Logarithmic CFT and the Verlinde formula

Def
- A CFT is called rational if it has only finitely many irreducible modules and if every module is completely reducible
- A CFT is called logarithmic if at least one module is not completely reducible.
- A logarithmic CFT is called logarithmic rational if it has only finitely many irreducible modules.

Examples:
- Logarithmic minimal models
- Admissible but non-integer level WZW theories
- WZW theories of Lie supergroups.

Applications:
- AdS_3 / CFT_2 correspondence
- Statistical Physics (Percolation, ...

1. Indecomposability and logarithmic singularities

Consider a CFT with Virasoro field

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \quad ,$$

$$T(z)\, T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}$$

and primary field ϕ ,

$$T(z)\, \phi(w) \sim \frac{h\,\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{(z-w)}$$

modes act as

$$[L_n, \phi(w)] = \operatorname{Res}_z T(z)\, \phi(w)\, z^{n+1}\, dz$$

e.g.

$$[L_{-1}, \phi(w)] = \partial\phi(w)$$

$$[L_0, \phi(w)] = h\,\phi(w) + w\,\partial\phi(w)$$

$$[L_1, \phi(w)] = 2hw\phi(w) + w^2\,\partial\phi(w)$$

The vacuum is sl_2-invariant: $L_n |0\rangle = 0 \quad n=-1,0,1. \rightsquigarrow$

$$(\partial_z + \partial_w) \langle \phi(z) \phi(w) \rangle = 0$$

$$(z\partial_z + w\partial_w + 2h) \langle \phi(z)\phi(w)\rangle = 0$$

$$(z^2\partial_z + w^2\partial_w + 2h(z+w)) \langle \phi(z)\phi(w)\rangle = 0$$

$$\rightsquigarrow \quad \langle \phi(z)\phi(w)\rangle = \frac{A}{(z-w)^{2h}}$$

, A might be zero.

Consider a second field x with

$$T(z)\, x(w) \sim \frac{h\,x(w) + \phi(w)}{(z-w)^2} + \frac{\partial x(w)}{(z-w)} \rightsquigarrow$$

$$[L_{-1}, x(w)] = \partial x(w)$$

$$[L_0, x(w)] = h\,x(w) + w\,\partial x(w) + \phi(w)$$

$$[L_1, x(w)] = 2hw\, x(w) + w^2 \partial_w x(w) + 2w\,\phi(w)$$

$\rightsquigarrow$ inhomogeneous differential equations for

$\langle \phi(z)\phi(w)\rangle$, $\langle \phi(z)\, x(w)\rangle$, $\langle x(z)\, x(w)\rangle$

with solution

$$\langle \phi(z)\phi(w)\rangle = 0$$

$$\langle \phi(z)\, x(w)\rangle = \frac{B}{(z-w)^{2h}}$$

$$\langle x(z)\, x(w)\rangle = \frac{C - 2B\log(z-w)}{(z-w)^{2h}}$$

Summary

The fields ϕ and χ generate a Virasoro representation V that is indecomposable but reducible. ϕ is the primary field of a subrepresentation V_ϕ and the structure is summarized as

$$0 \to V_\phi \to V \to V/V_\phi \to 0$$

and leads to log-singularities in correlation functions.

Problem

What is the algebraic structure, e.g. correlation functions and fusion, in a logarithmic CFT?

Verlinde formula

Let $V = V_0$ be a ~~VOA with~~ rational CFT with modules V_i, $i \in I$ and characters $ch_i = tr_{V_i}(q^{L_0 - \frac{c}{24}})$, $q = e^{2\pi i \tau}$.

Characters carry a representation of $SL(2;\mathbb{Z})$, especially

$$ch_i(-\tfrac{1}{\tau}) = \sum_{j \in I} S_{ij}\, ch_j(\tau)$$

Define

$$N_{ij}{}^{k} = \sum_{l \in I} \frac{S_{il}\, S_{jl}\, \overline{S_{lk}}}{S_{0l}}$$

then

$$ch_i \times ch_j := \sum_{k \in I} N_{ij}{}^{k}\, ch_k = ch_{V_i \times V_j}$$

2. The free boson: The Verlinde formula in a non-rational non logarithmic CFT

Consider a free boson X with OPE

$$X(z)\, X(w) \sim \frac{1}{(z-w)^2}$$

and

Virasoro field $T(z) = \frac{1}{2} :X(z)\, X(z):$ of central charge $c = 1$.

Primary fields are ϕ_λ, $\lambda \in \mathbb{R}$ with

$$X(z)\, \phi_\lambda(w) \sim \frac{\lambda\, \phi_\lambda(w)}{(z-w)} \qquad T(z)\, \phi_\lambda(w) \sim \frac{\lambda^2/2\; \phi_\lambda(w)}{(z-w)^2} + \frac{\partial \phi_\lambda(w)}{(z-w)}$$

$$\phi_\lambda(z)\, \phi_\mu(w) \sim \frac{\phi_{\lambda+\mu}(w)}{(z-w)^{-\lambda\mu}} + \dots$$

and fusion ring

$$V_\lambda \times V_\mu = V_{\lambda+\mu} .$$

Characters:

$$\mathrm{ch}[V_\lambda] = y\, \mathrm{tr}_{V_\lambda}\left(q^{L_0 - \frac{1}{24}}\, z^{X_0}\right) = \frac{y\, z^\lambda\, q^{\lambda^2/2}}{\eta(q)}$$

$q = e^{2\pi i \tau}$, $z = e^{2\pi i \mu}$, $y = e^{2\pi i t}$.

$$\mathrm{ch}[V_\lambda]\left(t - \frac{u^2}{2\tau}, \frac{u}{\tau}, -\frac{1}{\tau}\right) = \int_{\mathbb{R}} \underbrace{e^{-2\pi i \lambda\mu}}_{S_{\lambda\mu}} \, \mathrm{ch}[V_\mu](t,u,\tau)\, d\mu$$

Properties of the S-matrix

- symmetric: $S_{\lambda\mu} = S_{\mu\lambda}$
- unitary: $$\int_{\mathbb{R}} S_{\lambda\mu}\, \bar{S}^t_{\mu\nu}\, d\mu = \int_{\mathbb{R}} e^{-2\pi i(\lambda-\nu)\mu}\, d\mu = \delta(\cancel{\mu}\lambda-\nu)$$
- continuum Verlinde formula:

$$N_{\lambda\mu}{}^{\nu} = \int_{\mathbb{R}} \frac{S_{\lambda\rho}\, S_{\mu\rho}\, \bar{S}_{\rho\nu}}{S_{0\rho}}\, d\rho$$

$$= \int_{\mathbb{R}} e^{-2\pi i(\lambda+\mu-\nu)\rho}\, d\rho = \delta(\lambda+\mu-\nu)$$

$$\rightsquigarrow \mathrm{ch}[V_\lambda] \times \mathrm{ch}[V_\mu] := \int_{\mathbb{R}} N_{\lambda\mu}{}^{\nu}\, \mathrm{ch}[V_\nu]\, d\nu = \mathrm{ch}[V_{\lambda+\mu}] = \mathrm{ch}[V_\lambda \times V_\mu]$$

The compact free boson:

"Fusion of a rational CFT from its non-rational ~~cousin~~ parent"

$\mathcal{F}_0 = \bigoplus_{j \in \mathbb{Z}} \mathcal{V}_{jr}$ is closed under fusion,

for $r^2 \in 2\mathbb{Z}$ all conformal dimensions are integers ⇝ extended algebra.

Modules are ($\lambda \in \frac{1}{r}\mathbb{Z}$)

$\mathcal{F}_\lambda = \bigoplus_{j \in \mathbb{Z}} \mathcal{V}_{\lambda + jr}$

with fusion rules

$\mathcal{F}_\lambda \times \mathcal{F}_\mu = \mathcal{F}_{\lambda+\mu}$

Fusion from the non-rational parent:

choose a representative of $\mathcal{F}_\lambda$, e.g. $\mathcal{V}_\lambda$ ⇝

$\mathcal{V}_\lambda \times \mathcal{F}_\mu = \mathcal{V}_\lambda \times \bigoplus_{j \in \mathbb{Z}} \mathcal{V}_{\mu + jr} = \bigoplus_{j \in \mathbb{Z}} \mathcal{V}_{\lambda+\mu+jr} = \mathcal{F}_{\lambda+\mu} = \mathcal{F}_\lambda \times \mathcal{F}_\mu$

Strategy:

Compute fusion in a non-rational theory and deduce the inherited fusion rules for the extended rational theory.

3. The Verlinde formula in a non-rational logarithmic CFT

The $\mathcal{M}(1,2)$ singlet algebra is strongly generated by a Virasoro field T of central charge $c=-2$ and a dimension 3 primary.

Standard modules $\mathcal{F}_\lambda$ are parameterized by $\lambda \in \mathbb{R}$ with character

$$\mathrm{ch}[\mathcal{F}_\lambda] = \frac{q^{(\lambda-\frac{1}{2})^2}}{\eta(q)}$$

$\mathcal{F}_\lambda$ irreducible $\Leftrightarrow$ $\lambda \notin \mathbb{Z}$, otherwise

$$0 \to \mathcal{M}_{r+1} \to \mathcal{F}_{r+1} \to \mathcal{M}_r \to 0$$

splicing:

$$\cdots \to \mathcal{F}_{r+3} \to \mathcal{F}_{r+2} \to \mathcal{F}_{r+1} \to \mathcal{M}_r \to 0$$

$$\rightsquigarrow \mathrm{ch}[\mathcal{M}_r] = \sum_{j=0}^{\infty} \mathrm{ch}[\mathcal{F}_{r+2j+1}] - \mathrm{ch}[\mathcal{F}_{r+2j+2}]$$

$$\rightsquigarrow \mathrm{ch}[\mathcal{M}_r](-\tfrac{1}{\tau}) = \sum_{j=0}^{\infty} (-1)^j \,\mathrm{ch}[\mathcal{F}_{r+j+1}](-\tfrac{1}{\tau})$$

$$= \sum_{j=0}^{\infty} (-1)^j \int_{\mathbb{R}} e^{-2\pi i (r+j+\frac{1}{2})(\mu-\frac{1}{2})} \,\mathrm{ch}[\mathcal{F}_\mu](\tau)\, d\mu$$

$$= \int_{\mathbb{R}} \frac{e^{-2\pi i (r+\frac{1}{2})(\mu-\frac{1}{2})}}{1+e^{-2\pi i(\mu-\frac{1}{2})}} \,\mathrm{ch}[\mathcal{F}_\mu](\tau)\, d\mu$$

$$= \int_{\mathbb{R}} \underbrace{\frac{e^{-2\pi i r(\mu-\frac{1}{2})}}{2\cos\pi(\mu-\frac{1}{2})}}_{\mathcal{S}_{r\mu}} \,\mathrm{ch}[\mathcal{F}_\mu](\tau)\, d\mu$$