Fermionic structure of form factors.

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Preliminaries. We consider sG model

$$\mathcal{A}^{\mathrm{sG}} = \int \Big\{ \Big[\frac{1}{16\pi} (\partial_\mu \boldsymbol{\varphi}(x))^2 + \frac{\boldsymbol{\mu}^2}{\sin \pi\beta^2} e^{-i\beta\boldsymbol{\varphi}(x)} \Big] + \frac{\boldsymbol{\mu}^2}{\sin \pi\beta^2} e^{i\beta\boldsymbol{\varphi}(x)} \Big\} d^2x \,.$$

We shall use the parameter

$$\nu = 1 - \beta^2 \,,$$

We are interested in local fields which do not change the topological charge, namely in the primary fields

$$\Phi_\alpha(z,\bar z) = e^{i\alpha \ \frac{\nu}{2\sqrt{1-\nu}} \ \boldsymbol{\varphi}(z,\bar z)} \,,$$

and their descendants.
For irrational α, β the correspondence with CFT fields is fixed by dimensional considerations.

We shall consider only the fundamental particles: solitons and antisolitons.

S-matrix. Let β_j be rapidities. We shall use

$$B_j = e^{\beta_j}, \quad \mathfrak{b}_j = e^{\frac{2\nu}{1-\nu}\beta_j}, \quad \mathfrak{q} = e^{\pi i \frac{1}{1-\nu}} .$$

The two soliton S-matrix is given by

$$S_{i,j}(\beta_i - \beta_j) = S_0(\beta_i - \beta_j)\widetilde{S}_{i,j}(\mathfrak{b}_i/\mathfrak{b}_j) ,$$

$$S_0(\beta) = \exp\left(-i \int\limits_0^\infty \frac{\sin(2k\nu\beta)\sinh((2\nu-1)\pi k)}{k\cosh(\pi\nu k)\sinh(\pi(1-\nu)k)} dk \right) ,$$

and

$$\widetilde{S}_{i,j}(\mathfrak{b}_i/\mathfrak{b}_j) = \tfrac{1}{2}(I_i \otimes I_j + \sigma_i^3 \otimes \sigma_j^3) + \frac{\mathfrak{b}_i - \mathfrak{b}_j}{\mathfrak{b}_i\mathfrak{q}^{-1} - \mathfrak{b}_j\mathfrak{q}} \cdot \tfrac{1}{2}(I_i \otimes I_j - \sigma_i^3 \otimes \sigma_j^3)$$
$$+ \sqrt{\mathfrak{b}_i\mathfrak{b}_j}\frac{\mathfrak{q}^{-1} - \mathfrak{q}}{\mathfrak{b}_i\mathfrak{q}^{-1} - \mathfrak{b}_j\mathfrak{q}} \cdot (\sigma_i^+ \otimes \sigma_j^- + \sigma_i^- \otimes \sigma_j^+) .$$

Form factor axioms.

Symmetry axiom.

$$S_{j,j+1}(\beta_j - \beta_{j+1}) f_{\mathcal{O}_\alpha}(\beta_1, \cdots, \beta_j, \beta_{j+1}, \cdots, \beta_{2n})$$
$$= f_{\mathcal{O}_\alpha}(\beta_1, \cdots, \beta_{j+1}, \beta_j, \cdots, \beta_{2n}) \,.$$

Riemann-Hilbert problem axiom.

$$f_{\mathcal{O}_\alpha}(\beta_1, \cdots, \beta_{2n-1}, \beta_{2n} + 2\pi i) = e^{-\frac{\pi i \nu}{1-\nu}\alpha\sigma^3_{2n}} f_{\mathcal{O}_\alpha}(\beta_{2n}, \beta_1, \cdots, \cdots, \beta_{2n-1}) \,.$$

Residue axiom.

$$2\pi i \ \mathrm{res}_{\beta_{2n}=\beta_{2n-1}+\pi i} f_{\mathcal{O}_\alpha}(\beta_1, \cdots, \beta_{2n-2}, \beta_{2n-1}, \beta_{2n}) =$$
$$\Big(1 - e^{-\frac{\pi i \nu}{1-\nu}\alpha\sigma^3_{2n}} S_{2n-1,1}(\beta_{2n-1} - \beta_1) \cdots S_{2n-1,2n-2}(\beta_{2n-1} - \beta_{2n-2})\Big)$$
$$\times f_{\mathcal{O}_\alpha}(\beta_1, \cdots, \beta_{2n-2}) \otimes (e^+_{2n-1} \otimes e^-_{2n} + e^-_{2n-1} \otimes e^+_{2n}) \,.$$

Important observation: The form factor equations for all the fields with $\alpha + 2m\frac{1-\nu}{\nu}$ coincide.
The structure of the form factor:

$$\begin{aligned} & f_{\mathcal{O}_\alpha}(\beta_1, \cdots, \beta_{2n}) \\ & \quad = \sum_{\epsilon_1, \cdots, \epsilon_{2n} = \pm} w^{\epsilon_1, \cdots, \epsilon_{2n}}(\beta_1, \cdots, \beta_{2n}) \ \cdot \ \mathcal{F}_{\mathcal{O}_\alpha, n}(\beta_{I^-} | \beta_{I^+}) \,, \end{aligned}$$

where $I^{\pm} = \{j \mid 1 \leq j \leq 2n, \ \epsilon_j = \pm\}$, and the sum over ϵ_j's is such that $\sharp(I^+) = \sharp(I^-)$, $\beta_I = \{\beta_{i_1}, \cdots, \beta_{i_n}\}$.
The basis satisfies

$$\begin{aligned} & S_{i,i+1}(\beta_i - \beta_{i+1}) w^{\epsilon_1, \cdots, \epsilon_i, \epsilon_{i+1}, \cdots, \epsilon_{2n}}(\beta_1, \cdots, \beta_i, \beta_{i+1}, \cdots, \beta_{2n}) \\ & \qquad = w^{\epsilon_1, \cdots, \epsilon_{i+1}, \epsilon_i, \cdots, \epsilon_{2n}}(\beta_1, \cdots, \beta_{i+1}, \beta_i, \cdots, \beta_{2n}) \,. \end{aligned}$$

Integrals.

We introduce integration variable σ. We use the notations

$$S=e^{\sigma},\ \ B_j=e^{\beta_j},\ \ Q=e^{\pi i\frac{1-\nu}{\nu}},\ \ A=e^{\pi i\alpha}$$

$$\mathfrak{s}=e^{\frac{2\nu}{1-\nu}\sigma},\ \ \mathfrak{b}_j=e^{\frac{2\nu}{1-\nu}\beta_j},\ \ \mathfrak{q}=e^{\pi i\frac{1}{1-\nu}},\ \ a=e^{\pi i\frac{\nu}{1-\nu}\alpha}\,.$$

Introduce the function $\chi(\sigma|\beta_1,\cdots,\beta_{2n})$ which satisfies

$$\chi(\sigma+2\pi i)p(\mathfrak{s}\mathfrak{q}^4)=\chi(\sigma)p(\mathfrak{s}\mathfrak{q}^2)$$

$$\chi(\sigma+\tfrac{1-\nu}{\nu}\pi i)P(SQ)=\chi(\sigma)P(-S)\,,$$

where

$$P(S)=\prod_{j=1}^{2n}(S-B_j)\,,\quad p(\mathfrak{s})=\prod_{j=1}^{2n}(\mathfrak{s}-\mathfrak{b}_j)\,.$$

For $\beta_j\in\mathbb{R}$ no singularities in the strip $0>\mathrm{Im}(\sigma)>-\pi$.

Asymptotic behaviour for $\sigma \to \pm\infty$:

$$\chi(\sigma|\beta_1, \cdots, \beta_{2n}) \simeq_{\sigma\to\infty} e^{-2n\frac{1}{1-\nu}\sigma}, \qquad \chi(\sigma|\beta_1, \cdots, \beta_{2n}) \simeq_{\sigma\to-\infty} = 1\,.$$

Our main tool is the integral

$$I_\alpha(\beta_1, \cdots, \beta_{2n}) = \int\limits_{\mathbb{R}-i0} \chi(\sigma|\beta_1, \cdots, \beta_{2n}) e^{\frac{\nu\alpha}{1-\nu}\sigma} d\sigma\,.$$

This integral converges for

$$0 < \mathrm{Re}(\alpha) < \frac{2n}{\nu}\,,$$

and can be continued analytically with respect to α. The result is a meromorphic function in $\mathbb{C}$ with simple poles at

$$\alpha = 2(n+m) + (2n+l)\frac{1-\nu}{\nu}, \quad l, m \geq 0; \qquad \alpha = -2m - \frac{1-\nu}{\nu} l, \quad l, m \geq 0\,.$$

Definition. *Consider two Laurent polynomials $\ell(\mathfrak{s})$ and $L(S)$. We define their pairing $(\ell, L)_\alpha$ by the following two requirements:*
1. The pairing is bilinear.
2. If $\ell(\mathfrak{s}) = \mathfrak{s}^m$, $L(S) = S^l$ then

$$(\ell, L)_\alpha = I_{\alpha + 2m + \frac{1-\nu}{\nu} l} \, .$$

$\mathfrak{q}$-exact forms and Q-exact forms.

$$\text{For } \; D_a[z](\mathfrak{s}) = a^{-2} p(\mathfrak{s}) z(\mathfrak{s}) - p(\mathfrak{s}\mathfrak{q}^2) z(\mathfrak{s}\mathfrak{q}^4) \, , \quad (D_a[z], L)_\alpha = 0, \quad \forall L \, .$$

$$\text{For } \; D_A[Z](S) = Z(S)P(S) - AZ(SQ)P(-S) \, , \quad (\ell, D_A[Z])_\alpha = 0, \quad \forall \ell \, .$$

For antisymmetric Laurent polynomials of k variables:

$$\ell^{(k)}(\mathfrak{s}_1,\cdots\mathfrak{s}_k)=\big(\ell_1\wedge\cdots\wedge\ell_k\big)(\mathfrak{s}_1,\cdots\mathfrak{s}_k),$$

$$L^{(k)}(S_1,\cdots,S_k)=\big(L_1\wedge\cdots\wedge L_k)(S_1,\cdots,S_k)\,.$$

define

$$(\ell^{(k)},L^{(k)})_\alpha=\det\left(\,(\ell_i,L_j)_\alpha\,\right)_{i,j=1,\cdots,k}\ .$$

Form factors and BBS fermions. We need $\mathcal{F}(\beta_{I^-},\beta_{I^+})$. Introduce

$$\begin{aligned}&c_{I^-\sqcup I^+}(t,s)\\&=p_{I^+}(\mathfrak{q}^2\mathfrak{t})p_{I^-}(\mathfrak{s})\left(\frac{a^{-1}\mathfrak{q}^2\mathfrak{t}}{\mathfrak{q}^2\mathfrak{t}-s}-\frac{a^{-1}\mathfrak{t}}{\mathfrak{t}-\mathfrak{s}}\right)+p_{I^-}(\mathfrak{t})p_{I^+}(\mathfrak{q}^2\mathfrak{s})\left(\frac{a^{-1}\mathfrak{t}}{\mathfrak{t}-\mathfrak{s}}-\frac{a\mathfrak{t}}{\mathfrak{t}-\mathfrak{q}^2\mathfrak{s}}\right)\\&+\frac{a\mathfrak{t}}{\mathfrak{t}-\mathfrak{q}^2\mathfrak{s}}p(\mathfrak{q}^2\mathfrak{s})-\frac{a^{-1}\mathfrak{q}^2\mathfrak{t}}{\mathfrak{q}^2\mathfrak{t}-\mathfrak{s}}p(\mathfrak{s}):=\sum_{i=0}^{n-1}(\mathfrak{q}^2\mathfrak{t})^{n-i}\ell_{I^-\sqcup I^+,i}(\mathfrak{s})\,,\end{aligned}$$

and

$$\ell^{(n)}_{I^-\sqcup I^+}(\mathfrak{s}_1,\cdots,\mathfrak{s}_n)=\big(\ell_{I^-\sqcup I^+,0}\wedge\cdots\wedge\ell_{I^-\sqcup I^+,n-1}\big)\,(\mathfrak{s}_1,\cdots,\mathfrak{s}_n)$$

Then the Riemann-Hilbert axiom is satisfied if

$$\mathcal{F}_{\mathcal{O}_\alpha,n}(\beta_{I^-}|\beta_{I^+}) = (\ell^{(n)}_{I^-\sqcup I^+}\ ,\ L^{(n)})_\alpha\ ,$$

where $L^{(n)} = L^{(n)}(S_1,\cdots,S_n|B_1,\cdots,B_{2n})$ is an arbitrary Laurent polynomial which is anti-symmetric in S_i's and symmetric in B_j's.
Definition of tower. *Polynomials*

$$L^{(\star)} = \{L^{(n)}(S_1,\cdots,S_n|B_1,\cdots,B_{2n})\}\ .$$

constitute a tower if

$$L^{(n)}(S_1,\cdots,S_{n-1},B|B_1,\cdots,B_{2n-2},B,-B)$$
$$= (-1)^c B \prod_{p=1}^{n-1}(B^2 - S_p^2)\cdot L^{(n-1)}(S_1,\cdots,S_{n-1}|B_1,\cdots,B_{2n-2})$$

Every tower of charge 0 defines a local operator. Descendants of the local integrals are obtained multiplying by

$$\mathbf{i}_{2k-1} = C_k M^{2k-1} \sum B_j^{2k-1}, \quad \bar{\mathbf{i}}_{2k-1} = C_k M^{2k-1} \sum B_j^{-(2k-1)} \,.$$

The primary field Φ_α corresponds to the tower

$$M_0^{(n)}(S_1, \cdots, S_n) = \langle \Phi_\alpha \rangle \cdot S \wedge S^3 \wedge \cdots \wedge S^{2n-1} \,.$$

It is possible to pass to odd degrees only introducing the polynomials

$$C_{\pm,n}(S_1, S_2) = \frac{1}{2} \sum_{\epsilon_1, \epsilon_2 = \pm} \epsilon_1 \epsilon_2 P_n(\epsilon_1 S_1) P_n(\epsilon_2 S_2) \tau_\pm(\epsilon_2 S_2 / \epsilon_1 S_1) \,,$$

where

$$\tau_+(x) = -\sum_{l=0}^{\infty} (-x)^l \tfrac{i}{2\nu} \cot \tfrac{\pi}{2}(\alpha + \tfrac{l}{\nu}) \,, \quad \tau_-(x) = \sum_{l=-\infty}^{-1} (-x)^l \tfrac{i}{2\nu} \cot \tfrac{\pi}{2}(\alpha + \tfrac{l}{\nu}) \,.$$

Then the towers are created by action of four fermions

$$\psi^*(Z) = \sum_{j=1}^{\infty} Z^{-2j+1} \psi^*_{2j-1}, \quad \chi^*(X) = \sum_{j=1}^{\infty} X^{-2j+1} \chi^*_{2j-1}\,,$$
$$\bar{\psi}^*(Z) = \sum_{j=1}^{\infty} Z^{2j-1} \bar{\psi}^*_{2j-1}, \quad \bar{\chi}^*(X) = \sum_{j=1}^{\infty} X^{2j-1} \bar{\chi}^*_{2j-1}\,.$$

$$\begin{aligned}
&\Big(\psi^*(Z_1)\cdots\psi^*(Z_p)\bar{\psi}^*(Z_{p+1})\cdots\bar{\psi}^*(Z_k) \\
&\quad \times \bar{\chi}^*(X_{k'})\cdots\bar{\chi}^*(X_{q+1})\chi^*(X_q)\cdots\chi^*(X_1) M_0^{(n)}\Big)(S_1,\cdots,S_n) \\
&= \langle\Phi_\alpha\rangle(-)^{kk'} \frac{1}{\prod_{j=1}^{k}\sqrt{P_n(Z_j)P_n(-Z_j)}\ \prod_{j=1}^{k'}\sqrt{P_n(X_j)P_n(-X_j)}} \cdot \begin{vmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{vmatrix}\,,
\end{aligned}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ are respectively $k \times k'$, $k \times n$, $n \times k'$, $n \times n$ matrices.

$$\mathcal{B} = \begin{pmatrix} C_+(Z_1, S_1) & \cdots & C_+(Z_1, S_n) \\ \vdots & & \vdots \\ C_+(Z_p, S_1) & \cdots & C_+(Z_p, S_n) \\ C_-(Z_{p+1}, S_1) & \cdots & C_-(Z_{p+1}, S_n) \\ \vdots & & \vdots \\ C_-(Z_k, S_1) & \cdots & C_-(Z_k, S_n) \end{pmatrix},$$

$$\mathcal{C} = \begin{pmatrix} X_1 & \cdots & X_{k'} \\ \vdots & & \vdots \\ X_1^{2n-1} & \cdots & X_{k'}^{2n-1} \end{pmatrix}, \quad \mathcal{D} = \begin{pmatrix} S_1 & \cdots & S_n \\ \vdots & & \vdots \\ S_1^{2n-1} & \cdots & S_n^{2n-1} \end{pmatrix}.$$

$\chi^*(X_k)$, $\bar{\chi}^*(X_k)$ create holes, $\psi^*(Z_k)$, $\bar{\psi}^*(Z_k)$ crate particles.

The matrix $\mathcal{A}$ is needed in order to avoid interference between two chiralities.

$$\mathcal{A} = \begin{pmatrix} 0 & \cdots & 0 & C_+(Z_1, X_{q+1}) & \cdots & C_+(Z_1, X_{k'}) \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & C_+(Z_p, X_{q+1}) & \cdots & C_+(Z_p, X_{k'}) \\ C_-(Z_{p+1}, X_1) & \cdots & C_-(Z_{p+1}, X_q) & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ C_-(Z_k, X_1) & \cdots & C_-(Z_k, X_q) & 0 & \cdots & 0 \end{pmatrix} .$$

Take large n. The fermions χ^* and ψ^* are doing something at the right end of Fermi zone, while $\bar{\chi}^*$ and $\bar{\psi}^*$ are doing something at the left end. There is an important integer (weight)

$$m = \tfrac{1}{2}(\#(\psi^*) - \#(\chi^*) + \#(\bar{\chi}^*) - \#(\bar{\psi}^*)) .$$

For given weight m we describe descendants of $\Phi_{\alpha+2m\frac{1-\nu}{\nu}}$.

In particular,

$$\psi_1^* \cdots \psi_{2m-1}^* \bar{\chi}_{2m-1}^* \cdots \bar{\chi}_1^* \ M_0^{(\star)}$$

$$= \frac{\langle \Phi_\alpha \rangle}{\langle \Phi_{\alpha + 2m\frac{1-\nu}{\nu}} \rangle} \cdot \left(\tfrac{i}{\nu}\right)^m \prod_{j=1}^{m} \cot \tfrac{\pi}{2\nu}(\alpha\nu + (2j-1)) \ M_m^{(\star)} \,,$$

$$\bar{\psi}_1^* \cdots \bar{\psi}_{2m-1}^* \chi_{2m-1}^* \cdots \chi_1^* \ M_0^{(\star)}$$

$$= \frac{\langle \Phi_\alpha \rangle}{\langle \Phi_{\alpha - 2m\frac{1-\nu}{\nu}} \rangle} \cdot \left(\tfrac{i}{\nu}\right)^m \prod_{j=1}^{m} \cot \tfrac{\pi}{2\nu}(\alpha\nu - (2j-1)) \ M_{-m}^{(\star)} \,.$$

Later we shall use

$$\psi(Z) = \sum_{j=1}^{\infty} Z^{2j-1} \psi_{2j-1} \,, \qquad \bar{\psi}(Z) = \sum_{j=1}^{\infty} Z^{-2j+1} \bar{\psi}_{2j-1},$$

$$\chi(Z) = \sum_{j=1}^{\infty} Z^{2j-1} \chi_{2j-1} \,, \qquad \bar{\chi}(Z) = \sum_{j=1}^{\infty} Z^{-2j+1} \bar{\chi}_{2j-1},$$

Null-vectors.

Something special happens when $\alpha = m\frac{1-\nu}{\nu}$ which means

$$e^{\frac{\nu\alpha}{1-\nu}} = S^m\,.$$

Consider the antisymmetric polynomial

$$c(\mathfrak{s}_1,\mathfrak{s}_2) = p(\mathfrak{s}_1)\frac{\mathfrak{s}_2\mathfrak{q}^2}{\mathfrak{s}_1-\mathfrak{q}^2\mathfrak{s}_2} - p(\mathfrak{q}^2\mathfrak{s}_1)\frac{\mathfrak{s}_2}{\mathfrak{s}_1\mathfrak{q}^2-\mathfrak{s}_2} - p(\mathfrak{s}_2)\frac{\mathfrak{s}_1\mathfrak{q}^2}{\mathfrak{s}_2-\mathfrak{q}^2\mathfrak{s}_1} + p(\mathfrak{q}^2\mathfrak{s}_2)\frac{\mathfrak{s}_1}{\mathfrak{s}_2\mathfrak{q}^2-\mathfrak{s}_1}\,.$$

Suppose

$$c(\mathfrak{s}_1,\mathfrak{s}_2) = \sum_{j=1}^{n-1}\left(r_i(\mathfrak{s}_2)s_i(\mathfrak{s}_1) - r_i(\mathfrak{s}_1)s_i(\mathfrak{s}_2)\right)\,.$$

Pairing:

$$r_i\circ r_j = s_i\circ s_j = 0,\quad r_i\circ s_j = \delta_{i,j}\,.$$

Quantum Riemann bilinear identity

$$(m_1 \wedge m_2, C_{\pm})_0 = 2\pi i (m_1 \circ m_2)\,.$$

It can be checked that

$$\ell_{I^- \sqcup I^+, i} \circ \ell_{I^- \sqcup I^+, j} = 0\,,$$

which implies vanishing of certain local operators. In particular, if we define

$$\mathcal{C}_{\mathrm{even}} = \oint \psi^*(D)\chi(D)\frac{dD}{2\pi i D^3}\,,$$

then for $\alpha = \frac{1-\nu}{\nu}$

$$\mathcal{C}_{\mathrm{even}}\, \psi^*_{I^+}\chi^*_{I^-} M_0^{(\star)} \simeq 0, \quad \#(I^+) = \#(I^-) - 2\,.$$

Simplest case: $\psi_1^*\chi_{2m-1}^*\Phi_{(2m-1)\frac{1-\nu}{\nu}} = 0.$

Integrable structure of CFT

We claim that the space

$$\bigotimes_{m=-\infty}^{\infty} \mathcal{V}_{\alpha+2m\frac{1-\nu}{\nu}} \otimes \overline{\mathcal{V}}_{\alpha+2m\frac{1-\nu}{\nu}}\ ,$$

is created by action on $\Phi_\alpha(0)$ of $\mathbf{i}_{2j-1}$, $\bar{\mathbf{i}}_{2j-1}$ and fermionic creation operators:

$$\boldsymbol{\beta}^*_{2j-1},\ \boldsymbol{\gamma}^*_{2j-1},\ \bar{\boldsymbol{\beta}}^*_{2j-1},\ \bar{\boldsymbol{\gamma}}^*_{2j-1}\ .$$

What do we know about descendants of these fermions to Virasoro descendants? For technical reasons exact formulae are known only on the quotient space by $\mathbf{i}_{2j-1}$, $\bar{\mathbf{i}}_{2j-1}$ which can be realized as created by $\mathbf{l}_{-2k}$.

Introduce the notations

$$I^+ = \{2i_1^+ - 1, \cdots, 2i_p^+ - 1\}, \quad \boldsymbol{\beta}^*_{I^+} = \boldsymbol{\beta}^*_{2i_1^+-1} \cdots \boldsymbol{\beta}^*_{2i_p^+-1}, \quad etc\,.$$

Then if $\#(I^+) = \#(I^-)$,

$$\boldsymbol{\beta}^*_{I^+} \boldsymbol{\gamma}^*_{I^-} \Phi_\alpha(0) = \prod_{2j-1 \in I^+} D_{2j-1}(\alpha) \prod_{2j-1 \in I^-} D_{2j-1}(2-\alpha)$$
$$\times \left[P^{\text{even}}_{I^+,I^-}(\{\mathbf{l}_{-2k}\}|\Delta_\alpha, c) + d_\alpha P^{\text{odd}}_{I^+,I^-}(\{\mathbf{l}_{-2k}\}|\Delta_\alpha, c) \right] \Phi_\alpha(0)\,,$$

where

$$D_{2j-1}(\alpha) = -\sqrt{\frac{i}{\nu}}\,\Gamma(\nu)^{-\frac{2j-1}{\nu}} (1-\nu)^{\frac{2j-1}{2}} \frac{\Gamma\left(\frac{\alpha}{2} + \frac{1}{2\nu}(2j-1)\right)}{(j-1)!\Gamma\left(\frac{\alpha}{2} + \frac{1-\nu}{2\nu}(2j-1)\right)}\,,$$
$$d_\alpha = \tfrac{1}{6}\sqrt{(25-c)(24\Delta_\alpha + 1 - c)}\,.$$

Similarly for other chirality changing $\alpha \to 2-\alpha$.

Our main theorem.

Consider

$$\boldsymbol{\beta}^*(\zeta) = \sum_{j=1}^{\infty} \boldsymbol{\beta}^*_{2j-1}(\zeta \cdot \boldsymbol{\mu})^{-\frac{2j-1}{\nu}} + \sum_{j=1}^{\infty} \bar{\boldsymbol{\beta}}^*_{2j-1}(\zeta / \boldsymbol{\mu})^{\frac{2j-1}{\nu}} \quad etc\,.$$

Then

$$\frac{\langle \boldsymbol{\beta}^*(\zeta_1) \cdots \boldsymbol{\beta}^*(\zeta_m) \boldsymbol{\gamma}^*(\xi_m) \cdots \boldsymbol{\gamma}^*(\xi_1) \Phi_\alpha(0) \rangle_R^{\mathrm{sG}}}{\langle \Phi_\alpha(0) \rangle_R^{\mathrm{sG}}} = \det \omega_R^{\mathrm{sG}}(\zeta_i, \xi_j | \alpha)\,,$$

and the function $\omega_R^{\mathrm{sG}}(\zeta, \xi | \alpha)$ can be described through the TBA data. This allows to compare with previously introduced fermions and to conclude that

$$\boldsymbol{\beta}^*_{2k-1} = \psi_{2k-1}, \boldsymbol{\gamma}^*_{2k-1} = \chi^*_{2k-1}, \bar{\boldsymbol{\beta}}^*_{2k-1} = \bar{\psi}_{2k-1}, \bar{\boldsymbol{\gamma}}^*_{2k-1} = \chi^*_{2k-1}\,.$$

Back to null vectors. In the simplest case we have the singular vectors:

$$\boldsymbol{\beta}_1^*\boldsymbol{\gamma}_{2m-1}^*\Phi_{1,2m} = 0\,.$$

What can be checked with data at hand? For $\phi_{1,2}$ not a lot

$$\boldsymbol{\beta}_1^*\boldsymbol{\gamma}_1^*\Phi_{1,2} \sim \mathbf{l}_{-2}\Phi_{1,2} \ \ (\text{mod } \textit{left action of } \mathbf{i}_{2j-1},\bar{\mathbf{i}}_{2j-1})\,.$$

Actual null-vector is contains $\mathbf{l}_{-1}^2$, but $\mathbf{l}_{-1} = \mathbf{i}_1$.
For $m = 2$ we have

$$\boldsymbol{\beta}_1^*\boldsymbol{\gamma}_3^*\Phi_{1,4} = 0\,.$$

On the other hand we have

$$\boldsymbol{\beta}_1^*\boldsymbol{\gamma}_3^*\Phi_{\alpha} \sim \mathbf{l}_{-2}^2 + \Bigl(\frac{2c-32}{9} + d_\alpha\frac{2}{3}\Bigr)\mathbf{l}_{-4} \ \ (\text{mod } \textit{left action of } \mathbf{i}_{2j-1},\bar{\mathbf{i}}_{2j-1})\,.$$

For $\alpha = 3\frac{1-\nu}{\nu}$ this coincides with the CFT null-vector factorised by the left action of th integrals of motion!