Does quantum violation of null-energy-condition invalidate holographic c-theorem?

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Einstein had a dream

Unified TOE = Geometry

I have a dream

Consistent Quantum gravity

=

Consistent Renormalization group flow

Einstein equation and energy condition 1

Einstein equation allows any geometry

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = T_{\mu\nu}$$

Prepare such an energy-momentum tensor!

- Superluminal propagation (warp)
- Decreasing entropy
- Time machine (CTC)
- Wormhole

  Any constraint on EM tensor?

Einstein equation and energy condition 2

Null energy condition seems very promising

$$T_{\mu\nu}k^\mu k^\nu \geq 0\,, \quad k^\mu k_\mu = 0$$

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = T_{\mu\nu} \quad \longrightarrow \quad R_{\mu\nu}k^\mu k^\nu \geq 0$$

The condition is sufficiently strong to exclude

- Superluminal propagation (warp)
- Decreasing entropy in black hole
- Time machine (CTC)
- Wormhole

Holographic c-theorem relies on it (see later)

Null energy condition violation

(Un)fortunately, null energy condition (NEC) seems violated from various quantum effects

- Casimir effects
- Weyl anomaly
- Squeezed quantum states
- Hawking radiation
- Orientifold

$$L_0 = T_{++} = -\frac{c}{24l}$$

Hard to say if these violations are good or bad.

Hard to say if quantum gravity must exclude "pathological" space-time unless...

Unless we have holography

Holography tells us something we have to protect in quantum gravity
(even with NEC violation)

- Unitarity
- Causality (of dual theory)
- Rigorously proved theorems in dual QFT
- In particular, c-theorem
- NEC violation is OK as long as it is consistent

If you don't believe in holography, it is still OK.

I give some mysterious (unexpected?) results.

Field theory c-theorem (d=2)

$$T^{\mu}_{\ \mu} = -\frac{c}{12}R$$

- $c_{UV} \geq c_{IR}$ (weak theorem) proved
- $$\frac{d\tilde{c}}{d\log\mu} = |x|^4 \langle T^{\mu}_{\ \mu}(x) T^{\mu}_{\ \mu}(0)\rangle = G_{IJ}B^I B^J$$
  (strong theorem) proved
- $$\partial_I \tilde{c} = (G_{IJ} + w_{IJ})B^J + (P^I g)_J w^J$$
  (gradient formula) proved?

- Scale invariance implies conformal invariance

Field theory a-theorem (d=4)

$$T^{\mu}_{\ \mu} = a\text{Euler} - c\text{Weyl}^2$$

- $a_{UV} \geq a_{IR}$ (weak theorem) proved
- $$\frac{d\tilde{a}}{d\log\mu} = G_{IJ}B^I B^J$$

  (strong theorem) perturbatively proved?
- $$\partial_I \tilde{a} = (G_{IJ} + w_{IJ})B^J + (P_I g)^J w_J$$

  (gradient formula) perturbatively proved??
- Scale invariance implies conformal invariance within perturbation theory

Summary of recent discussions

- Dilaton scattering approach by Komargodski and Schwimmer is convincing. Physical (non-perturbative) proof of weak a-theorem
- There was a debate over "counterexamples" of scale invariance without conformal invariance in d=4
- Counterexamples are all gone. Very subtle problem though. Gauge (scheme) artifact.
- Some progress in strong a-theorem as well as generic argument for scale = conf in d=4 (Luty-Rattazzi-Polchinski).
- Perturbative. Probably known by Osborn in 90's.

You can find my lecture notes at

https://sites.google.com/site/scalevsconformal/

Holographic c-theorem (macro)

Consider holographic flow $ds^2 = dr^2 + e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu$

Radial direction = renormalization scale

$$r \sim \log \mu$$

Define holographic c-function $a(r) = \frac{1}{((A)'(r))^{d-1}}$

Einstein equation:

$$\frac{da(r)}{dr} = -\frac{A''(r)}{(A'(r))^d} = -\frac{1}{(A'(r))^d}[T^t{}_t - T^r{}_r] \geq 0$$

Monotonically decreasing as long as NEC holds.

$$T_{\mu\nu}k^\mu k^\nu \geq 0\ , \quad k^\mu k_\mu = 0$$

Holographic c-theorem (micro)

When matter is given by NLSM (with potential)

$$T^t_{\ t} - T^r_{\ r} = g^{rr} G_{IJ} \partial_r \Phi^I \partial_r \Phi^J$$

Reminds us of strong c-theorem $\frac{da}{dr} \sim G_{IJ}\beta^I \beta^J$

Metric is positive (unitarity).

In (fake)supergravity, we can consider BPS flow

$$\partial_r \Phi^I = G^{IJ} \partial_J |W|$$

$$\partial_I \tilde{c} = (G_{IJ} + w_{IJ})\beta^J$$

Precisely the gradient formula

Why w = 0 ? (in many examples, w is exact)

Scale vs Conformal

Scale invariance: $T^{\mu}_{\ \mu} = \partial^{\mu} J_{\mu}$

Conformal invariance: $T^{\mu}_{\ \mu} = 0$

$$T^{\mu}_{\ \mu} = \beta^I O_I + \beta_a \partial^{\mu} J^a_{\mu} = \partial^{\mu} J_{\mu}$$

$$\partial^{\mu}(\bar{\psi}\gamma_{\mu}\psi) = Y\phi\psi\psi + c.c. + \cdots$$

Q: Is it easier to find scale invariant solution?

Note because of EOM $S_a \partial^{\mu} J^a_{\mu} = (Sg)^I O_I$ beta functions are ambiguous (gauge dependence)

$$\begin{aligned} \beta^I &\rightarrow \beta^I + (S \cdot \beta)^I \\ \beta_a &\rightarrow \beta_a + S_a \end{aligned}$$

Can fix gauge: $T^{\mu}_{\ \mu} = B^I O_I$ $\quad \frac{dg^I}{d\log\mu} = B^I$

Holographic scale = conf (micro)

Operator identity $S_a \partial^\mu J^a_\mu = (Sg)^I O_I$ is realized by gauging: $\Phi \to e^{i\Lambda}\Phi \ , \quad A \to A + d\Lambda$

- Scale but non-conformal solution

$$\begin{aligned} A &= a\frac{dz}{z} \\ \Phi &= c \end{aligned} \qquad\qquad \begin{aligned} A &= 0 \\ \Phi &= cz^{ia} \end{aligned}$$

- Two obstructions
  - Holographic c-theorem
  - no-potential for gauge direction

$$\frac{da}{dr} \sim g_{IJ} D^r \Phi^I D_r \Phi^J$$

$$(S_J \Phi)^I D_I W(\Phi) = 0$$

Again, (strict) null energy condition is crucial

Does this beautiful story miserably fail once NEC is violated?

Averaged Achronal NEC?

It is (un)fortunate string theory has not posed a good answer to violation of NEC

- GR people proposed averaged NEC

$$\int d\lambda T_{\mu\nu} k^\mu k^\nu \geq 0$$

- Can avoid Casimir energy example by restricting to achronal averaging
- Interesting proposal (given weak a-theorem)
- But may not explain the weight...

$$\frac{da(r)}{dr} = -\frac{A''(r)}{(A'(r))^d} = -\frac{1}{(A'(r))^d}[T^t{}_t - T^r{}_r] \geq 0$$

- And violation is known....

Anomalous NEC violation in AdS4

No general argument, but I'd like to focus on a universal violation due to Weyl anomaly

- Holographic RG metric is conformally flat

$$ds^2 = dr^2 + e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu$$

- Weyl anomaly $T^M_M = a\mathrm{Euler} - c\mathrm{Weyl}^2$ induces

$$\begin{aligned}
\bar{T}^M_N &= \Omega^{-4}T^M_N - 8c\Omega^{-4}\left((W^{AM}_{BN}\log\Omega)^{;B}_{;A} + \frac{1}{2}R^B_A W^{AM}_{NN}\log\Omega\right) \\
&- a\left(4\bar{R}^B_A\bar{W}^{AM}_{BN} - 2\bar{H}^M_N - \Omega^{-4}(4R^B_A W^{AM}_{BN} - 2H^M_N)\right) \\
H_{MN} &= -R^A_M R_{AN} + \frac{2}{3}RR_{MN} + \left(\frac{1}{2}R_{AB}R^{AB} - \frac{1}{4}R^2\right)g_{MN} \, .
\end{aligned}$$

Anomalous NEC violation in AdS4

$$\begin{aligned} \bar{T}^M_{\ N} &= \Omega^{-4} T^M_{\ N} - 8c\Omega^{-4}\left((W^{AM}_{\ \ BN}\log\Omega)^{;B}_{;A} + \frac{1}{2}R^B_A W^{AM}_{\ \ NN}\log\Omega \right) \\ &- a\left(4\bar{R}^B_A \bar{W}^{AM}_{\ \ BN} - 2\bar{H}^M_N - \Omega^{-4}(4R^B_A W^{AM}_{\ \ BN} - 2H^M_N)\right) \\ H_{MN} &= -R^A_M R_{AN} + \frac{2}{3}RR_{MN} + \left(\frac{1}{2}R_{AB}R^{AB} - \frac{1}{4}R^2\right) g_{MN} \ . \end{aligned}$$

Evaluate the anomalous contribution in

$$ds^2 = dr^2 + e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu$$

$$(\bar{T}^r_{\ r} - \bar{T}^t_{\ t})|_{\rm anom} = 4aA''(r)(A'(r))^2 \geq 0 \ ,$$

Anomalous part violates NEC

Still holographic c-theorem holds

Let us define modified holographic c-function

$$a(r) \equiv \frac{\pi^{3/2}}{\Gamma(3/2)(A'(r))^2} - 4a\frac{\pi^{3/2}}{\Gamma(3/2)} \log A'(r) \ ,$$

It is monotonically decreasing

$$a'(r) = -\frac{\pi^{3/2}}{\Gamma(3/2)(A'(r))^3)}(T^t{}_t - T^r{}_r)|_{\rm class} \geq 0$$

as long as the classical part satisfies the NEC.

(In this sense, due to the classical part, this is not a so-called consistent NEC violation)

Other corrections of the same order

We modified gravity EOM at $O(R^2)$

Higher curvature corrections

$$S = \int d^4x \sqrt{g} a_1 R^2_{MNIJ} + a_2 R^2_{MN} + a_3 R^2$$

give completely local modification of holographic RG-flow (Myers et al)

To preserve holographic c-theorem

- Have to avoid ghost: $a_1 + a_2 + 3a_3 = 0$
- We had to cancel $\delta T^{\mu}_{\mu} = b \Box R$ Weyl anomaly

In string theory, there may be a separation...

Log correction in c-function

I do not have a complete field theory interpretation of log correction...

- In d=3, presumably holographic c-function is related to S^3 partition function or entanglement entropy
- There is a very interesting recent computation by Bhattacharyya, Grassi, Marino, Sen
  $\delta F_{1-loop} = \log N$ from 1-loop SUGRA
- Reproduced ABJM computation (reasonable?)
- It should have same origin

Summary

- NEC is a holy grail in classical holography
- Can be (should be?) violated quantum mechanically
- "Reasonable violation" such as anomaly induced one does not invalidate the holographic c-theorem
- Nonlocal log correction (beyond local curvature corrections)

- What is "reasonable" violation?
- Full string computation?

Thank you and
a happy new year !