Large N Volume Independence
and
3D Bosonization

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University of Minnesota

at IPMU, February 5 2013

Based on arXiv:1208.1769, A.C. & Daniele Dorigoni

Organization of the talk

Part I (20 mins):

Introduction to volume independence in large N gauge theories

Part II (rest of talk:

How do we use it?

Idea: free bosonization from shackles of two dimensions

QCD and its large N limits

Talk is about large N gauge theories. Why care?

QCD: SU(3) gauge theory with quarks, underlies all of nuclear physics!

Despite 40 years of efforts, analytic solution has been far out of reach. No obvious expansion parameters!

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Non-obvious expansion parameter: $1/N$, number of colors. ‘t Hooft 1973

QCD at large N simplifies dramatically. Still no analytic solution, but some quantitative and qualitative insights developed.

$$\text{For phenomenology, good reason to think } 3 \gg 1$$

Remarkable property of (some) large N gauge theories:
volume independence.

Volume independence is an example of a common feature of large N gauge theories, *orbifold equivalence*.

Turns out different large N gauge theories equivalent to each other!

Simplification of QCD at large N

Large N limit: $N \to \infty$, keeping $g^2 N$ fixed, N_f fixed

But there is more than one such large N limit of QCD.

't Hooft 1973
Armoni, Shifman, Veneziano 2003

- Quarks in F representation
- Quarks in 2-index AS representation

Isomorphic at N=3!

In this talk, focus instead on QCD with adjoint quarks: QCD(Adj)

Surprising result of ASV: at large N and large volume, many observables in QCD(AS) = corresponding observables in QCD(Adj).

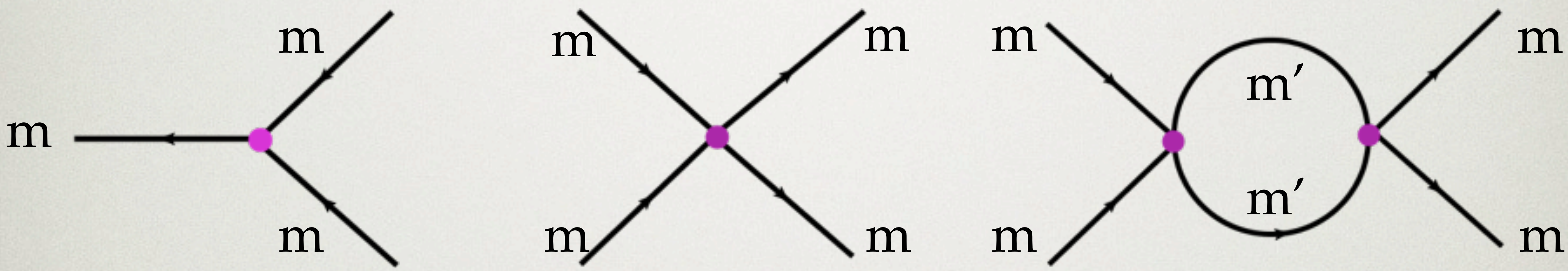
This will make at least part of what I say connected to real-world QCD

Simplification of QCD(Adj) at large N

Large N limit: $N \to \infty$, keeping $g^2 N$ fixed, N_f fixed

Still very strongly coupled in terms of quarks and gluons

But physical states interact only weakly, so theory is nearly free!



$\sim 1/N$ $\qquad$ $\sim 1/N^2$ $\qquad$ $\sim \frac{1}{N^2}\frac{N_f}{N^2} \sim \frac{1}{N^4}$

Meson (and glueballs) become stable and weakly-interacting at large N

In part II, we'll see that this gives insight on idea of orbifold equivalence...

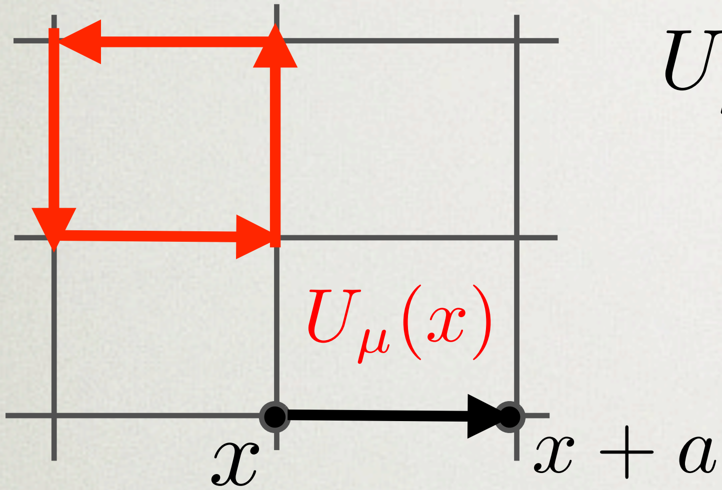
How does a large N gauge theory on e.g. T^4 depend on volume?

For pure YM, Eguchi & Kawai suggested answer: it doesn't.

Eguchi-Kawai Reduction

Eguchi & Kawai
1982

E&K studied pure 4D Yang-Mills theory on T^4 ,
discretized on Euclidean lattice.



$U_\mu(x) = e^{iaA_\mu(x)}$ K points in each direction

Action = sum over traces of all plaquettes
- closed loops of U's.

Non-perturbative definition of YM
with exact gauge invariance

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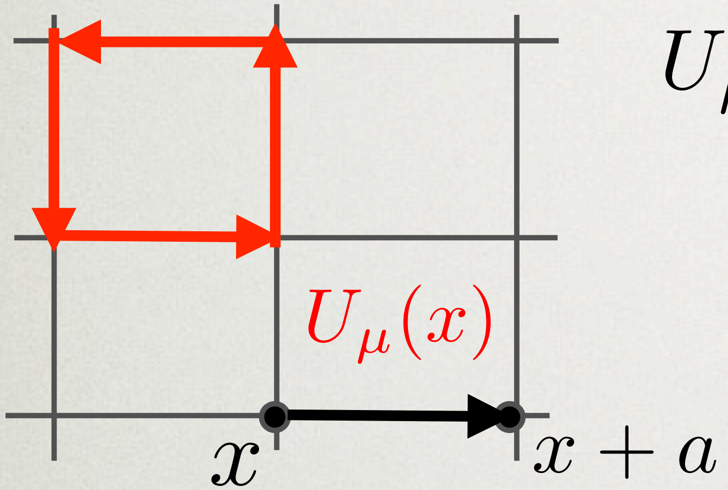

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Non-perturbative definition of YM
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All observables of theory can be extracted from correlation functions of closed loops of U's: the Wilson loops.

E&K showed correlation functions of Wilson loops in K^4 theory must be same as corresponding `loops' in single-plaquette theory at large N.

Equivalence between 4D theory and `0D' unitary matrix model!

Argument involved crucial assumption: unbroken `center symmetry'.

Death of EK reduction, and its resurrection

If EK reduction worked, would be amazing. At the least, would have been able to understand pure YM even back in 80s numerically!

So was the crucial assumption about center symmetry correct?

For *large* T^4, center symmetry in YM is **not** spontaneously broken, and its realization is an order parameter for confinement.

Order parameter $$\mathrm{Tr}\,\Omega_\mu = \mathrm{Tr}\,\mathcal{P} e^{i \int_{S^1_\mu} dx\, A_\mu}$$

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Order parameter
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Bhanot, Heller, Neuberger 1982

Unfortunately, EK reduction **fails** in pure YM theory for *small* T^4.

As any S^1 in T^4 is shrunk, there's 1st order phase transition, center symmetry breaks.

This is just the standard deconfinement transition as temperature = 1/(size of S^1) is increased.

Many attempts over the years to fix the problem; all failed.

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Roadblock for 25 years...

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Unsal,
Kovtun, Yaffe

2007: working version of large N volume independence found!

Two new ideas: consider QCD(Adj) instead of YM, **and**
use periodic boundary conditions for fermions on T^4

Fermions with PBCs preserve center symmetry no matter the size of T^4

No thermal interpretation of size of S^1's with PBCs: spatial compactification

Euclidean path integral computes $\tilde{Z} = \mathrm{tr}\,(-1)^F e^{-LH}$

Example: $N_f = 1$ QCD(Adj), which is $\mathcal{N}$=1 super YM theory. On T^4 with
SUSY-preserving BCs, known not to have phase transitions!

So QCD(Adj) independent of size of T^4, down to
arbitrarily small sizes in the large N limit.

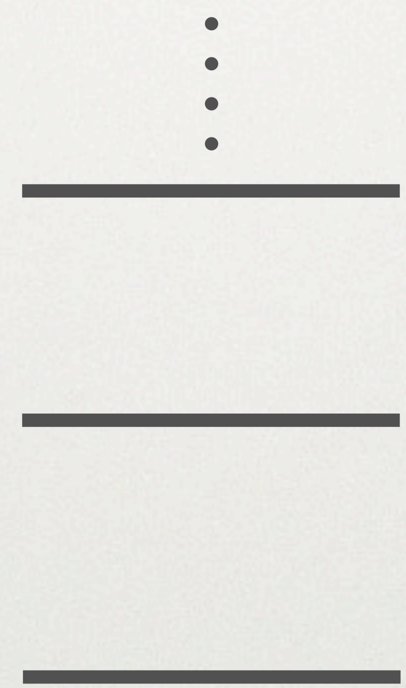
But how can a theory not see the volume of space it lives on?

Heuristic picture of volume (in)depedence

Take a scalar field theory on R^3xS^1, with circumference L, periodic BCs

Expand fields in Fourier modes on S^1

Result: KK spectrum



$$m = \frac{4\pi}{L}$$

$$m = \frac{2\pi}{L}$$

$$m = 0$$

Theory feels finiteness of L: gives a gap in the spectrum, etc.

As $L \to \infty$, KK spacing vanishes, momenta become continuous

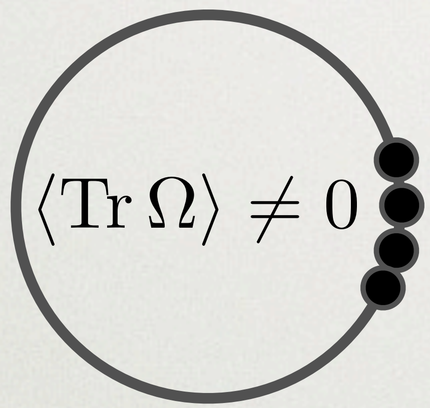
Gauge theories are more subtle!

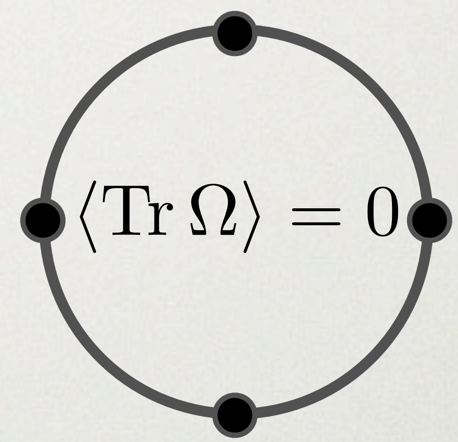
Heuristic picture of volume (in)depedence

Take SU(N) gauge theory on $R^3 x S^1$, circumference L, periodic BCs

Suppose $L \ll \Lambda_{\mathrm{QCD}}$. Then asymptotic freedom $\Rightarrow$ weak coupling

Physics depends on expectation value of $\quad \mathrm{Tr}\,\Omega = \mathrm{Tr}\,\mathcal{P}e^{i\int_{S^1} dx A_{S^1}}$





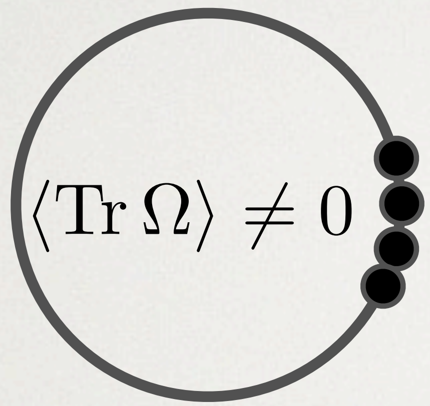
Center symmetry
broken

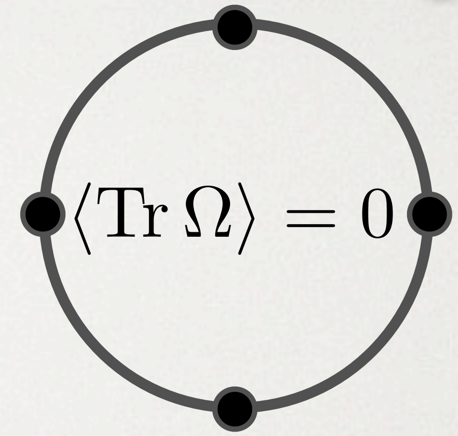
Center symmetry
preserved

Pure YM, YM+adjoint
fermions with AP BCs

YM+ adjoint fermions
with P BCs

Heuristic picture of volume (in)depedence

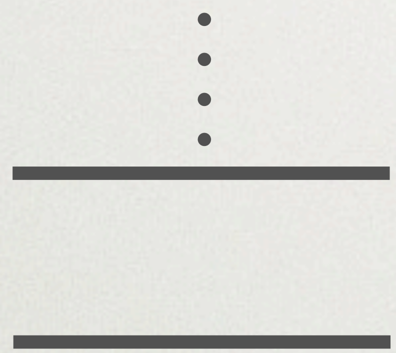


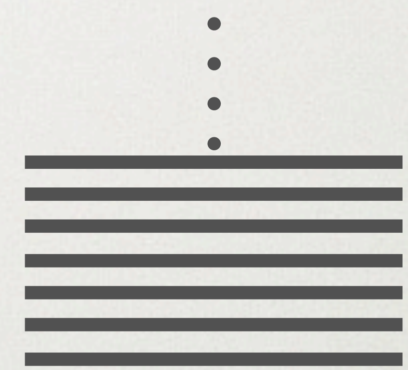


Broken center

Preserved center

$$\Delta m = \frac{2\pi}{L}$$





$$\Delta m = \frac{2\pi}{NL}$$

Non-trivial VEV for A_4 produces adjoint Higgs mechanism

Equal eigenvalue spacing for $\Omega \Rightarrow$
lightest W-boson mass $m_W = \frac{2\pi}{NL}$

Heuristic picture of volume (in)depedence

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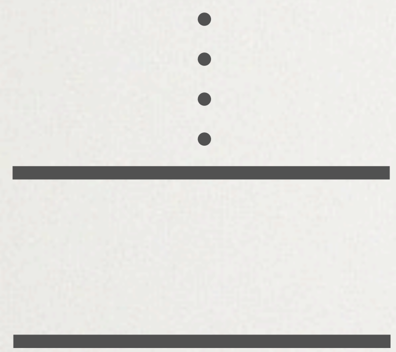
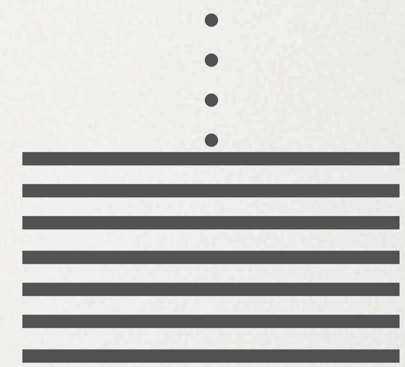


Preserved center



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Non-trivial VEV for A_4 produces adjoint Higgs mechanism

Equal eigenvalue spacing for $\Omega \Rightarrow$
lightest W-boson mass $m_W = \frac{2\pi}{NR}$

Fix N, small L: weakly coupled confining phase, **volume dependence**

or

Large N, L~1/N: weakly coupled confining phase, **volume dependence**

Gives theorist-friendly deformation of theories on R^D;
with physics smoothly connected to R^D

Small-L center symmetric regime

Small-L regime gives window on tractable regime of supersymmetric *and* non-supersupersymmetric QFTs

Fruitful and active program with many very interesting results

Unsal, Argyres, Poppitz, Shifman, Dunne, Yaffe, Schaefer ...

Surprising and novel semiclassical realization of confinement, due to non-self-dual topological defects

First physical realization of ‘t Hooft’s IR renormalons

Possibility of defining QFTs non-perturbatively in the continuum using ideas of transseries and resurgence

Heuristic picture of volume (in)depedence

Broken center

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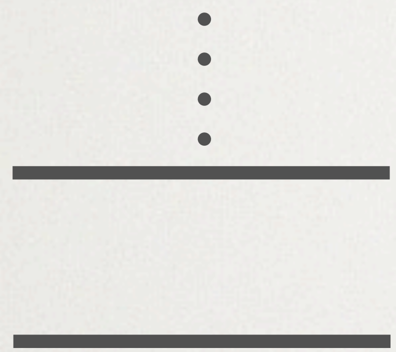

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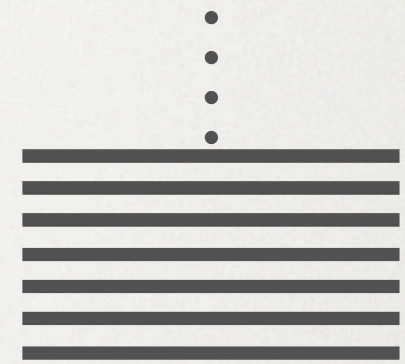

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Fix N, small L: weakly coupled confining phase, **volume dependence**

Large N, L~1/N: weakly coupled confining phase, **volume dependence**

Fix L, take large N: modes become continuum, **volume independence**

Large N gauge theories in center-symmetric
phase don't notice compactification!

Non-perturbative volume indepedence

What's the connection of Eguchi-Kawai lattice picture?

What we just described was continuum/weak coupling version of volume independence

Need to map continuum changes of volume to lattice

Non-perturbative volume indepedence

What's the connection of Eguchi-Kawai lattice picture?

Unsal, Kovtun, Yaffe realized:
volume independence = orbifold equivalence.

(More details in Part II)

$R^d x S^1$ gauge theory with discretized S^1 = quiver gauge theory on R^d

Dimensional deconstruction of Arkani-Hamed, Cohen, Georgi.

Change of volume = change of number of lattice sites
= change in number of nodes of quiver

UKY proved that if center symmetry is preserved,
one node theory is equivalent to many-node theory

QCD(adj) has center symmetry for any number of nodes.

This is a *working* version of EK reduction!

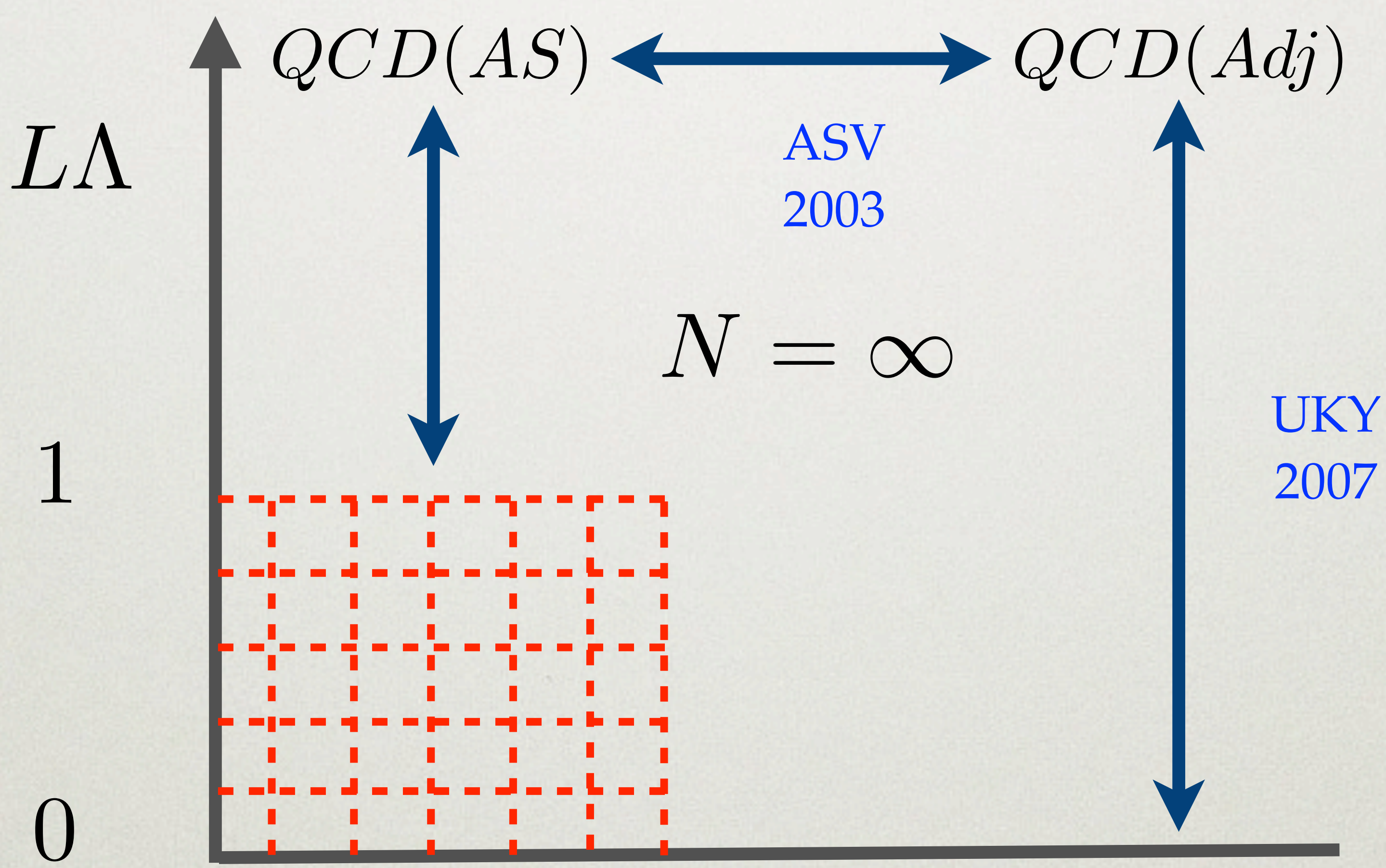
Gives equivalences between gauge theories on
R^d and lower-dimensional field theories.

Summary of Part I

Large N limit gives powerful insights on QCD, the theory underlying all of nuclear physics

Since 80s, dreams that large N theories might enjoy *volume independence*

Dreams realized recently: QCD(Adj) has desired property!



Part II: What can we do with this?

AC, Dorigoni 2012

Part II: Outline

Explain the modern perspective on volume independence as an orbifold equivalence

Result: equivalences between familiar gauge theories in dimension D, and other (less-familiar) gauge theories in lower d < D.

Correspondence between 3D QCD(Adj) on $R^2 x S^1$ and 2D theories

Illustrate matching between 3D and 2D theories at small L

Use this to extend bosonization from 2D to 3D theories!

Kachru, Silverstein 1998

Orbifold Projections and Equivalences

Kovtun, Unsal, Yaffe, 2003-4

First found in context of string theory and AdS/CFT, but the notion has natural home in QFT

Step 1: Define a projection mapping one gauge theory to another

Step 2: Understand conditions for the theories to be equivalent

Orbifold projection: algorithm to toss out d.o.f in \`mother' gauge theory to make \`daughter' theory

Involves identifying symmetry under which stuff being kept is neutral while stuff tossed is charged.

Orbifold equivalence

If projection symmetry is not spontaneously broken

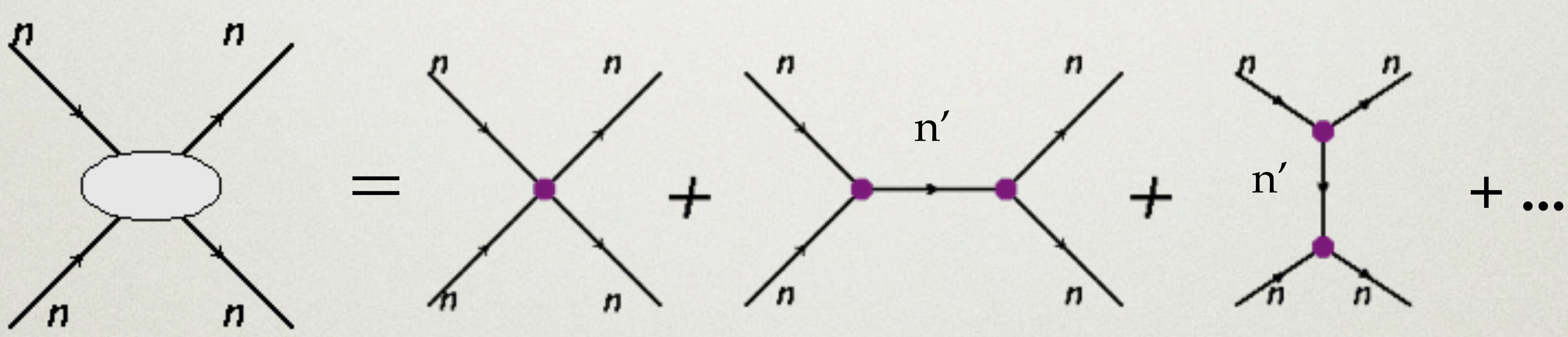
Then correlation functions of \`neutral' operators in mother and daughter theories will coincide in the large N limit.

Cartoon picture of orbifold equivalence

Consider mother and daughter gauge theories related by symmetry which acts non-trivially on color-singlet operators of mother.

By assumption, both mother and daughter have ‘n’ operators, but only mother has charged ‘c’ operators.

4-point correlator of uncharged color-singlet operators \`n’ in mother and daughter theories at large N:



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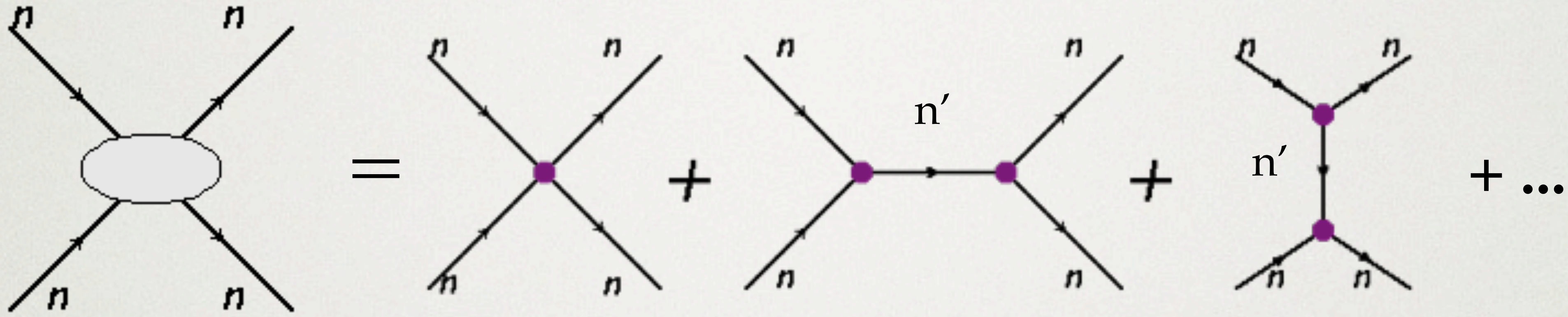

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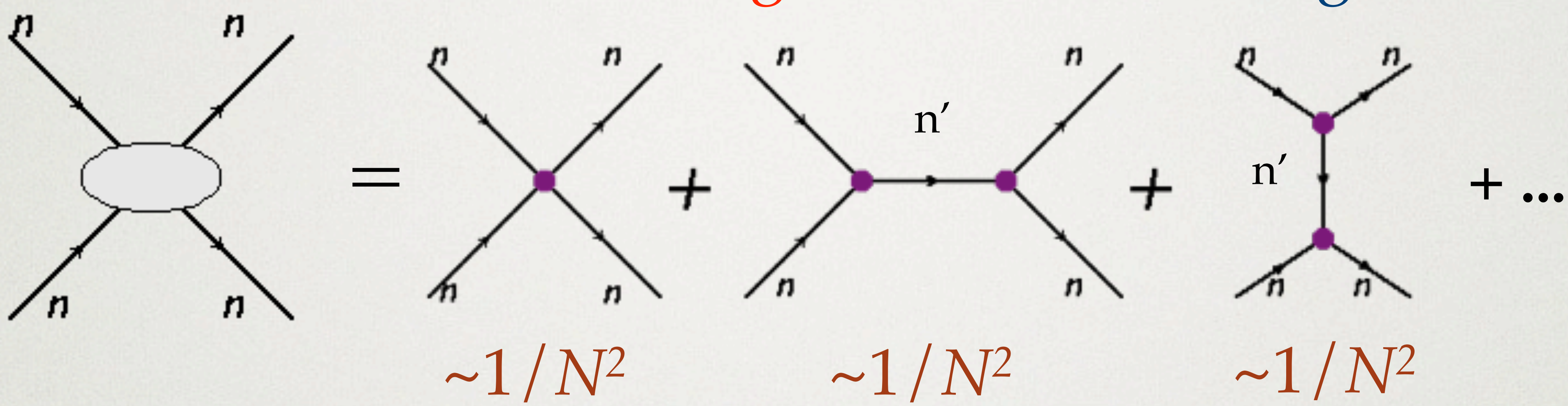
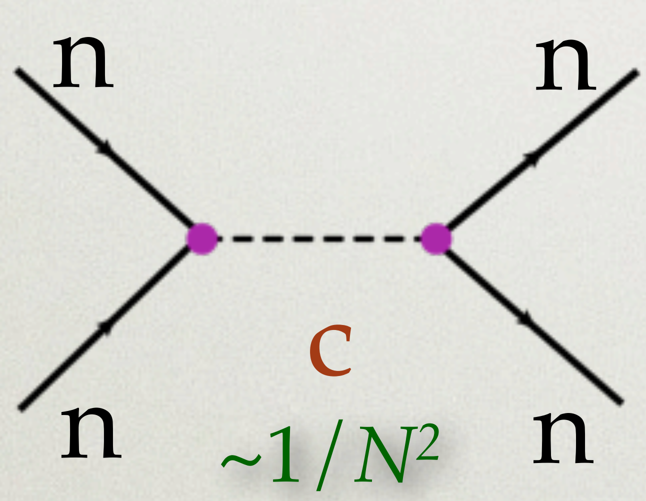


Meson processes in Mother not possible in Daughter:



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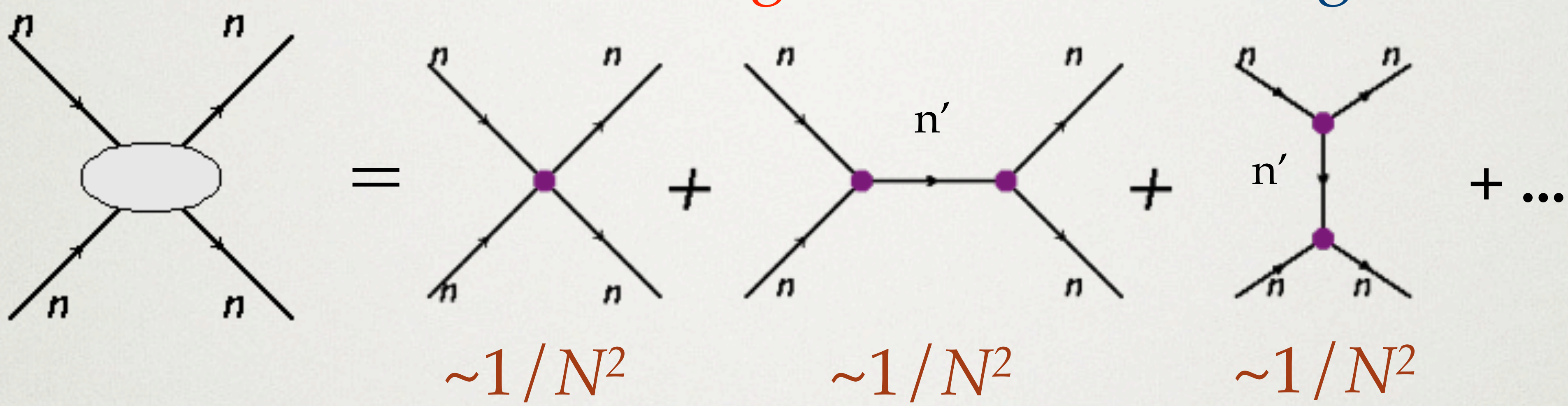
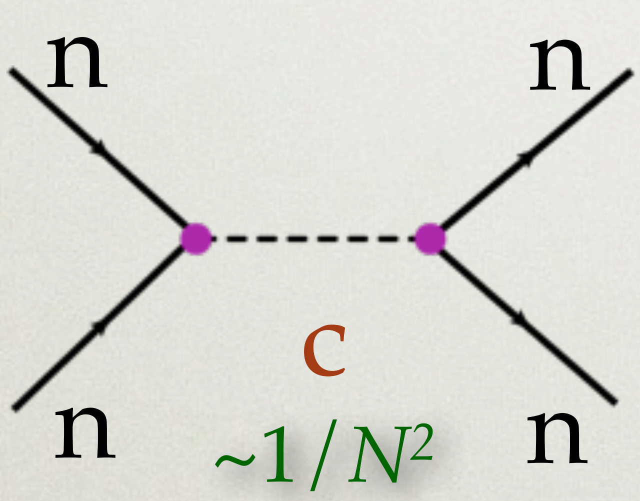


Meson processes in Mother not possible in Daughter:

Not allowed if symmetry is unbroken



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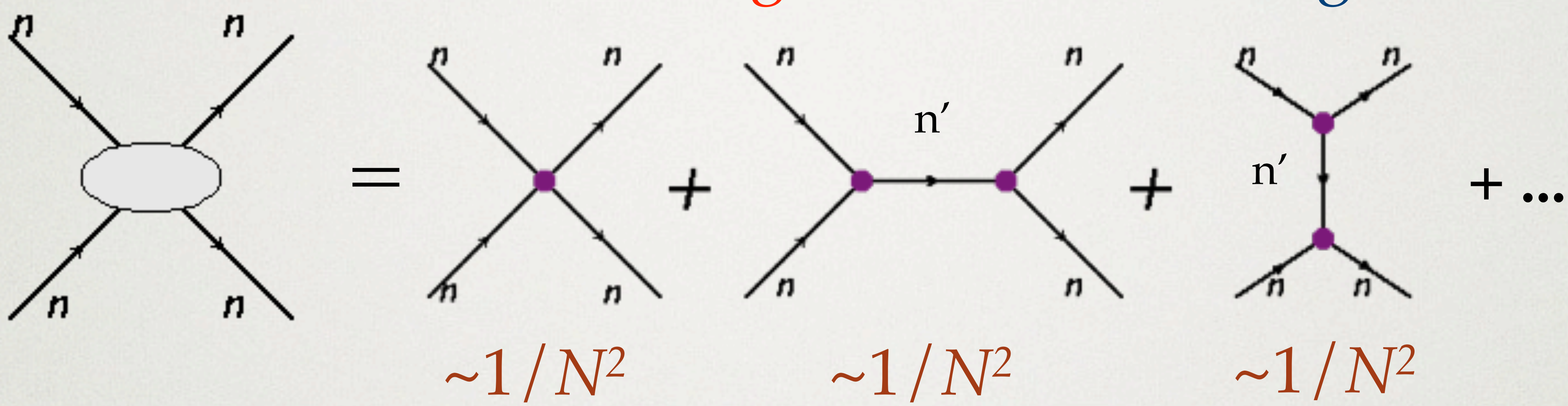

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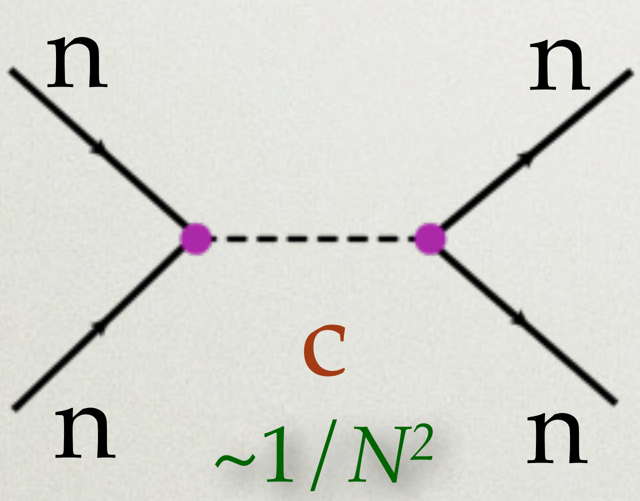

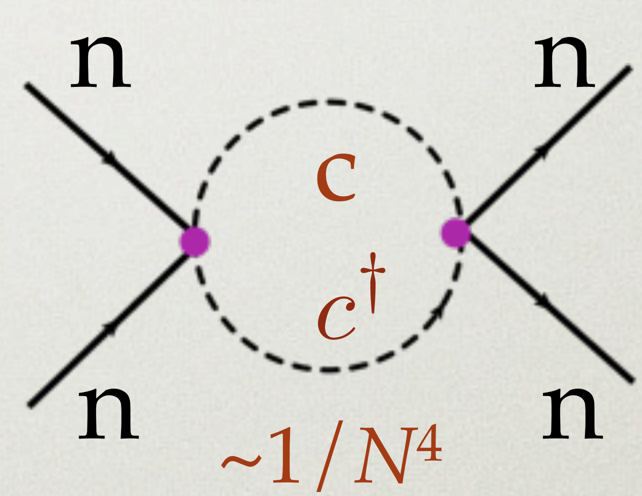

Allowed but suppressed

Cartoon picture of orbifold equivalence

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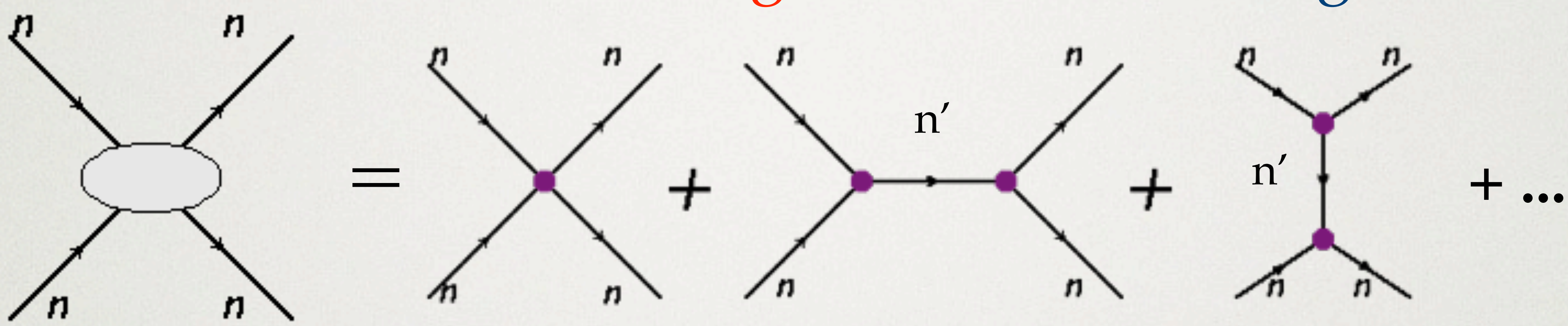

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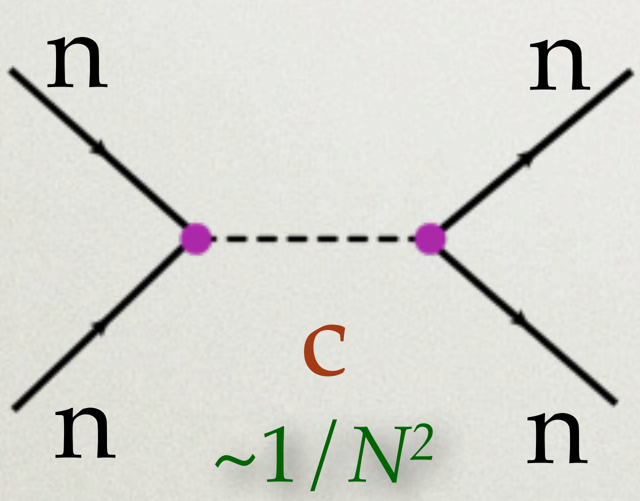
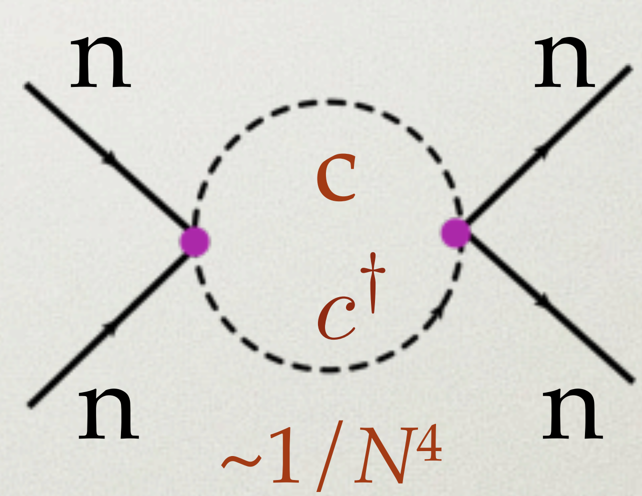




Allowed but suppressed

So if N is large, *and* the symmetry used to relate the parent and daughter is conserved, expect equivalence!

Volume independence as an orbifold equivalence

To view volume independence as an orbifold projection, need to define volume-changing orbifold projections.

Best-understood projections based on discrete groups

Looking for setting where discrete volume changes are natural

Volume independence as an orbifold equivalence

To view volume independence as an orbifold projection, need to define volume-changing orbifold projections.

Best-understood projections based on discrete groups

Looking for setting where discrete volume changes are natural

Obvious idea: discretize direction you want to shrink

Now translation group is discrete

Volume reduction: orbifold by subgroups of translation group

Another perspective: switch dimensional deconstruction of target theory

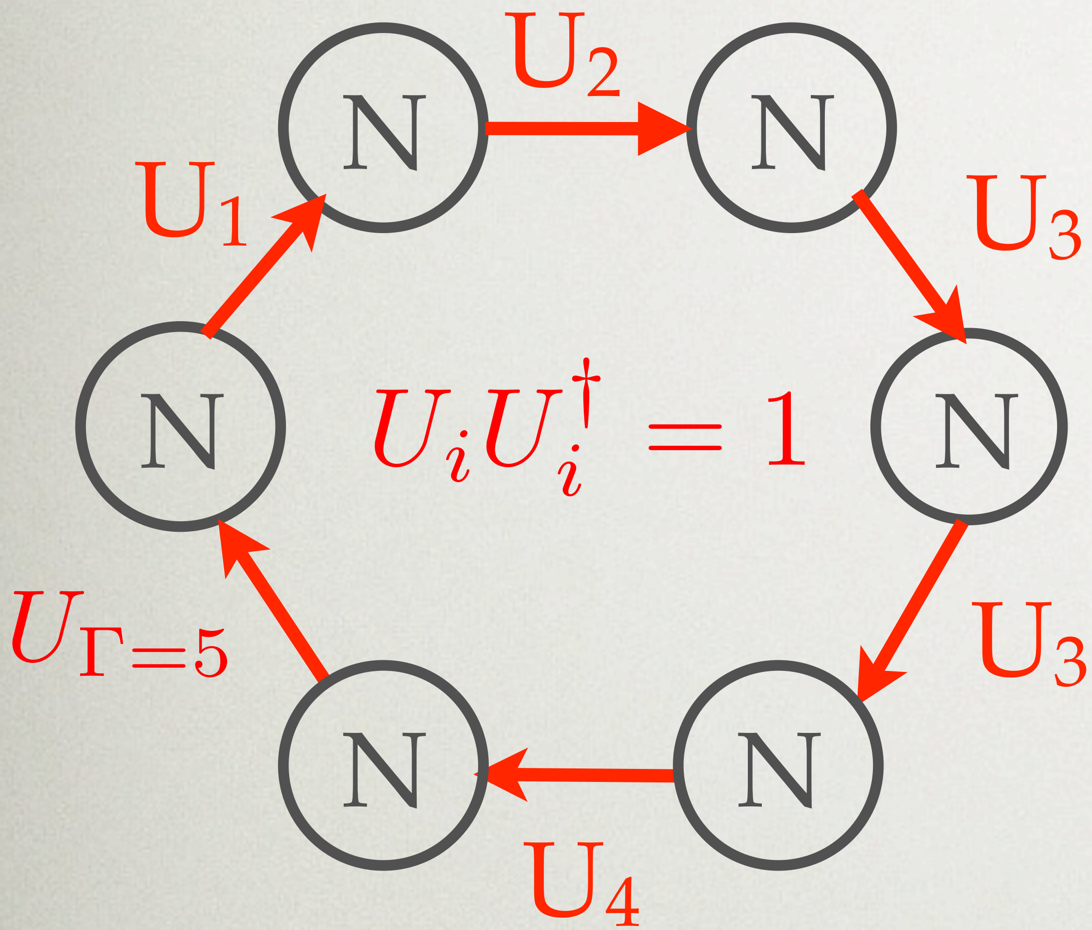
Result is a quiver gauge theory equivalent to continuum theory in IR

(This is just different words for the same thing...)

Volume expanding projections also possible, use center symmetry

Orbifolds and volume reduction

3D YM, deconstructed: 2D $SU(N)^\Gamma$ quiver gauge theory

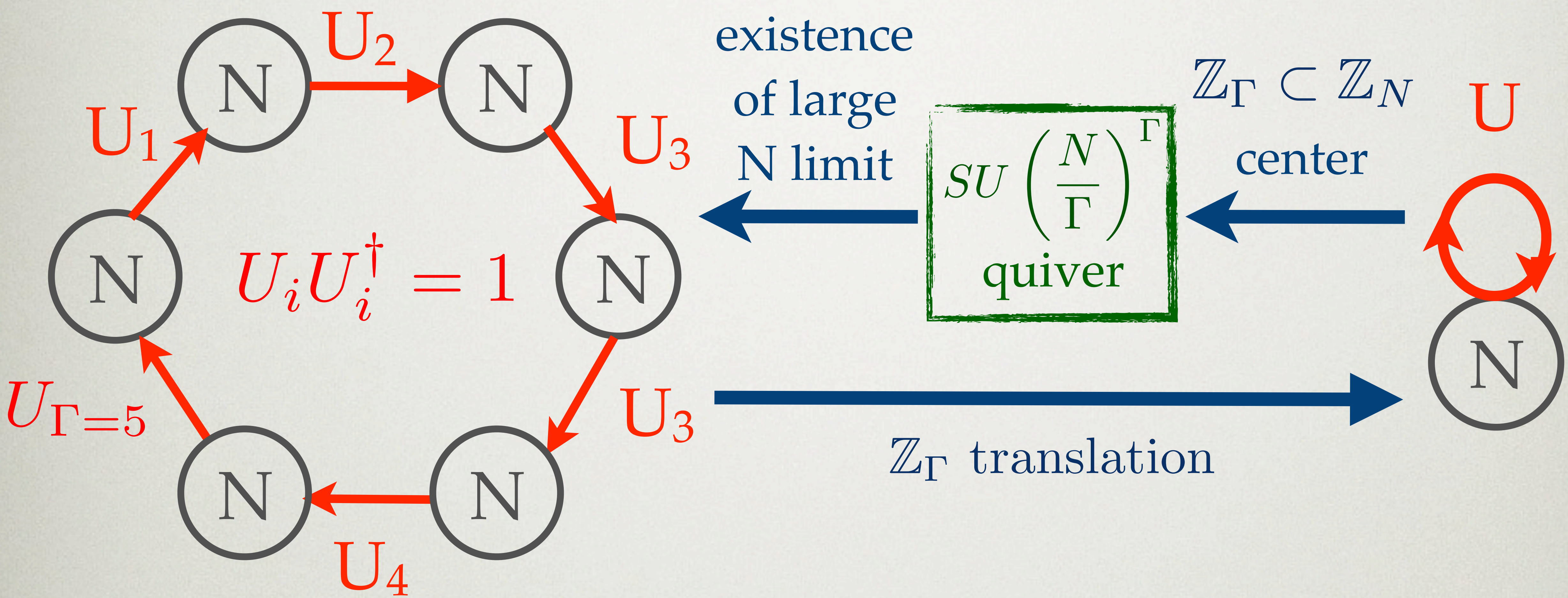


$$\mathcal{L} \ni \frac{1}{g^2 a^2} \mathrm{tr}\, |D_\mu U_i|^2$$

3D YM on $\mathbb{R}^2 \times S^1_{L=\Gamma a}$

Orbifolds and volume reduction

3D YM, deconstructed: 2D $SU(N)^\Gamma$ quiver gauge theory



$$\mathcal{L} \ni \frac{1}{g^2 a^2} \mathrm{tr}\, |D_\mu U_i|^2$$

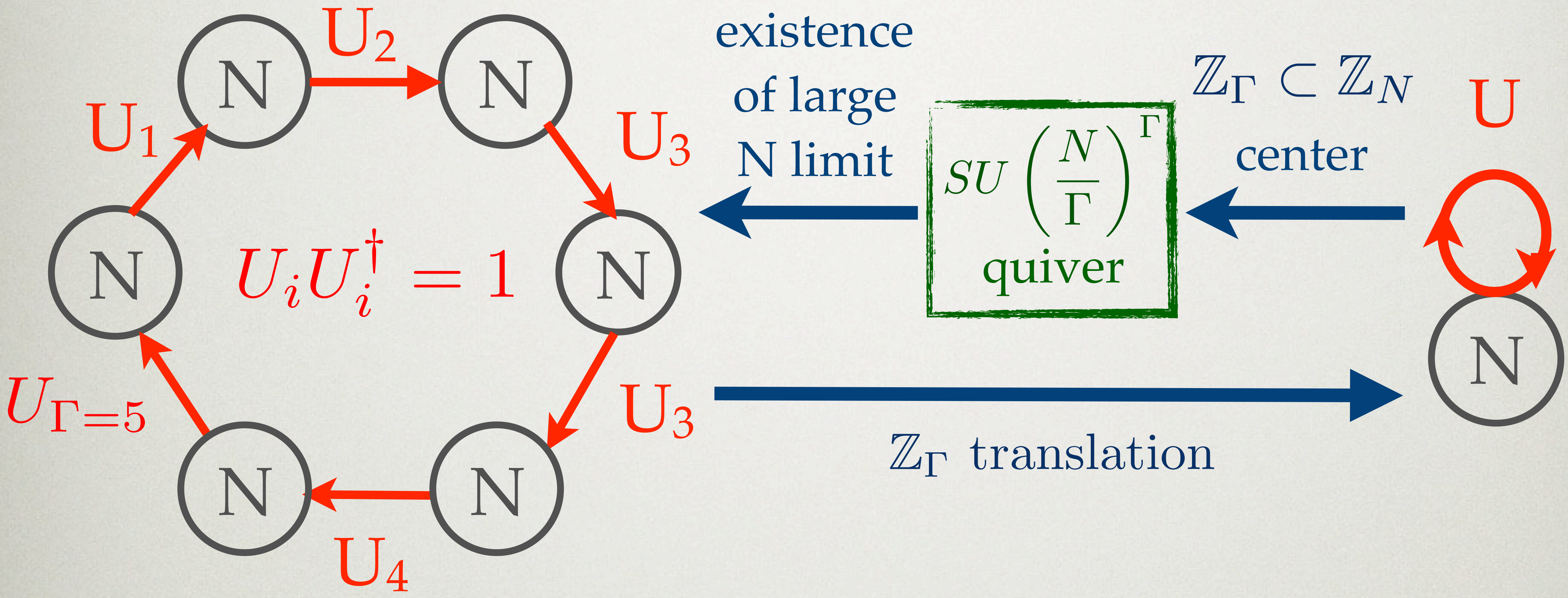
$$\mathcal{L} \sim \frac{1}{g^2} \mathrm{tr} \left[F_{\mu\nu}^2 + \frac{1}{a^2} |D_\mu U|^2 \right]$$

3D YM on $\mathbb{R}^2 \times S^1_{L=\Gamma a}$

2D gauged sigma model

Orbifolds and volume reduction

3D YM, deconstructed: 2D $SU(N)^\Gamma$ quiver gauge theory



3D YM on $\mathbb{R}^2 \times S^1_{L=\Gamma a}$

2D gauged sigma model

Large N equivalence between 3D gauge theory and 2D theory, so long as symmetries preserved.

Volume independence vs. symmetries

For volume independence, the critical symmetries are center symmetry, and translation symmetry.

Don't expect translation breaking in theories we consider

In SU(N) gauge theory on T^D, gauge transformations need only be periodic up to center of SU(N).

Gives rise to global symmetry: $\mathbb{Z}_N^D$ center symmetry

Sigma models have corresponding phase symmetry

Example order parameters

$$\begin{aligned} \operatorname{tr}\Omega &= \operatorname{tr}\mathcal{P}e^{i\int_{S^1} dx\, A_{S^1}} \\ &= \operatorname{tr} U_1 U_2 \ldots U_\Gamma \\ &\to \operatorname{tr} U \end{aligned}$$

Unsal, Kovtun, Yaffe, 2003-2004

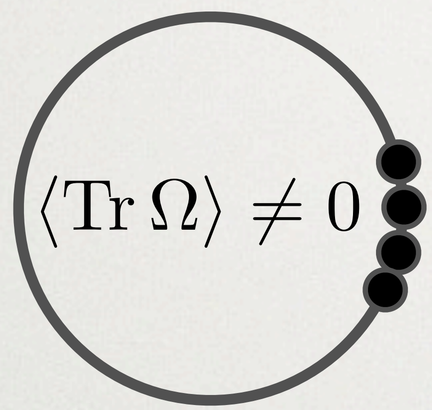
KUY: if and only if such symmetries preserved, orbifold equivalence holds

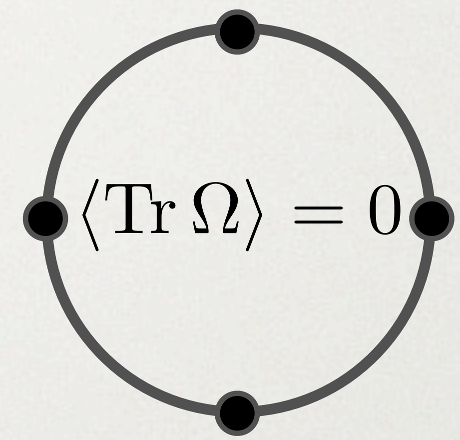
When does equivalence hold?

Can check symmetry realization at weak coupling/small L

Gluons give attractive interaction between eigenvalues

Adjoint fermions give repulsive interaction between eigenvalues





Center symmetry **broken**

Center symmetry **preserved**

Pure YM, YM + adjoint fermions with AP BCs

YM + adjoint fermions with P BCs

For larger L, support from lattice simulations for center preservation

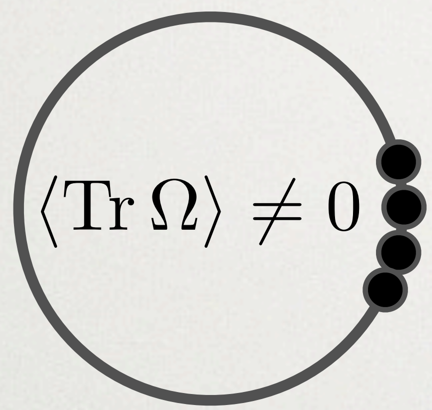
Bringoltz, Sharpe 2011;
Hietanen,Narayanan 2010

When does equivalence hold?

Can check symmetry realization at weak coupling/small L

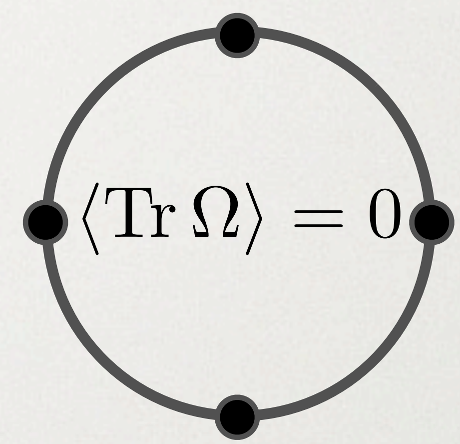
Gluons give attractive interaction between eigenvalues

Adjoint fermions give repulsive interaction between eigenvalues



Center symmetry
broken

Pure YM, YM + adjoint fermions with AP BCs



Center symmetry
preserved

YM + adjoint fermions with P BCs

Unsal, Kovtun, Yaffe 2007

For larger L, support from lattice simulations for center preservation

Bringoltz, Sharpe 2011;
Hietanen,Narayanan 2010

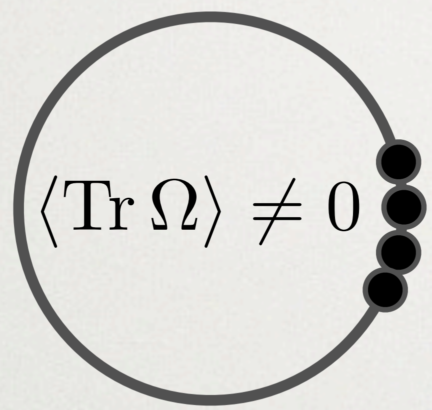
When does equivalence hold?

Can check symmetry realization at weak coupling/small L

Gluons give attractive interaction between eigenvalues

Adjoint fermions give repulsive interaction between eigenvalues

Unsal, Kovtun, Yaffe 2007



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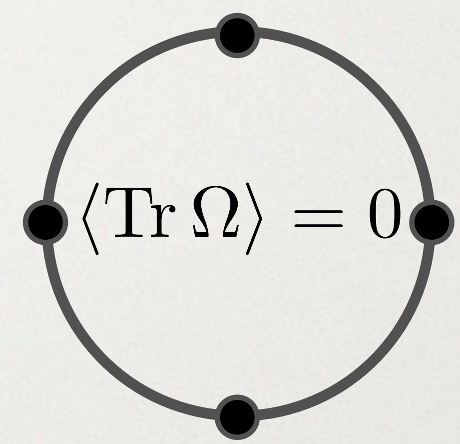

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AC, Dorigoni 2012

3D YM+1 adjoint Majorana fermion

$$\mathcal{L}_{3D} = \mathrm{tr}\left[\frac{-1}{2g^2}F_{\mu\nu}^2 + \bar{\psi}(\not{D} - M)\psi\right]$$

$$\mathcal{L}_{2D} = ?$$

$M = 0$

$\Rightarrow \mathcal{N} = 1$

spontaneously broken...

Need a deconstruction/discretization.

For gauge fields, usual lattice gauge theory setup. Fermions are subtle.

AC, Dorigoni 2012

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$$\boxed{\begin{array}{c} M = 0 \\ \Rightarrow \mathcal{N} = 1 \\ \text{spontaneously} \\ \text{broken...} \end{array}}$$

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Naive discretizations give fermion doubling.

Keep it simple: use Wilson fermions to kill doublers ~~SUSY~~

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kinetic term Wilson term

Typical `anisotropic lattice' subtlety: need to tune `c' to get Lorentz right..

AC, Dorigoni 2012

Small L check: bosonic sector

Looks a bit baroque. How is 3D physics encoded in 2D theory?

Consider L~1/N large N limit: theory becomes weakly coupled.

Spectrum in center-symmetric phase: adjoint Higgs mechanism...

W bosons: $$p_0^2 = p_1^2 + \frac{4}{a^2}\sin^2\left(\frac{\pi(n-m)}{N}\right)$$

n, m: color indices

Is this a 3D dispersion relation? Yes, a discretized one.

$$p_0^2 = p_1^2 + \frac{4}{a^2}\sin^2\left(\frac{ap_2}{2}\right), p_2 \in (-\pi/a, \pi/a]$$

Extra dimension encoded in color space.

AC, Dorigoni 2012

Small L check: bosonic sector

To see 3rd dimension emerge, look at e.g. bosonic contribution to vacuum energy

$$E_{\rm b} = \sum_{n,m}^{N_c} \int dp_1 \sqrt{p_1^2 + \frac{4}{a^2}\sin^2\left(\frac{\pi(n-m)}{N}\right)}$$

Continuum limit: small sine, which is small $p_2 a$

Write $N = N_c\Gamma$, then $p_2 = \frac{2\pi}{\Gamma a}\frac{n-m}{N_c} = \frac{2\pi}{L}\frac{n-m}{N_c}$

$$E_{\rm b} \to \sum_k \int dp_1 \sqrt{p_1^2 + \left(\frac{2\pi k}{N_c L}\right)^2}$$

$\boxed{\text{Looks like theory on a circle of size } N_c L}$

Note: integral over extra dimension coming from color trace

AC, Dorigoni 2012

Small L check: fermionic sector

Do the same for the fermions. As always, a bit more complicated.

$$p_0^2 = p_1^2 + \frac{c^2}{a^2}\sin^2\left(2\times\frac{\pi(n-m)}{N}\right) + \left[m + \frac{2rc}{a}\sin^2\left(\frac{2\pi(n-m)}{N}\right)\right]^2$$

This is the expected dispersion relation for a 3D fermion on a lattice.

Continuum limit is small sine limit, but note factor of 2 in argument

If r=0, both corners of Brilloin zone contribute: fermion doubling!

Finite r gives doubler mode mass m~1/a

For small L, can see c = 1 at order to which we're working

m = 0, r > 0: perturbative cancellation of vacuum energy in continuum limit

SUSY

For small L large N limit, can see equivalence explicitly.

What is all this good for?

In general, orbifold equivalences relate strongly coupled theories to other strongly coupled theories.

But some strongly coupled theories are more tractable than others!

Ex: ASV equivalence links a non-SUSY theory to SUSY theory for $N_f=1$

Ex: use equivalences to dodge fermion sign problem in finite-density lattice QCD

AC, Hanada, Robles-Llana

Rest of the talk: use large N 2D/3D equivalence to bosonize 3D QCD(Adj) using known 2D techniques

Bosonization has been very useful in study of 2D theories.
Ex: allows simple calculation of string tensions in 2D gauge theories

Having a working version for 3D theories might be useful.

What is bosonization?

$$Z[J] = \int \mathcal{D}B\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\;\; e^{i\int d^Dx\,\left(\bar{\psi}F(B,J)\psi + i\mathcal{L}(B,J)\right)}$$

$$\Downarrow$$

$$Z[J] = \int \mathcal{D}B\,\det(F[B,J])\;e^{iS(B,J)}$$

$$= \int \mathcal{D}B\,e^{i\tilde{S}_B(B,J)}$$

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This is **not** what is usually meant by bosonization.

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$$\Downarrow$$

$$\begin{aligned} Z[J] &= \int \mathcal{D}B\,\det(F[B,J])\;e^{iS(B,J)} \\ &= \int \mathcal{D}B\,e^{i\tilde{S}_B(B,J)} \\ &= \int \mathcal{D}B\,\underline{\mathcal{D}B'\,e^{iS'_B(B,B',J)}}. \end{aligned}$$

local action

In 2D, quite general recipes for bosonization known

For D>2, bosonization possible only in a very few special cases

Ex: QCD vs Chiral Lagrangian

But we have 2D/3D large N equivalence...

Bosonization for 3D QCD(Adj) at large N

$\boxed{\text{QCD(Adj) on } R^3}$

AC, Dorigoni 2012

Bosonization for 3D QCD(Adj) at large N

AC, Dorigoni 2012

Volume independence

$\boxed{\text{QCD(Adj) on } R^3}$

$\boxed{\text{3D QCD(Adj) on } R^2 x S^1}$

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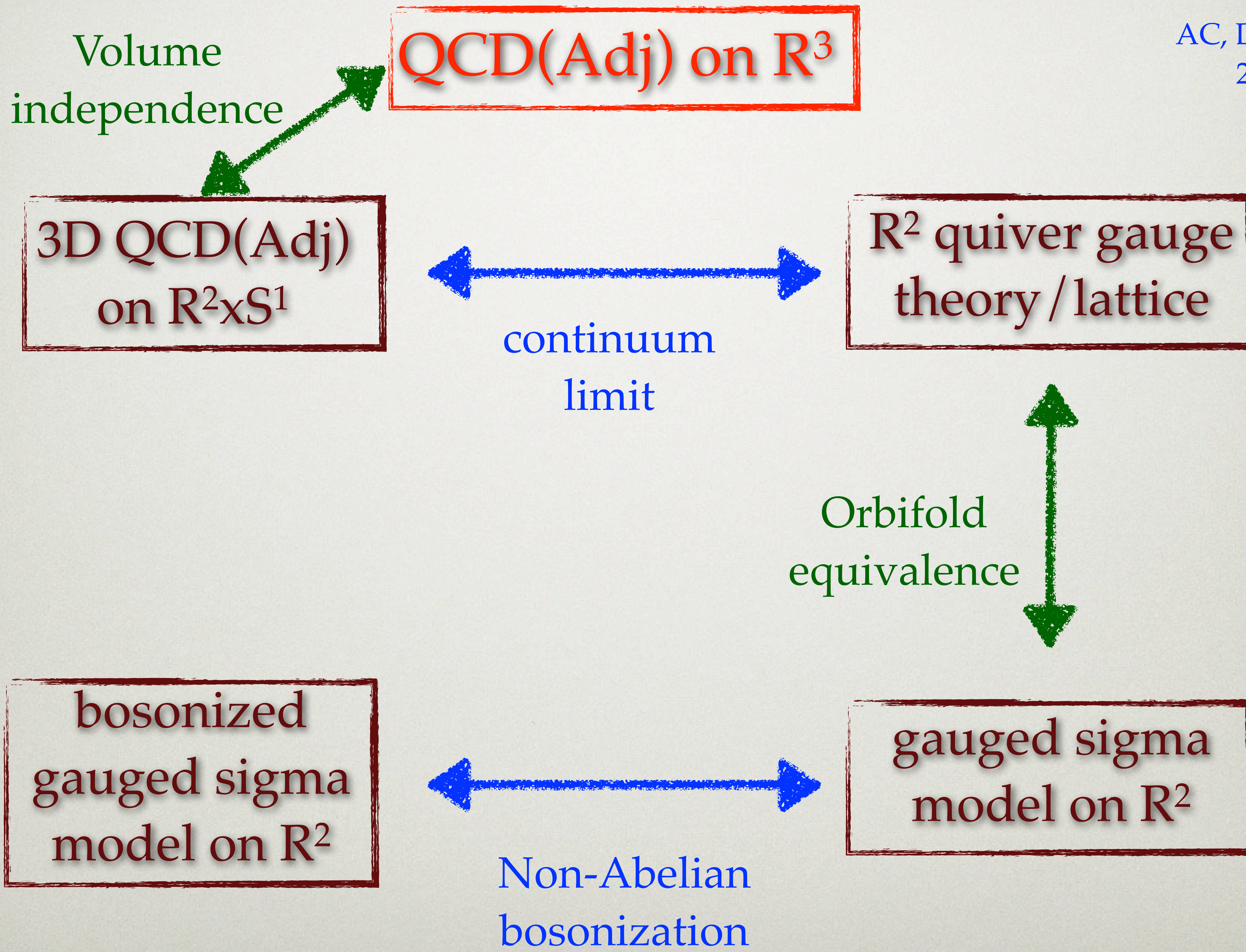
AC, Dorigoni 2012



Bosonization for 3D QCD(Adj) at large N

AC, Dorigoni 2012

QCD(Adj) on R^3

Volume independence

3D QCD(Adj) on $R^2 x S^1$

continuum limit

R^2 quiver gauge theory / lattice

Orbifold equivalence

bosonized gauged sigma model on R^2

Non-Abelian bosonization

gauged sigma model on R^2

Non-Abelian Bosonization

The fermionic 2D theory:

$$\mathcal{L}_{2D} = \mathrm{tr}\left[-\frac{1}{2g_2^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{a^2 g_2^2}|D_\mu\phi|^2 + \bar{\psi}\left(i\not{D} - m\right)\psi \right.$$
$$\left. - \frac{c}{2a}[\bar{\psi},\phi]\gamma_*\{\psi,\phi^\dagger\} + \frac{rc}{2a}[\bar{\psi},\phi][\psi,\phi^\dagger]\right]$$

How do we bosonize this fermionic action?

Use standard approach of Polyakov-Wiegmann, Quevedo-Burgess, ...

Non-Abelian Bosonization

Witten; Polyakov & Wiegmann; Quevedo & Burgess

$$Z[A_\mu] = \int \mathcal{D}\psi\, e^{i\int d^2x\, \mathrm{tr}\, \bar{\psi} \not{D}(A_\mu)\psi}$$

$$= \det[C(i\gamma^\alpha\partial_\alpha + A_\mu)]^{1/2}$$

$$= \exp i\, [\mathrm{tr}\ \log W(A_\mu)]$$

How to rewrite W as a local action?

$$\int d^2x\, [\bar{\psi}^i \not{\partial}\psi_i + A^a_\mu \bar{\psi}^j \underbrace{T^a_{ij}\gamma^\mu\psi_j}_{J^a_\pm}]$$

$$[J^a(x), J^b(y)] = i f^{abc} J_c \delta(x-y) + \frac{k}{2\pi}\delta'(x-y)$$

Bosonized currents:

$$J_+ = \frac{ik}{2\pi} g^{-1}\partial_+ g$$

$$J_- = -\frac{ik}{2\pi} g^{-1}\partial_- g$$

Non-Abelian Bosonization

Witten; Polyakov & Wiegmann; Quevedo & Burgess

For $N_f = 1$ adjoint fermions coupled to SU(N) gauge fields, need k = N

$$g \in SU(N)$$

Dynamics of g described by famous Wess-Zumino-Witten action

$$S_{\rm WZW} = \frac{N}{8\pi}\left[\int d^2x\,\mathrm{tr}\,(\partial_\mu g^{-1}\partial_\mu g) + \frac{2}{3}\int_B d^2x\,d\xi\,\epsilon^{ABC}\,\mathrm{tr}\,(g^{-1}\partial_A g g^{-1}\partial_B g g^{-1}\partial_C g)\right]$$

Coupling to gauge fields given by gauging:

$$Z(A) = \int dg \exp\left(\frac{iN}{8\pi}\int d^2x\,\mathrm{tr}\,(D_\mu g^{-1} D^\mu g) + i\,N\tilde{\Gamma}(g, A)\right)$$

Gauged WZW term

Non-Abelian Bosonization

AC, Dorigoni 2012

What about bosonized versions of the Yukawa couplings?

Use symmetries to work it out, by using spurion analysis

Fermion action invariant under $SU(N)_L \times SU(N)_R \times \mathbb{Z}_N$

Generic single-trace fermion bilinear with spurion fields A, B

$$\mathrm{tr}\,\bar{\psi} A \psi B + h.c.$$

Transformation Properties

Symmetry	ψ_L	ψ_R	ϕ	g
$SU(N_c)_V$	$V\psi_L V^\dagger$	$V\psi_R V^\dagger$	$V\phi V^\dagger$	$VgV^\dagger$
$SU(N_c)_A$	$A\psi_L A^\dagger$	$A^\dagger\psi_R A$	ϕ	AgA
$SU(N_c)_L$	$L\psi_L L^\dagger$	ψ_R	ϕ	Lg
$SU(N_c)_R$	ψ_L	$R\psi_R R^\dagger$	ϕ	$gR^\dagger$
$\mathbb{Z}_{N_c}$	ψ_L	ψ_R	$\omega\phi$	g

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Other conceivable term ruled out by spurion-sector phase symmetry

$$\mu' \left[\mathrm{tr}\,(gAgB^\dagger) + h.c.\right]$$

Numerical value of μ scheme dependent, undetermined by procedure

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Non-Abelian Bosonization: dictionary

Bosonization Dictionary

J^a_+	$\mathrm{tr}\,\bar{\psi}\gamma_+ t^a \psi$	$\frac{iN_c}{2\pi}\mathrm{tr}\, g^\dagger \partial_+ g\, t^a$
J^a_-	$\mathrm{tr}\,\bar{\psi}\gamma_- t^a \psi$	$\frac{iN_c}{2\pi}\mathrm{tr}\, \partial_- g^\dagger g\, t^a$
$\mathcal{O}_{\mathrm{Yukawa}}$	$\mathrm{tr}\,\bar{\psi} A \psi B + \mathrm{h.c.}$	$\mu\left[\mathrm{tr}\,(gB)\mathrm{tr}\,(g^\dagger A) + h.c.\right]$

What's the deal with scheme dependence?

Bosonization involves doing the integral over fermions, rewriting resulting determinant in terms of a local action.

Relation between currents in bosonic and fermionic language constrained by Ward identities, no room for scheme dependence

For e.g. mass terms, there's always a `scheme-dependent' multiplicative parameter in bosonization dictionary

Same goes for our Yukawa terms

Non-Abelian Bosonization: dictionary

Bosonization Dictionary

J_+^a	$\mathrm{tr}\,\bar\psi\gamma_+ t^a\psi$	$\frac{iN_c}{2\pi}\mathrm{tr}\,g^\dagger\partial_+ g\,t^a$
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Can now write the bosonized action. Simplest form for r=1:

$$\boxed{\begin{aligned} S^b_{2D} = \int d^2x \Big\{ &\mathrm{tr}\left(\frac{-1}{2g_2^2}F^2 + \frac{1}{a^2g_2^2}|D_\mu\phi|^2 + \frac{N}{8\pi}|D_\mu g|^2\right) \\ &+\tilde m^2\,\mathrm{tr}\ g\,\mathrm{tr}\,g^\dagger - \frac{\tilde c}{a^2}\mathrm{tr}\,(g\phi)\mathrm{tr}\,(g^\dagger\phi^\dagger)\Big\} + N\tilde\Gamma(g,A)\end{aligned}}$$

$\tilde m, \tilde c$ related to m, c in fermionic theory, but relation is scheme dependent.

3D vs 2D

Started in 3D with

$$\mathcal{L}_{3D} = \mathrm{tr}\,\left[\frac{-1}{2g^2}F_{\mu\nu}^2 + \bar{\psi}(\not{D} - M)\psi\right]$$

Standard lore: there shouldn't be an equivalent local 3D bosonic action

Indeed, found bosonized action, but it is **local in 2D, not 3D**.

$$\boxed{\begin{aligned} S_{2D}^b = \int d^2x \Big\{ &\mathrm{tr}\,\left(\frac{-1}{2g_2^2}F^2 + \frac{1}{a^2 g_2^2}|D_\mu \phi|^2 + \frac{N}{8\pi}|D_\mu g|^2\right) \\ &+\tilde{m}^2\,\mathrm{tr}\; g\,\mathrm{tr}\, g^\dagger - \frac{\tilde{c}}{a^2}\mathrm{tr}\,(g\phi)\mathrm{tr}\,(g^\dagger\phi^\dagger)\Big\} + N\tilde{\Gamma}(g,A) \end{aligned}}$$

Note appearance of double-trace terms, vs idea that color traces give rise to spatial integrals in orbifold projections

Would be nice to understand volume-expanding projection to 3D, but expect result to be non-local

Conclusions

Modern view of volume independence as orbifold equivalence yields large N equivalences between gauge theories in different number of dimensions

Working version involves QCD(Adj), equivalent at large L & N to QCD(AS)

We understand lower D theories better than higher D theories; can we learn something by reducing QCD(Adj) to e.g. 0+1D QM?

Focused on large N map between 3D QCD(Adj) and 2D theory

Large N equivalence implies possibility of bosonization of 3D theory!

Lots left to do!

What can we learn using the bosonized description?

What about 4D to 2D? Argyres, Shapere, Unsal

Can we go from bosonized 2D theory back to 3D directly?

Volume independence is an amazing feature of large N. It has to be useful for something...