW invariants?